THE THEORY OF STRUCTURES

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BY

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PREFACE TO THE FOURTH EDITION

This edition has been revised to include the developments in the theory and practice of structural engineering that have occurred since the last revision in 1928. During this period the Committee on Steel Column Research of the American Society of Civil Engineers completed its investigations and made a final report which resulted in the general adoption by American engineers of revised column formulas; the American Association of State Highway Officials issued revised specifications for highway bridges which are now in general use by public authorities; provisions for impact on railroad bridges were radically changed as a result of investigations conducted by the American Railway Engineering Association and others; the New York City building laws were materially modified with respect to loads, materials, and methods of computing, and other building laws are in process of amendment; and the theoretical treatment of framed bents was enriched by the addition of the moment-distribution method proposed by Prof. Hardy Cross. All these modifications in theory and practice have been given consideration in the revised text. In addition, some of the topics treated have been amplified and clarified.

A table giving the scaled weights of a considerable number of highway-bridge trusses, designed by the author and his associates and built in the last ten years, varying in length from 100 to 600 ft., including both simple and continuous spans of various widths, has been added. These data should be useful for highway-bridge engineers engaged in making preliminary designs and preliminary estimates of cost.

The revised text is based upon over forty years of teaching and practice, the latter involving responsibility for the design and supervision of construction of numerous bridges and other structures; hence, the theorems and methods presented in the book have been tested, not only in the classroom, but also by

application to actual structures. A thorough study under competent instructors of the principles given in the text and the solution of the accompanying problems will furnish an adequate foundation for engineering students wishing to engage in the fascinating field of structural engineering.

CHARLES M. SPOFFORD.

August, 1939.

PREFACE TO THE FIRST EDITION

The purpose of this book is to present in a thorough and logical manner the fundamental theories upon which the design of engineering structures is based and to illustrate their application by numerous examples. No attempt has been made to treat of the design of complete structures, but the design of the more important elements of which all structures are composed is fully considered.

The subject matter is confined almost entirely to the treatment of statically determined structures, it being the writer's purpose to deal with indeterminate cases in another volume; the commonly used approximate methods for some of the more ordinary types of indeterminate structures are, however, included.

While the theories presented are for the greater part only such as have been in common use for many years, the method of treatment frequently differs considerably from that found in other books. Special attention may be called to the early introduction of the influence line and to its use in deriving and illustrating analytical methods, as well as to the chapter upon deflections.

The author wishes particularly to acknowledge his indebtedness to Professor George F. Swain for the logical and inspiring instruction received from him as a student.

Charles M. Spofford.

July 18, 1911.

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THEORY OF STRUCTURES

CHAPTER I

OUTER AND INNER FORCES

1. Definitions.—A structure as defined in the "Century Dictionary" is "a production or piece of work artificially built up, or composed of parts joined together in some definite manner." As used in this book, however, its meaning will be restricted to a part or combination of parts constructed to hold *in equilibrium* definite forces, with special reference to bridges and buildings.

Structures may be either *statically determined* or *statically undetermined.* Statically determined structures are those in which the reactions and primary stresses can be computed by statics. Structures for which these functions cannot be obtained by statics belong to the second class.

A *bridge* is a structure crossing a natural or artificial obstacle, such as a river, ravine, street, or railroad. The term comprehends the substructure of masonry piers and abutments and the superstructure of wood, metal, or masonry. The superstructure may consist of simple beams supported at the ends directly on the masonry or, in the case of long spans, supported on crossbeams which are themselves supported at the ends by girders, trusses, or arches.[1] In the latter case the longitudinal beams are known as *stringers* and the crossbeams as *floor beams.* As a clear conception of the function of the stringers and floor beams is essential to the understanding of the matter that follows, the student is advised to study carefully Figs. 1, 2, 3, and 4 and to examine some of the bridges in his vicinity.

Deck bridges are those in which the floor is at the top of the main superstructure, as in the simple I-beam bridge shown in

[1] For a clear understanding of girders and trusses, see Figs. 1 to 4 and Arts. 60 and 78.

Fig. 2. Deck truss bridges sometimes have the floor supported directly on the top chords, but usually it is carried by stringers and floor beams, the floor beams being connected to the truss verticals.

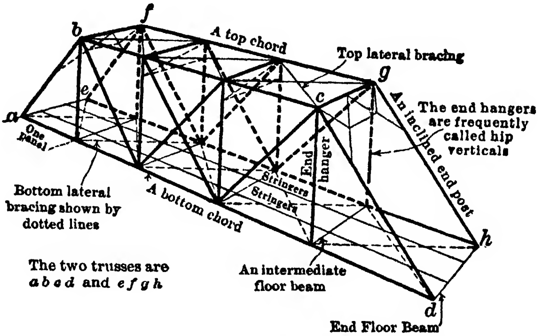


FIG. 1.—Framework of a through railroad truss bridge.

Half-through bridges are those having the greater part of the superstructure above the floor level but with insufficient depth to permit the use of overhead bracing. Lateral stability

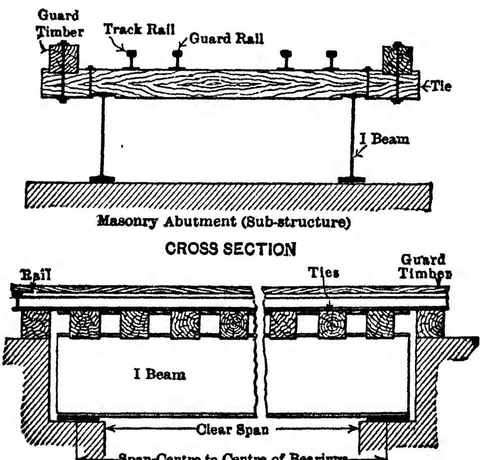


FIG. 2.—I-beam bridge for single-track railroad. (A deck structure.)

in such bridges may be obtained by the use of knee braces, as shown in Fig. 3. Shallow trusses called *pony* trusses are sometimes used for such bridges, the verticals being designed to trans-

mit to the floor beams sufficient lateral forces to carry the wind forces on the top chord and to prevent the top chord from buckling sideways.

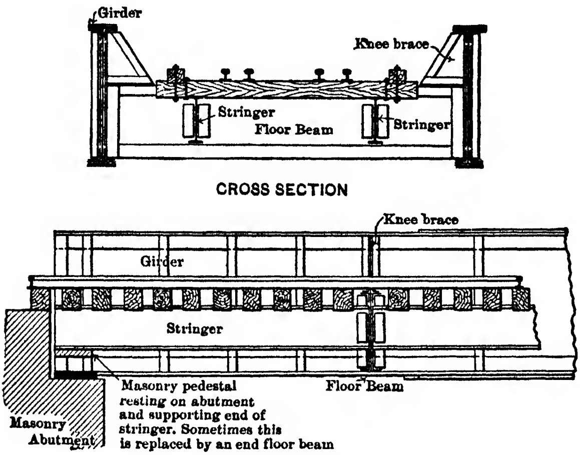


Fig. 3.—Half-through single-track plate-girder railroad bridge.

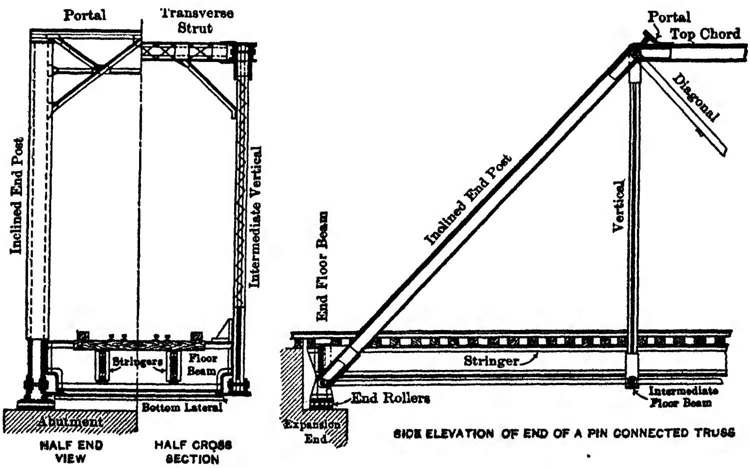


Fig. 4.—A single-track through railroad bridge.

Through bridges are those in which the greater part of the superstructure is above the floor level and in which overhead lateral bracing may be used between the trusses to obtain lateral stability. Such a bridge is shown in Fig. 4.

Whether a deck, through, or half-through bridge should be used for a given location depends upon the external conditions. In general, bridges of considerable span are built as through structures unless the approaches on either side are at a considerable elevation above the obstacle to be crossed. The solution of this question for a given case is usually obvious and will not be considered here.

2. Live and Dead Loads.—The forces to be considered may be divided into two classes: *outer* and *inner.* The outer forces consist of the applied loads and the resultant reactions and may be divided into two distinct types: live or moving loads and dead or quiescent loads. The inner forces are the molecular forces which are brought into action by the outer forces and hold them in equilibrium. The dead load includes the weight of the structure itself and all its permanent quiescent load such as the pavement on highway bridges; the rails and other track appurtenances on railroad bridges; and the floors, walls, roofs, and partitions in buildings. The live load consists of all forces that are applied intermittently. For bridges, these may be locomotives, cars, and other vehicles, pedestrians, and snow and wind; for buildings, people, snow, wind, and office furnishings and partitions; for dams and retaining walls, water, ice, and earth pressure. The word *kip* is in frequent use by structural engineers to represent a force of 1,000 lb., and the author has used the term where convenient. Changes in temperature also cause stress in certain forms of indeterminate structures such as arches.

In any structure in which the displacements of all points are proportionate to the loads causing them, the effect upon the structure of all the forces applied simultaneously is the sum of the effects due to the loads when applied separately. It follows from this principle, which is called the principle of *superposition,* that for such structures the effect of live and dead loads may be determined separately and then combined. This principle applies to all the structures treated in this book.

3. Outer Forces.—The determination of the intensity, distribution, and point of application of the outer forces is often difficult and requires mature judgment based upon extensive experience. For structures of great magnitude the question is particularly complicated and of vital importance; the design of such structures should never be attempted without a thorough

study of this problem in its relation to the structure in question. In the following articles, some of the difficulties in the way of an exact solution of the question will be presented and data given for use in the solution of the more common cases.

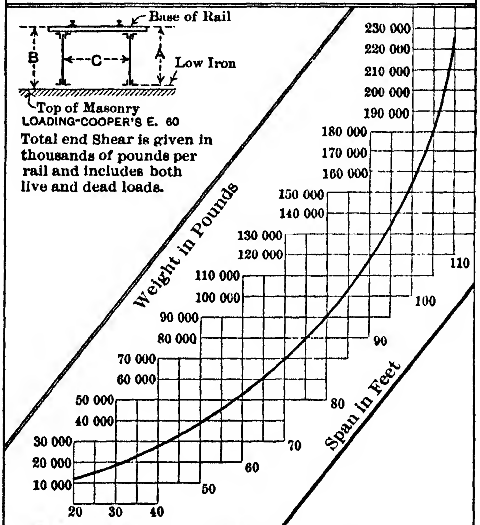
4. Weight of Structure.—It is impossible to determine accurately the weight of a given structure before the completion of the design. It is equally impossible to design the structure with precision until its weight is known. It is therefore necessary in all cases to make use of approximate methods of solution, first assuming the weight, next designing with the assumed data, then computing the weight and revising the design in the light of the new information thus obtained. For the more common types of structures, data accumulated by experience may be used by the designer and the first assumption made with sufficient accuracy to make revision unnecessary. For structures out of the ordinary, and particularly those in which the weight of the structure itself is a large percentage of the total load, several revisions are sometimes necessary, and a final computation of the weight after the completion of the detailed drawings and before the commencement of shopwork should never be omitted. The failure to do this for the huge Quebec bridge, which failed during erection in 1907, resulted in serious errors in the stresses for which the structure was designed.

In all cases the designer should first design completely the minor portions of the structure and determine their weight carefully so as to eliminate as much uncertainty as possible. For example, in the design of a bridge, the stringers, and the floor slab if a highway bridge, should first be figured and their weights carefully determined; the floor beams may then be designed, and finally the lateral bracing, considerable information being thus given as to the total weight of the bridge and the uncertainty being thrown into the main girders or trusses.

5. Weight of Railroad Bridges.—It is possible to make a more accurate preliminary estimate of the weight of such bridges than can be done for other types of important structure, since there is less variation in loads and other conditions. Current practice on first-class American railroads differs but little, and it is believed that the diagrams given in Figs. 5 to 10 give reasonable values for the *total weight of the steel in such structures.* The total weight of the bridge includes also the weight of the

ties, rails, and other accessories, which should be added to the values given in the diagrams. For the ordinary railroad-bridge floor with wooden ties, this weight may be taken, in the absence of a specific design, as 400 to 450 lb. per linear foot. For solid ballasted floors, this weight is of course much greater.

DECK PLATE GIRDER SPANS
WEIGHTS, DIMENSIONS, ETC.



Span	Total End S.	A	B	C
30 ft.	102.2	4′11″	4′11″	7′0″
35 "	112.6	5-3	5-3	"
40 "	124.2	5-10	5-10	"
45 "	135.6	6-4	6-4	"
50 "	145.7	6-10	6-10	"
55 "	155.1	7-1	7-1	"
60 "	165.2	7-4	7-11	"
65 "	181.2	8-6	9-1	8′0″
70 "	193.3	9-4	9-11	"
75 "	207.5	9-6	10-1	"
80 "	221.4	9-9	11-4	"
85 "	236.5	10-0	11-7	9-0
90 "	249.7	10-3	11-9	"
95 "	264.7	10-3	11-9	"
100 "	283.0	10-5	11-11	"
110 "	321.0	10-10	12-4	9-6

Erector's Note:
W = Total weight of one single track span. Steel only.

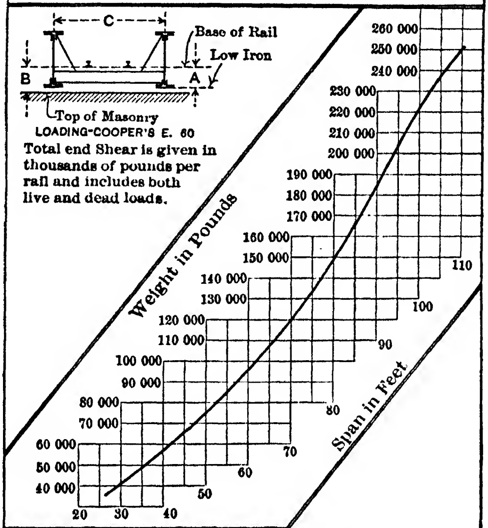
Span	Weight of one girder	Weight of cross frames, etc.
25′ to 60′	.45 W	.10 W
65′ " 110′	.465 W	.07 W

FIG. 5.

These diagrams were furnished by the Heath & Milligan Mfg. Company, paint and color makers, of Chicago, Ill., for whom they were prepared by consulting engineers connected with one of the large railroad systems of the country, and are for carbon-steel bridges designed for the typical locomotive shown in

Fig. 11. Where other loadings or alloy steel are to be used, these weights may be changed approximately in the ratio of the locomotive weights, or the allowable steel stresses.[1]

THROUGH PLATE GIRDER SPANS, I-BEAM FLOOR
WEIGHTS, DIMENSIONS, ETC.



Span	Total End Sh.	A	B	C
30 ft.	106.2	2′-2″	2′-2″	15′-0″
35 ,,	117.1	,,	,,	16-0
40 ,,	128.7	,,	,,	17-0
45 ,,	141.1	2-2½	2-2½	17-6
50 ,,	152.2	,,	,,	,,
55 ,,	162.1	2-3	2-9	,,
60 ,,	173.2	,,	,,	,,
65 ,,	186.2	,,	,,	,,
70 ,,	199.3	2-3½	3-9	,,
75 ,,	212.5	,,	3-10½	,,
80 ,,	226.4	,,	,,	,,
85 ,,	240.5	,,	,,	,,
90 ,,	257.4	,,	,,	,,
95 ,,	268.7	,,	,,	,,
100 ,,	284.0	2-4	,,	,,
110 ,,	306.0	2-4½	,,	,,

Erector's Note:
W = Total weight of one single track span with two light girders. Steel only.

Span	Weight of one light girder	Weight of one heavy girder	Weight of one floor system
30′ to 50′	.22 W	.39 W	.56 W
55′ ,, 80′	.27 W	.48 W	.47 W
85′ ,, 110′	.31 W	.57 W	.38 W

FIG. 6.

6. Approximate Truss Weights.—Bridges differing materially from those previously considered and for which other data are not available may be estimated by the following rule devised by Clarence W. Hudson, Consulting Engineer:

[1] For further data on weights of trusses the reader is referred to the paper by Dr. J. A. L. Waddell entitled Weights of Metal in Steel Trusses, published in *Trans. Am.Soc.C.E.*, Vol. 101, 1936.

Let L = maximum live stress in bottom chord.

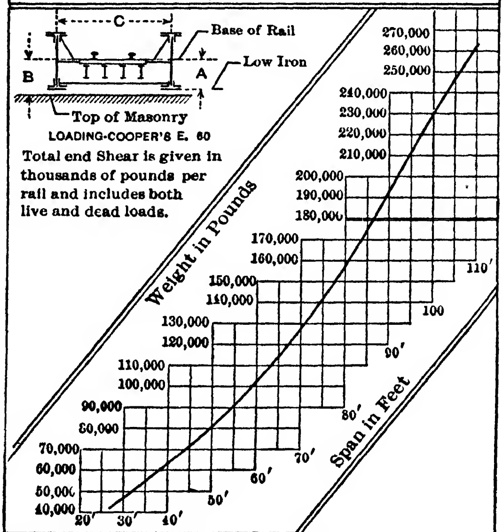
I = impact in member in which L occurs.

D_1 = dead stress in the same member due to known weight of floor.

D_2 = dead stress in same member due to weight of truss and bracing (guessed).

f_t = allowable unit stress in tension.

THROUGH PLATE GIRDER SPANS, STRINGER FLOOR.
WEIGHTS, DIMENSIONS, ETC.



Span	Total End Sh.	A	B	C
30 Ft.	108.7	3′3″	3′3″	15′6″
35 "	121 0	3′3½″	3′3½″	16′6″
40 "	133.2	"	"	17′6″
45 "	144.7	"	"	"
50 "	156.0	"	"	"
55 "	167.5	3′4″	3′11″	"
60 "	176 3	"	"	"
65 "	191 1	3′4½″	"	"
70 "	204.7	"	"	"
75 "	220.4	"	5′0″	"
80 "	232.8	"	"	"
85 "	248.6	"	"	"
90 "	262.3	3′5″	"	"
95 "	277 4	"	"	"
100 "	291 2	3′6″	"	"
110 "	319.6	"	"	"

Erector's Note:
W = Total weight of one single track span with two light girders. Steel only.

Span	Weight of one light girder.	Weight of one heavy girder.	Weight of one floor system.
30′ to 50′	.24 W	.42 W	.52 W
55′ to 80′	.25 W	.46 W	.50 W
85′ to 110′	.28 W	.51 W	.44 W

FIG. 7.

Let A_1 = area, sq. in., of member in which L occurs.

A_2 = area, sq. in., per linear unit of one truss.

W = weight per linear foot of one truss and its bracing.

Then,

$$A_1 = \frac{L + I + D_1 + D_2}{f_t},\ A_2 = 5A_1,$$

and

$$W = \frac{50}{3}A_1 \tag{1}$$

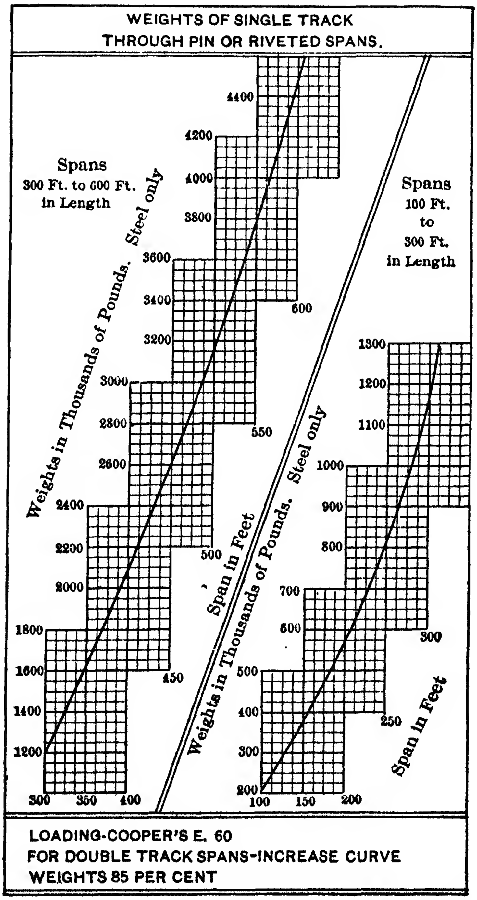


FIG. 8.

The above equation is based upon an allowance of $1.25A_1$ for the upper chord, $1.25A_1$ for the web members, A_1 for details, and

$0.5A_1$ for bracing. The weight of steel is used in round figures as 10 lb. per square inch of cross section for a bar 1 yd. in length.

This method is said to give a very close approximation to the actual weight.

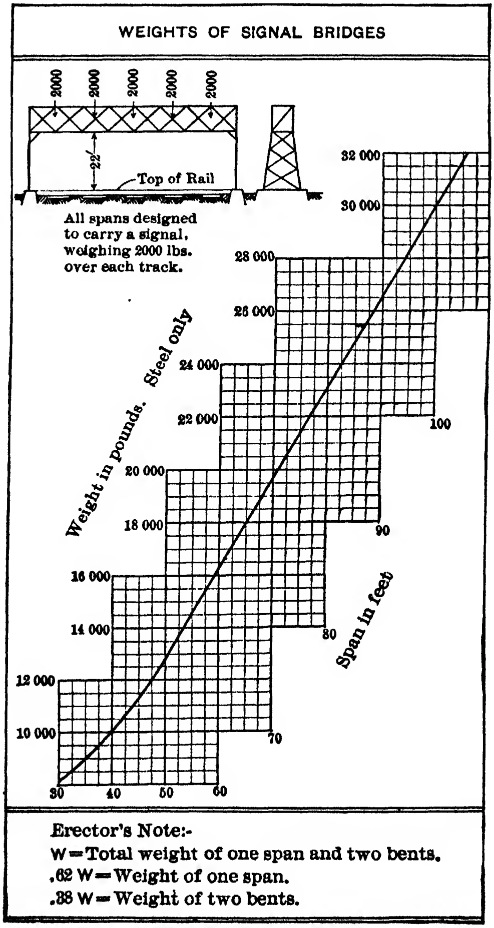


FIG. 9.

7. Weight of Highway Bridges.—In view of the wide variation in type of floor construction, width of roadway, number and width of sidewalks, type of wearing surface, and character of live loading, it is impossible to develop satisfactory formulas or diagrams for the weight of highway bridge trusses. The data at the end

of this article for weights of steel highway bridge trusses of moderate spans designed under the author's supervision should be helpful in making preliminary designs, as should also the table giving weights of roadway material.

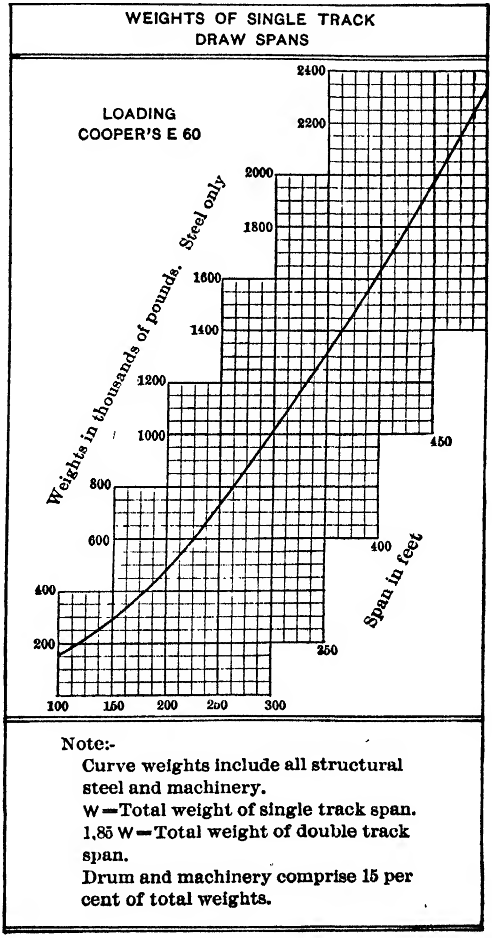


FIG. 10.

The loadings and the unit stresses used correspond to those given in the 1935 specifications of the American Association of State Highway Officials. Except those in classification *C*, all steel is carbon steel, and all weights are scale weights. In all the spans in classification *A*, the width of roadway between

curbs is 22 ft. 0 in. If sidewalks are used, they have a clear width of 5 ft. 0 in. The roadway has a 3-in. concrete wearing surface supported by a 6½-in. reinforced-concrete slab. The sidewalk consists of a 4½-in. concrete slab. Roadway and sidewalk are supported on carbon-steel stringers, floor beams, and brackets.

All spans in classification *B* have 24-ft. roadways and 3-ft. 3-in. sidewalks. The roadway has a 1½-in. concrete wearing

WEIGHT OF HIGHWAY STEEL BRIDGE TRUSSES

Type of bridge	Classification	Number of sidewalks	Live loading	Spans, feet	Weight per foot of bridge of both trusses, lateral bracing, and bearings, in pounds
End-supported through	*A*	1	*H*-20	139	1,060
End-supported through	*A*	1	*H*-15	184	1,210
End-supported through	*A*	1	*H*-15	220	1,250
End-supported through	*A*	None	*H*-15	210	1,060
End-supported through	*A*	None	*H*-15	260	1,330
Continuous through	*A*	1	*H*-15	1 at 168; 1 at 112	1,070
Continuous through	*A*	1	*H*-20	1 at 234; 2 at 146½	1,360
Continuous through	*A*	1	*H*-20	1 at 224; 2 at 140	1,330
End-supported deck	*B*	2	*H*-15	102	1,320
Continuous deck	*B*	2	*H*-15	1 at 125; 1 at 163	1,260
Continuous; partially through and partially deck	*B*	2	*H*-15	2 at 200; 1 at 275	1,740
End-supported deck	*C*	1	*H*-20	208	2,050
End-supported deck	*C*	1	*H*-20	240	2,120
Continuous; partially through and partially deck	*C*	1	*H*-20	2 at 396; 1 at 616	4,550
Continuous through	*D*	2	*H*-20	2 at 184	1,975
End-supported pony truss	*E*	None	*H*-15	120	1,350

surface supported on a reinforced-concrete slab having a depth of 7 in. The sidewalk has a 6-in. reinforced-concrete slab. Roadway and sidewalk are supported on carbon-steel stringers, floor beams, and brackets.

All spans in classification *C* have a 40-ft. roadway and a 6-ft. sidewalk. The roadway has a 2-in. bituminous surface supported on a 7-in. reinforced-concrete slab, the latter being composed of light-weight concrete. The sidewalk has a 7-in. reinforced-concrete slab, also of light-weight concrete. The truss members are nearly all of silicon steel, as are the stringers and the floor-beam flanges in the continuous spans. The allowable unit stresses for silicon steel were taken as 35 per cent in excess of those for carbon steel in these bridges. All other material is carbon steel.

WEIGHT OF ROADWAY MATERIAL

	Pounds per Cubic Foot
Hard (yellow) pine, 4 lb. per ft. b.m. (where protected by waterproofing and always dry. Otherwise, use 4½ lb.)	48
Creo-resinate yellow-pine paving blocks	65
Spruce and white pine, 2½ lb. per ft. b.m	30
Bricks, pressed and paving	150
Portland cement concrete	144 to 155
Tar concrete (base for asphalt walks, etc.)	125
Silician rock (Simpson Bros.)	140
Trinidad asphalt (Barber Asphalt Co.):	
Refined	74
As laid	140
Granolithic or artificial stone	150
	Pounds per Square Foot
Pavements (exclusive of sand cushion):	
6-in. granite block	80
4-in. brick	50
4-in. wood block (creo-resinate)	22
Roadway waterproofing, 1¼ in. thick (felt, roofing pitch, sand, and road pitch)	12
Buckle plates	10 to 20
Bituminous surfacing, 2 in. thick	20
Haydite concrete, 7 in. thick with reinforcing trusses and bituminous surfacing	90
Open grating with subpurlins	23
I-beam-lok, 4½ in. thick, filled with concrete	60

NOTE: b.m. = board measure.

The span in classification D has a 40-ft. roadway and a 5-ft. 9-in. sidewalk. The roadway has a 2-in. bituminous surface supported on a 7-in. reinforced-concrete slab. The sidewalk has a 6-in. reinforced-concrete slab. Many of the truss members are of silicon steel. All other members are of carbon steel.

The span in classification E has a 20-ft. roadway. The roadway has an 8-in. concrete slab, which includes a 2-in. concrete wearing surface. Roadway and curbs are supported on carbon-steel stringers. All steel is carbon steel, and all weights are actual scaled weights.

8. Weight of Roof Trusses.—The weight of roof trusses depends upon the span, distance apart of trusses, roof covering and roof pitch. The conditions are somewhat more uniform than for highway bridges, and formulas for approximate weights may be used with some degree of success.

The formula[1] that follows was deduced by N. Clifford Ricker from the weight of 50 roof trusses designed for spans 20 to 200 ft. in length. The distance apart of trusses varied from 10 to 30 ft. and the rise from one-tenth to one-fourth the span. The roofs were assumed to be covered with ⅞-in. wooden sheathing carrying painted tin. The snow load was taken as 20 lb. per horizontal square foot and wind as 30 lb. per square foot on a vertical plane, properly reduced to allow for pitch of roof. Trusses of yellow pine with steel verticals, of white pine with steel verticals, and entirely of steel were included.

Let W = weight of truss, lb. per square foot of horizontal projection of roof.

S = span, ft.

$$W = \frac{S}{25} + \frac{S^2}{6{,}000} \tag{2}$$

Steel trusses of shorter spans than 100 ft. probably weigh somewhat more than the value given by the formula, and white-pine trusses are somewhat lighter than those of yellow pine.

For roof trusses under other conditions, the value given by the formula may be modified or the designer may use his judgment.

[1] See Ricker, A Study of Roof Trusses, Univ. Ill. Eng. Expt. Sta., *Bull.* 16, 1907.

The following table gives the approximate weights per square foot of roof surface for some of the roof coverings in common use:

	Pounds per Square Foot
Pine shingles	2
Corrugated iron, without sheathing	1 to 3.75
Felt and asphalt, without sheathing	2
Felt and gravel, without sheathing	5 to 10
Slate, ¼ in. thick	9
Tin, without sheathing	0.75
Skylight glass, 3/16 to ½ in., including frames	4 to 10
White-pine sheathing, ⅞ in. thick	2
Yellow-pine sheathing, ⅞ in. thick	4
Tiles:	
Flat	15 to 20
Corrugated	8 to 10
On concrete slabs	30 to 35
Plastered ceiling	10 to 15

Weights of other materials may be obtained from dealers or from manufacturers' publications. The Architects' and Builders' Handbook, Kidder-Parker, ed. 18, 1931, John Wiley & Sons, Inc., New York, contains much detailed information.

9. Weight of Steel-frame Buildings.—The weight of such buildings is largely dependent upon the weight of walls, floors, partitions, and fireproofing, and these can be estimated in detail from the architect's plans and in many cases obtained from manufacturers' handbooks. The weight of the steel is, however, so variable that no attempt will be made to give values for it; but no difficulty need arise in designing, since the weight of the steel, in any given member, forms but a very small percentage of the load that it has to carry.

Concrete for building work may be made with cinders, broken stone, or gravel, and its weight may be taken as follows:

Cinder concrete, 112 lb. per cubic foot. Traprock or gravel concrete, 144 to 155 lb. per cubic foot.

For concrete reinforced with steel, add for preliminary computation 4 lb. per cubic foot to the foregoing weights.

In practice the minimum weight of a fireproof floor may be taken as 75 lb. per square foot except for office buildings where 10 lb. should be added to provide for movable partitions.

Fireproofing for columns or beams may be either of terra-cotta or of concrete. The thickness should be not less than

2 in. The weight per foot depends upon the size of the member to be protected.

10. Live Loads for Railroad Bridges.—It is possible to determine definitely the weights of the locomotives and cars used upon a given railroad. In consequence the actual live loads crossing a given bridge can be ascertained with considerable exactness, though it is necessary to make due allowance for the effect of high speed, irregularities in track, and other dynamic effects that do not occur when the loads are at rest. These dynamic forces are considered in Arts. 15 to 17 and will be neglected for the present.

In the design for a new bridge, it is also desirable to make due allowance for possible increase in weight of locomotives and cars, hence, the loads for which bridges are designed may be somewhat heavier than those which are in actual use at the time of construction, though the factor of safety (see Art. 19) provides to some extent for such increase.

As to the type and number of locomotives and character of train loading, American practice is fairly uniform.

Two combinations are usually considered:

a. Two consolidation locomotives followed by a uniform load per foot.

b. A pair of axles with loads somewhat heavier than those of the consolidation engine and no uniform load.

The former loading gives the maximum stresses for most cases, but the latter is sometimes the controlling factor for stringers, short beam spans, and minor truss members.

In designing, the effect of rails and ties in distributing the locomotive load is usually neglected, the wheel loads being considered as applied at points.

As the actual variation in wheel spacing and loads for locomotives of different makes is often slight, it has become in recent years the custom of most railroads to specify the typical locomotives, first proposed by the late Theodore Cooper, Consulting Engineer, and these loadings are given in the specifications for steel railway bridges, published in Chicago, by the American Railway Engineering Association. In these locomotives the distances between axles are in even feet and the wheel loads in even thousands of pounds. Although these loads and spaces may not represent actual cases, they agree closely with average locomo-

tives and are much simpler to deal with than loadings in odd hundreds of pounds and axle spacings in feet and inches. Moreover, the uncertain developments of the future and the unknown effect of impact make the use of such typical loads but little less accurate than the use of actual wheel loads.

Figure 11 shows Cooper's E_{60} locomotive, which is suitable for roads carrying heavy traffic. For other conditions, types known as E_{72}, E_{50}, E_{40}, E_{30}, etc., are used, these differing from the E_{60} type in weight only; for example, in the E_{40} type the driving-wheel axle loads are 40,000 lb. instead of 60,000 lb., and the other loads are proportionate. This has the advantage of allowing tables of moments and shears made for one type to be readily used for another type by multiplying by a simple ratio.

Loads are axle loads.

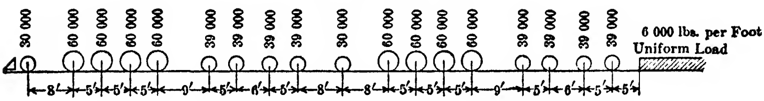


FIG. 11.—Diagram of Cooper's E_{60} standard loading.

Such tables are now incorporated in numerous handbooks and specifications.

For members receiving loads from more than one track, it is customary to use full live load on each of two tracks, one-half full load on third track, and one-fourth full load on fourth track, with all loads moving in either the same or opposite directions as may be required to give maximum stress.

Before leaving the subject a few words as to other methods of loadings are desirable. Some years ago there was considerable agitation in favor of adopting a uniform load, in order to simplify computation. The advantage is obvious to those who are familiar with such work, but the disadvantage is that in order to obtain properly proportioned trusses this load must necessarily vary for different spans and for different members in the same span. This offsets to a considerable extent the advantage gained. Moreover, the adoption of the standard locomotive loadings has further simplified the labor of computation for actual wheel loads, so that at the present time it is believed that for ordinary structures the advantage in using a uniform load is too small to consider. For complicated trusses a combination of the two methods is perhaps best, *viz.*, the use of

wheel concentrations for web members and of a uniform load for the chords, since the approximation for the chords by using a uniform load is less than for the web members.

11. Live Loads for Highway Bridges.—The live loads in common use in the United States for highway bridges of ordinary size are those recommended by the American Association of State Highway Officials and contained in their specifications.[1] The highway loadings in the current specifications for spans less than 60 ft. are divided into three classes designated as H_{20}, H_{15}, and H_{10}, corresponding, respectively, to 20-, 15-, and 10-ton trucks preceded and followed by trucks weighing three-fourths that of the designated truck. For spans over 60 ft. a uniform load plus a concentrated load is specified. For bridges carrying electric railroads, five classes of loadings varying from 60- to 20-ton cars are designated.

The sidewalk load per square foot given in these specifications varies with the width and loaded length of the sidewalk.

For bridges in large cities and in districts where heavy manufactured articles or heavy bulk materials are to be transported, special consideration should be given to the maximum loads to be used; in the case of especially long and wide spans the magnitude, extent, and method of combining the live loads on different lanes should be given careful consideration.

Snow and ice load must also be considered in computing the stresses in drawspans when open since such stresses may attain considerable importance. The magnitude of these loads in the vicinity of New York will probably not exceed 10 lb. per square foot. For fixed-span bridges, snow need not be taken into account since the maximum wagon and other loads will not occur simultaneously with the snow load.

12. Live Loads for Buildings.[2]—The live floor load to be used in designing a building depends on the purpose for which the building is to be used and, in the larger cities, is generally prescribed by law.

[1] Standard Specifications for Highway Bridges, American Association of State Highway Officials, Washington, D. C., 1935.

[2] For further information upon live loads for buildings the student is referred to the article by C. C. Schneider, *Trans. Am.Soc.C.E.*, Vol. LIV, 1905, pp. 371 *et seq.*, with the ensuing discussion; also to *New York City Record*, Vol. LXV, No. 19507, 1937.

The following may be adopted for cases not specified otherwise by the building laws controlling the building under consideration.

	Pounds per Square Foot
Spaces where groups of people are likely to assemble	100
Residences and sleeping quarters	40
Office floors, including corridors	50
Theaters and assembly halls with fixed seats	75
Halls without fixed seats, museums, ground floors and basements of hotels, stores, restaurants, shop and office buildings, fire escapes, etc	100
Loads for industrial and commercial purposes, including garages, shall be the maximum caused by the uses to which the structure is to be subjected and not less than the following:	
Display and sale of light merchandise	75
Factory work, wholesale stores, stock rooms in libraries	120
Garages for private passenger cars only	75
Garages for all types of cars	175
with provisions for the heaviest concentrated load to which the floor may be subjected and not less than 6,000 lb. concentrated	
Trucking spaces and driveways within the limit of a structure	175
with provisions for the heaviest concentrated load to which the building is to be subjected and not less than 24,000 lb.	
Reductions for columns, piers or walls, and foundations in storage buildings	15 % of full assumed live load
In other buildings	No reduction for roofs 15 % of live load on top floor 20 % of live load on next floor 25 % of live load on next floor On each successive lower floor increasing 5 % up to 50 %
Girders except in roofs carrying over 200 sq. ft. of floor area	Reduction of 15 %
Trusses and girders supporting columns and for footings	Reductions as permitted above

13. Wind Pressure.—Wind pressure is a subject upon which little exact information exists, although many experiments have been made and much study given to the subject by engineers and scientists. Among the unsettled questions are:

a. The relation between pressure and velocity.

b. The variation of pressure with size and shape of exposed plane surfaces.

c. The direction and intensity of pressure upon nonvertical surfaces.

d. The intensity of pressure upon nonplanar surfaces.

e. The total pressure upon a number of parallel bars or other members placed side by side.

f. The decrease of pressure upon leeward surfaces.

g. The lifting powers of the wind.

a. In comment upon this subject, it may be said that the pressure varies about as the square of the velocity and that the results given by different experimenters vary from

$$P = 0.005V^2 \qquad \text{to} \qquad P = 0.0032V^2$$

of which the latter value represents the result of unusually careful experiments by Stanton[1] upon the intensity of pressure on plates varying in size from 25 to 100 sq. ft. and is probably more nearly correct than the higher value. In these formulas

$$P^* = \text{pressure, lb. per square foot}$$
$$V\dagger = \text{velocity, miles per hour}$$

b. The variation of pressure with size and shape of exposed surface is important and is not well understood, although it is sure that the resultant pressure on a large surface may be taken as less per square foot than that on a small surface, since the

[1] See *Minutes Proc. Inst. C. E.*, Vols. CLVI 1903–04 and CLXXI 1907–08.

* Pressure specified for a bridge in the Philippines some time ago was as high as 160 lb. per square foot, according to Howard C. Baird, Consulting Engineer, New York, N. Y.

† For observed wind velocities in various portions of the United States and Canada, see Transmission Towers, American Bridge Company, New York, 1925.

Wind velocity at Blue Hill Observatory, near Boston, Mass., on Sept. 21, 1938, was reported as reaching a point as high as 186 miles per hour. The Boston Weather Bureau reported at this time a maximum velocity of 88 miles per hour.

maximum intensity of the wind is due to gusts of comparatively small cross section.

c. The pressure upon vertical plane surfaces may be taken as normal to the surface and equal in intensity to the assumed wind pressure. Upon surfaces that are not vertical, the presure is usually considered to be normal to the surface but lower in intensity than upon vertical surfaces. The variation in pressure with respect to the slope is not well understood and a number of empirical formulas are in use, among which may be noted the much used Duchemin formula

$$P_n = P\frac{2 \sin i}{1 + \sin^2 i} \tag{3}$$

and the Hutton formula

$$P_n = P(\sin i)^{(1.84 \cos i - 1)} \tag{4}$$

and also the empirical formula

$$P_n = P\frac{i}{45} \tag{5}$$

for i less than 45°

in which P_n = intensity of normal pressure upon the given surface

P = intensity of normal pressure upon the vertical surface

i = angle made by surface with the horizontal

Values as determined by the above equations are given on page 22 for an assumed wind pressure on a vertical surface of 30 lb. per square foot.

An equation is also sometimes deduced on the unwarranted assumption that the wind always blows in horizontal lines and that if the pressure is resolved into normal and tangential components the tangential component may be neglected.

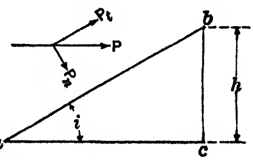


FIG. 12.

The demonstration is as follows:

Let the wind be assumed as blowing horizontally against the surface *ab* of height *bc* and making an angle i with the horizontal. Let the length of the surface be 1 ft., perpendicular to the plane of the paper (see Fig. 12).

Let P = intensity per square foot of the horizontal wind force on a vertical surface.

P_n = intensity per square foot of the normal force acting on surface ab.

P_t = intensity per square foot of the tangential force acting on surface ab.

The total horizontal pressure on surface ab then equals Ph.

The normal component of this pressure $= Ph \sin i$

The intensity of the normal component $= \dfrac{Ph \sin i}{ab}$

But

$$ab = \frac{h}{\sin i}.$$

Therefore

$$P_n = P \sin^2 i$$

This equation gives lower values than the empirical formulas, for it makes no allowance for the reduction in pressure on the leeward side that is known to exist, and that may in part be attributed to the influence of the tangential component, or for the suction on the windward side that laboratory experiments also indicate as probably existing. It should also be noted that the

TABLE FOR WIND PRESSURE

$i°$	$P\frac{i}{45}$	$P\frac{2 \sin i}{1 + \sin^2 i}$ Duchemin formula	$P(\sin i)^{(1.84 \cos i - 1)}$ Hutton formula
5	0	5.2	3.9
10	6.7	10.1	7.3
15	10.0	14.6	10.5
20	13.3	18.4	13.7
25	16.7	21.5	16.9
30	20.0	24.0	19.9
35	23.3	25.8	22.6
40	26.7	27.3	25.1
45	30.0	28.3	27.0
50	Above 45°, use 30 lb.	29.0	28.6
55	....	29.4	29.7
60		30.0	30.0
65	....	Above 60°, use 30 lb.	Above 60°, use 30 lb.
70			

NOTE: Tabular values are for P = 30 lb. per square foot.

wind does not blow uniformly in horizontal lines but may deviate considerably from this direction.

In the absence of further date upon this phase of wind pressure, it would seem wise to use one of the empirical formulas instead of the theoretical one, and formula (5) is frequently used by structural engineers in the United States.

The following theorem relating to the wind pressure upon plane surfaces is particularly useful in determining reactions upon roof trusses:

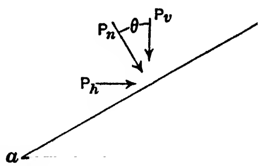


FIG. 13.

The *horizontal* component of the total normal pressure upon a plane surface equals the intensity of the normal pressure multiplied by the area of the *vertical* projection of the surface, and the *vertical* component of the total normal pressure equals that intensity multiplied by the area of the *horizontal* projection of the surface.

This theorem applies to any surface subjected to a uniformly distributed *normal* pressure and may be proved as follows:

Let P_n = intensity of the normal force acting upon surface ab.
P_h = horizontal component of total normal force upon ab.
P_v = vertical component of total normal force upon ab.
bc = vertical projection of ab.
ac = horizontal projection of ab.
θ = angle between ab and horizontal.

Assume surface ab to be of length unity perpendicular to paper.

Then total normal pressure on $ab = P_n \times ab$.

Hence,

$$P_h = P_n \times ab \times \sin\theta = P_n \times ab \times \frac{bc}{ab} = P_n \times bc$$

and

$$P_v = P_n \times ab \times \cos\theta = P_n \times ab \times \frac{ac}{ab} = P_n \times ac$$

Since bc and ac are the vertical and horizontal projections of ab, the theorem is proved.

It will be observed that if θ is greater than 45°, P_h will be larger than P_v; if less, P_v will be the larger. It is obvious that

this is correct since the steeper the roof the greater the horizontal component. When $\theta = 90°$, $P_v = 0$; and when $\theta = 0$, $P_h = 0$.

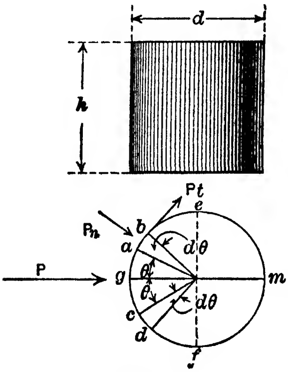


FIG. 14.

The application of this method to the solution of a problem is given in Art. 25.

d. The pressure upon nonplanar surfaces is important in the case of chimneys, stand pipes, and similar objects.

If the assumptions previously made in the deduction of formula (5) are also made for curved surfaces, the total pressure upon such surfaces can be easily figured. The following demonstration shows the solution for the case of a vertical cylinder.

Let P = intensity of pressure on a vertical plane.

P_n = intensity of pressure on a plane making an angle of $90° - \theta$ with the direction of the wind $= P \sin^2 (90° - \theta)$.

P_t = tangential component of pressure on same plane.

The normal pressure on the differential area ab subtended by the angle $d\theta = P \sin^2 (90° - \theta)\frac{hd}{2}d\theta$.

As the tangential component P_t is neglected by hypothesis and as the component of P_n acting upon surface ab in a direction parallel to ef is balanced by an equal and opposite component upon cd, the force tending to overturn the cylinder is the summation of the components of P_n parallel to gm and is given by the following expression:

$$\int_{-\pi/2}^{\pi/2} P \sin^2 (90° - \theta)h\frac{d}{2}d\theta \cos \theta = \frac{Phd}{2}\int_{-\pi/2}^{\pi/2} \cos^3 \theta d\theta$$

$$= \frac{Phd}{2}\int_{-\pi/2}^{\pi/2} \cos \theta \,(1 - \sin^2 \theta)d\theta = \frac{Phd}{2}\left(\frac{4}{3}\right) = \frac{2}{3}Phd$$

= two-thirds of the total pressure on a plane diametrical section.

The foregoing equation for the total pressure on a cylinder is apparently confirmed by experiments made in 1927 in the wind tunnel of the Massachusetts Institute of Technology by J. A.

Herlihy and R. S. Hatch upon a model of a chimney in the shape of a slightly tapered frustrum of a cone.

In a similar manner the pressures on a spherical or conical surface may be computed and are found to be as follows:

The horizontal component on an entire sphere equals half the pressure on a plane diametrical section. On a half sphere the horizontal component is one-half that on an entire sphere. The vertical component on either half of a sphere equals one-eighth of the pressure on its plane diametrical section. The point of application of the vertical and horizontal component on any portion of a spherical surface must be such that the resultant pressure passes through the center of the sphere.

For a cone having height h and radius of base r, the horizontal component $= \frac{2}{3} \frac{prh^3}{r^2 + h^2}$, acting at distance of $\frac{1}{3} h$ from base, and the vertical component $= \frac{\pi}{4} \frac{pr^2h^2}{r^2 + h^2}$, acting at distance of $\frac{16}{9} \frac{r_1}{\pi}$ from center line of base.

The foregoing values for spheres and cones lack experimental confirmation and should probably be modified by a constant, the value of which would have to be determined by experiment and might be either greater or less than unity.[1]

e. With respect to the total pressure upon a number of parallel bars placed side by side, it may be stated that experiments previously refered to indicate that the total pressure on a pair of circular plates placed $1\frac{1}{2}$ diameters apart is less than that on one plate, from which the conclusion is drawn that the pressure on the leeward plate is in a direction opposite to the current. When plates were placed 2.15 diameters apart, the resultant pressure on the two plates was found to equal that of a single plate and the shielding effect was found to be well maintained with wider spacing, since at a distance of 5 diameters the total pressure was only 1.78 that on a single plate.

f. The pressure upon the windward side of an exposed surface is a function of the density and velocity of the air currents. The pressure on the leeward side is also a function of the shape of the surface and has been shown by numerous experiments to be

[1] For further theoretical considerations of the wind pressure on spheres, see the paper entitled Aerodynamics of the Perisphere and Trylon at World's Fair, by Alexander Klemin, *Proc. Am. Soc. C. E.*, May, 1938.

less than the static pressure of the air current. The resultant total pressure upon a surface is in consequence a function not only of the direct pressure on the windward side but also of the pressure on the leeward side, which in turn is a function of the form of the surface. It is therefore doubtful that an algebrical formula can be deduced which will give the pressure on surfaces of varying shape with any considerable degree of precision.

g. In the case of a very rapid reduction of atmospheric pressure, as in a tornado, it is often observed that building roofs are lifted and walls blown outward. This phenomenon is due to the air in the building, which is under more or less restraint, changing pressure less rapidly than the outside air and thereby producing a difference in pressure. This lifting action doubtless occurs to a greater or less degree whenever the external pressure is reduced and should be guarded against by anchoring roofs securely to the walls.

The many uncertainties connected with wind pressure make worthless the attempts to specify with precision its magnitude and direction. In the absence of additional information and further theoretical studies there seems to be no reason for deviating from the rules now in common use for bridge design in the United States. These rules may be stated as follows:

The wind force shall be considered as a moving load acting in *any* horizontal direction. The wind force on the bridge shall be taken at 30 lb. per square foot of

1. 1½ times the vertical projection of the floor system and girders,
2. the vertical projection of all trusses, omitting the portion of leeward trusses shielded by floor system, but not less than 200 lb. per linear foot in the plane of the loaded chord or flange for railroad bridges and 300 lb. for highway bridges, and not less than 150 lb. per linear foot in the plane of the unloaded chord.

The wind force due to the live load shall be taken as 300 lb. per linear foot on trains and electric cars, applied 8 ft. above the top of the rail for trains and 6 ft. above the top of the rail for electric cars on highway bridges.

In case the wind stress, as computed in any member from the foregoing loadings, combined with the stress due to dead load, live

load, impact, and centrifugal force, is less than that due to a wind force of 50 lb. per square foot on the unloaded structure acting as a moving load combined with the dead stress and computed as given under (1) and (2), the latter loading shall be used.

On railroad bridges in addition to the wind forces a live concentrated horizontal force should be used, applied at the top of the rail in either direction at any point of the span and equal to 20,000 lb., to cover the effect of locomotive sidesway.

The allowable wind pressure on roofs and buildings in large cities is determined by building laws, extracts from which are given in Chap. XXI.

14. Snow Load.—The weight per foot of snow and ice varies greatly with climatic conditions. The following rule suggested by C. C. Schneider in the paper referred to in Art. 12 gives reasonable results for conditions similar to those existing in Boston and New York:

"Use for all slopes up to 20° with the horizontal 25 lb. per square foot of horizontal projection of roof. Reduce this value by one pound for each additional degree of slope up to 45°, above which no snow need by considered."

To determine the maximum stresses in a truss member, wind and snow must be properly combined. The following combinations may exist and should be considered:

Dead load with snow on both sides.

Dead load with snow on one side and wind on the other.

Dead load with ice at 10 lb. per square foot, properly reduced according to slope, on both sides, and wind on one side.

The maximum stress as determined by either of these combinations should be used.

For roofs, it is frequently the custom to combine the snow and wind loads by using one load sufficient to cover them both. The following loads are given in the Boston Building Laws:

Roofs with pitch 4 in. per foot or less, a vertical load of 40 lb. per square foot of horizontal projection applied either to half or whole roof. Roofs with pitch of more than 4 in. and not more than 8 in. per foot, a vertical load of 15 lb. per square foot of horizontal projection and a wind load of 10 lb. per sq. ft. of surface acting at right angles to one slope, these two loads to be taken as acting either separately or together. For roofs with pitch between 8 and 12 in. per foot, change above vertical load to 10 lb. and wind

load to 15 lb., and for pitch of more than 12 in., change the vertical load to 5 lb. and the wind load to 20 lb.

15. Centrifugal Force and Friction.—For railroad bridges on curves the effect of centrifugal force must be considered. This may be computed by the following formula:

$$C = 0.00117S^2D \tag{6}$$

where C = centrifugal force percentage of live load
S = speed, miles per hour
D = degree of curve

On trestle towers and similar structures the longitudinal thrust of the train must be considered. This may be taken as either

(*a*) 15 per cent of the total live load without impact due to braking

or

(*b*) 25 per cent of the weight on the driving wheels without impact due to accelerating.

These forces shall be taken on the track only and assumed to act at 6 ft. above top of rail.

16. Impact on Railroad Bridges.—It is easy to determine the weight per wheel applied to a railroad bridge by the locomotive or cars of a given train when at rest; but when in motion the effect of unbalanced locomotive drivers, roughness of track, flat, irregular, or eccentric wheels, rapidity of application, and centrifugal force induced by deflection of structure cannot be determined theoretically and has not yet been precisely determined by experiment. Nor is the distribution of these loads by rails and ties a matter that can be easily ascertained. In consequence the engineer is compelled either to use a low unit stress or to increase the live stresses by an allowance for "impact" sufficient to cover these uncertainties. The latter method is more scientific and is coming into general use. *Impact* is used in mechanics to mean the dynamic effect of a sudden applied load, but as used in bridge engineering it stands for the increased stress produced in a member not only by the rapid application of the load but also by the other causes just mentioned, and the term *coefficient of impact* is given to the factor by which the live stress must be multiplied to obtain the impact. No rational formula for determining this coefficient of impact has yet been

deduced, but several empirical formulas are in more or less common use.

It is proved in mechanics that a load when instantaneously applied to a bar produces a stress exactly double that caused by the same load when gradually applied. In the ordinary structure the maximum load is, however, never applied instantaneously, though in short railroad bridges the length of time required to produce maximum moment or shear is very small. In consequence, sudden application alone is never sufficient to double the live stresses as computed for quiescent loads. Many engineers, however, use for short spans a coefficient equal to unity, assuming that the effect of vibration and other uncertainties is balanced by the difference between the stress due to instantaneous application and that due to the very rapid but not instantaneous application caused by a railroad train. For longer spans the coefficient is generally reduced.

The method of determining the allowance for impact on railroad bridges,[1] as given in the 1935 specifications of the American Railway Engineering Association, is to divide the total impact into two parts:

a. That due to rolling of the live load from side to side, designated the *lurching effect* and assumed to increase the static load on one rail by 20 per cent and diminish it on the other rail by the same amount. If the rails are 5 ft. center to center and the trusses or girders are located a distance of S ft. apart, this is equivalent to increasing the truss or girder live panel loads by $100/S$.

b. The direct vertical effect:

With steam locomotives (hammer blow, track irregularities, and car impact), a percentage of the static live-load stress equal to:

For l less than 100 ft.,

$$100 - 0.60l \tag{7a}$$

For l 100 ft. or more,

$$\frac{1{,}800}{l - 40} + 10 \tag{7b}$$

[1] For full discussion of impact upon railroad bridges with experimental data, see paper by J. B. Hunley, Esq., in the *Am.R. R.Eng.Assoc., Bull.* 380, October, 1935.

With electric locomotives (track irregularities and car impact), a percentage of the static live-load stress equal to

$$\frac{360}{l} + 12.5 \tag{8}$$

l = length, feet, center to center of supports for stringers, longitudinal girders, and trusses (chords and other main members)

or

l = length of floor beams or transverse girders in feet, for floor beams, floor-beam hangers, subdiagonals of trusses, transverse girders, and supports for transverse girders

The impact shall not exceed 100 per cent of the static live load.

For members receiving load from more than one track, the impact percentage shall be applied to the static live load on the number of tracks shown below:

Load received from	
Two tracks:	
For l less than 175 ft..........	Full impact on 2 tracks
For l from 175 to 225 ft.......	Full impact on 1 track and a percentage of full impact on the other, as given by the formula $450 - 2l$
For l greater than 225 ft.......	Full impact on 1 track and none on the other
More than two tracks:	
For all values of l.............	Full impact on any 2 tracks

17. Impact on Highway Bridges and Buildings.—Allowance for impact upon these structures may usually be less than for railroad bridges.

The American Association of State Highway Officials specifies the following percentage of the live load to cover impact: $\frac{50}{l + 125}$, in which l is the length of structure loaded to produce maximum live stress in the member under consideration.

For buildings, it is customary to make no allowance for impact, except where moving cranes or other shock-producing machinery are used.

Before leaving the subject of impact, it should be noted that it is probable that the effect of impact upon wooden beams is less injurious than upon steel beams, owing to the greater elasticity of the wood, and that unit stresses for timber as generally specified are for use without impact.

18. Inner Forces.—The allowable working unit stresses for a given structure depend upon the material, the character of loading, the precision with which the stresses can be computed, and the uses to which the structure is to be put. If proper allowance for impact is made, the character of loading may, however, be neglected.

The following unit stresses represent good practice for ordinary structural steel structures, provided that *proper allowance for impact* is made.

WORKING STRESSES—STEEL

	Pounds per Square Inch
a. Structural and rivet steel	
Axial tension, structural steel, net section	18,000
Tension in extreme fibers of rolled shapes, girders, and built sections, subject to bending	18,000
Axial compression, gross section:	
For stiffeners of plate girders	18,000
For columns centrally loaded and with values of l/r not greater than 140:	
Riveted ends	$15,000 - \frac{1}{4}\frac{l^2}{r^2}$
Pin ends	$15,000 - \frac{1}{3}\frac{l^2}{r^2}$
l = length of member, in inches.	
r = least radius of gyration of member, in inches.	
Compression in extreme fibers of rolled shapes, girders, and built sections, subject to bending (for values of l/b not greater than 40)	$18,000 - 5\frac{l^2}{b^2}$
l = length, in inches, of unsupported flange between lateral connections or knee braces.	
b = flange width, in inches.	
Stress in extreme fibers of pins	27,000
Shear in plate girder webs, gross section	11,000
Shear in power-driven rivets[1] and pins	13,500
Shear in turned bolts and hand-driven rivets	11,000
Bearing on pins	24,000
Bearing on power-driven rivets,[1] milled stiffeners, and other steel parts in contact	27,000

[1] Rivets driven by pneumatically or electrically operated hammers are considered power driven.

Bearing on rocker pins	12,000
Bearing on turned bolts and hand-driven rivets	20,000
Bearing on expansion rollers and rockers, lb. per linear inch:	
For diameters up to 25 in	$\frac{p - 13,000}{20,000} 600d$
For diameters from 25 to 125 in	$\frac{p - 13,000}{20,000} 3,000\sqrt{d}$

d = diameter of roller or rocker, in inches
p = yield point in tension of the steel in the roller or the base, whichever is the lesser

b. Cast steel. For cast-steel shoes and pedestals, the allowable unit stresses in compression and bearing shall be the same as those for structural steel. Other allowable unit stresses shall be three-fourths of those for structural steel.

For alloy steels the foregoing values should be increased 33⅓ per cent for silicon steel and 50 per cent for nickel steel except for columns, shear in plate-girder webs, and bearing on expansion rollers and rockers. Allowable unit stresses for silicon and nickel steel are given in detail in the 1935 specifications for steel railway bridges of the American Railway Engineering Association, and these values may also be used for highway bridges.

WORKING STRESSES—TIMBER[1]

The strength of timber depends upon its species, moisture content, and freedom from knots and from other defects. The 1935 specifications of the American Association of Highway Engineers cover the effect of all these factors upon the strength of the timber and give working stresses for various grades and species for highway bridges that are identical with those recommended in the 1935 *Bulletin* of the American Railway Association for use in rail bridges.

It is impractical to include in this text the allowable stresses for all the various species and grades, but the following range covers the allowable stresses in pounds per square inch for structural beams and stringers with loads applied to the narrow face and for short columns for different grades of longleaf Southern pine:

Fiber stress in bending or tension	1,400 to 1,800
Maximum horizontal shear	100 to 200
Compression perpendicular to grain	380
Modulus of elasticity E	1,600,000

For solid columns the allowable unit stress per square inch in compression parallel to grain varies from 1,000 to 1,200 lb. for columns with a value of

[1] For further data on timber design, see Wood Structural Design Data, Vol. I, published by the National Lumber Manufacturers' Association, Washington, D. C., 1st ed., 1934.

l/d not in excess of 11 in which l equals length of column in inches, and d equals least side or diameter in inches. For columns of greater values of l/d the following equations are recommended in the Standard Specifications for Highway Bridges adopted by the American Association of State Highway officials, 1941, page 146, *et seq.*

For solid columns in which P/A equals or exceeds one-third of the stress permitted for short columns,

$$\frac{P}{A} = c\left[1 - \frac{1}{3}\left(\frac{l}{Kd}\right)^4\right]$$

in which c equals the allowable unit stress in a short column and

$$K = \frac{\pi}{2}\sqrt{\frac{E}{6c}} = 0.64\sqrt{\frac{E}{c}}$$

For long columns with $\frac{l}{d}$ greater than K but not to exceed 50, determine the safe load by the following formula:

$$\frac{P}{A} = \frac{\pi^2 E}{36(l/d)^2}$$

The following values for bearing on masonry represent good practice:

WORKING STRESSES—BEARING ON MASONRY

	Pounds per Square Inch
Granite	800
Portland cement concrete	600
Sandstone and limestone	400

NOTE: Live load to be increased to allow for impact.

19. Factor of Safety.—The unit stresses given in the previous article are all much less than the breaking strength of the material, the ratio between the breaking strength and the allowable unit stresses being known as the *factor of safety*. The necessity for using such a low value is due to the following facts:

1. Material cannot be stressed with safety above the elastic limit, which is generally not more than one-half the breaking strength.
2. The magnitude, point of application, and distribution of the live loads as well as the allowance for impact are approximate.
3. The material is variable in quality and may be injured in fabrication.
4. The effect of changing conditions cannot be predicted. This applies to character and amount of loading and to deterioration of material through rust or rot.

5. The common theories give primary stresses only and neglect the secondary stresses due to distortion of the structure, these additional stresses being sometimes of considerable importance.

Problems

1. Using the Hutton formula, determine the horizontal and vertical components of the total wind force on the side L_0U_2 of roof truss A, for an assumed wind pressure of 30 lb. per square foot on a vertical surface. Direction of wind shown by arrows.

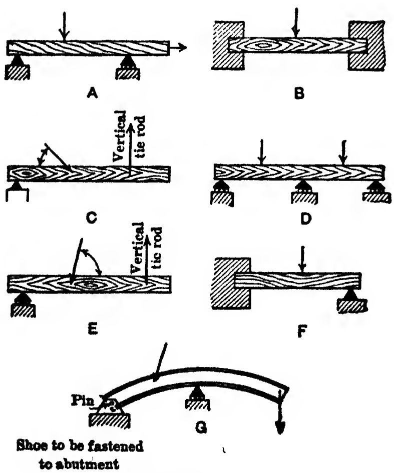
2. Determine the horizontal wind pressure per square foot required to tip the car, assuming the direction of the wind to be perpendicular to its side. Neglect wind pressure on trucks.

3. Compute the impact in pounds and the total stress to be used in design for a railroad-bridge truss member subjected to the following conditions:

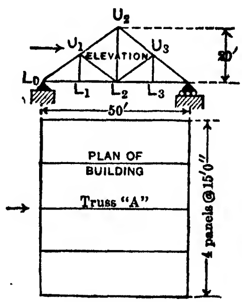
Distance center to center of trusses, 18 ft.
Maximum live stress, 200,000 lb.
Dead stress, 100,000 lb.
Loaded length when stress is a maximum, 100 ft.

4. State whether each of the beams shown is statically determined with respect to the outer forces, and give reasons. Assume that the magnitude and position of applied loads, and position of points of support are known in all cases.[1]

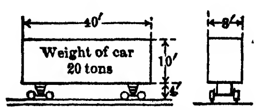
5. Determine weight per lineal foot of track of a railroad-bridge floor made up as follows:



PROB. 4.



PROB. 1.



PROB. 2. Standard Gauge.

[1] Read Arts. 20 and 21 before solving this problem.

Ties, yellow pine, 8 in. by 8 in. by 10 ft. 0 in.—12 in., center to center. Guard timber, yellow pine, 8 by 6 in. Rails and fastenings, 150 lb. per lineal foot of track.

a. Estimate from the diagram of Fig. 8 the total weight of carbon steel in a double-track through pin bridge of 150-ft. span, and determine its total cost, assuming the steel to cost 6 cents per pound, erected and painted.

b. Determine weight of ballast per lineal foot of a single-track solid-floor bridge, assuming average depth and width of ballast to be 14 in. and 13 ft., respectively. Weight of ballast per cubic foot to be assumed as 100 lb.

CHAPTER II

LAWS OF STATICS, REACTIONS, SHEARS, AND MOMENTS; INFLUENCE LINES

20. Laws of Statics.—The theory of structures is based upon the fundamental principles of statics, and these the student must thoroughly understand.

For the present, *structures in a plane and with the applied loads acting in the same plane alone will be considered.* Such structures will be in equilibrium if the following conditions are satisfied:

1. The algebraic sum of the components of all the forces acting parallel to any axis must equal zero.
2. The algebraic sum of the moments of all the forces about any axis at right angles to the plane of the forces must equal zero.

If the forces are resolved into components parallel to two rectangular axes OX and OY and if the algebraic sum of the forces parallel to OX is designated as ΣX and of those parallel to OY as ΣY, the first of the conditions above will be fulfilled when $\Sigma X = O$ and $\Sigma Y = O$; hence, the two principles stated above are fully comprehended by the three following equations:

$$\Sigma X = O, \qquad \Sigma Y = O, \qquad \Sigma M = O$$

If the forces acting upon a body do not satisfy all three equations, then the body cannot be in equilibrium. For example, if $\Sigma X = O$ and $\Sigma Y = O$, but ΣM does not, the body is in a condition of rotation about a stationary axis. If $\Sigma Y = O$ and $\Sigma M = O$, but ΣX does not, then the body has a motion of translation in a direction parallel to the X axis.

It is often advantageous to use $\Sigma M = O$ more than once, different axes being employed. If used three times without the other equations, it will satisfy all three equations unless the traces of the three axes with the plane of the forces lie in the line of action of the resultant.

In practice, it is common to use horizontal and vertical axes, for which case the first two equations may be written

$$\Sigma H = O \qquad \text{and} \qquad \Sigma V = O$$

21. Reactions.—Each of the reactions upon a structure may have three unknown properties, *viz.*, magnitude, direction, and point of application. Usually, however, the point of application of each reaction is fixed in position and the direction of one of the reactions is known. If such is the case when there are two points of support, *i.e.*, two reactions, as is the case in most structures, there remain but three unknown properties of the

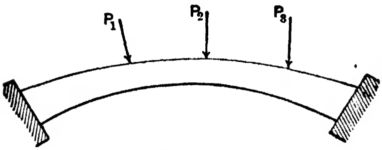


FIG. 15.

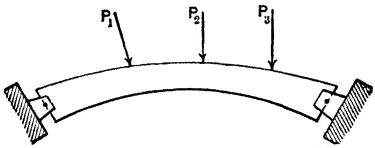


FIG. 16.

reactions, all of which may be computed by the three equations of statics unless the line of action of one of the reactions acts through the two points of support, and hence the structure is statically determined with respect to the outer forces, whether it is or is not possible to determine the inner stresses by statics. If there are more than three unknown properties of the reactions, *e.g.*, if in the case of a structure on two points of support only the points of application are fixed, or if the structure is supported on more than two points, then it is statically undetermined with

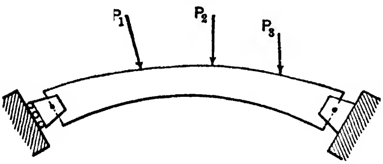


FIG. 17.

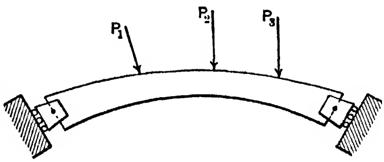


FIG. 18.

respect to the outer forces, unless some special form of construction is adopted, as in the three-hinged arches and cantilever bridges considered later. If there are fewer than three unknowns, then the structure is in general unstable and will tend to move bodily under the applied loads unless these fulfill certain special conditions.

Illustrations of the foregoing conditions are afforded by the structures shown in Figs. 15, 16, 17, and 18, for all of which the position and magnitude of the applied loads and all of the dimensions of the structure are supposed to be known.

Figure 15 represents the ordinary masonry arch in which each reaction is unknown in direction, magnitude, and point of application. In consequence the structure is indeterminate with respect to the outer forces in a threefold degree. Figure 16 shows a two-hinged arch that has the points of application of both reactions determined, but the magnitude and direction of neither reaction known; hence, it is indeterminate in the first degree with respect to the outer forces. In Fig. 17 a set of rollers is shown at one end. The function of these rollers is to make the reaction at that end perpendicular to the supporting surface since the rollers, if in good condition, can offer but little resistance to motion along this surface. This structure is, therefore, statically determined with respect to the outer forces since the points of application of both reactions and the direction of one are known. In Fig. 18, rollers are shown at both ends; hence the direction of each reaction is known. Unless these reactions meet on the line of action of the resultant of the applied loads, equilibrium cannot exist, the structure will move, and therefore the structure is unstable.

22. Computation of Reactions. Method of Procedure.—It is evident that if the horizontal and vertical components of a reaction that is unknown in direction and magnitude are determined or if either component of a reaction that is known in direction but not in magnitude is determined, the reaction itself may at once be obtained. In consequence the determination of the reactions in a structure that is statically determined with respect to the outer forces and hence has but three unknowns may be accomplished by computing the horizontal and vertical components of one reaction and either component of the other.

This method often, though not always, simplifies the solution of reaction problems and will be used hereafter. Its adoption makes it desirable to use the horizontal and vertical components of the outer forces, and these, also, can frequently be computed more easily than the actual forces. With these components of the outer forces known the solution of the problem may be accomplished by the application of the three statical equations.

The following mode of procedure is suggested for the use of the beginner, who is advised to follow it exactly until he has mastered the method thoroughly. For structures in which the reactions are not parallel to the forces or in which the character

of the unknown reactions cannot be easily predicted, even the experienced computor should not omit any of the steps in the process:

1. Draw a careful sketch of the structure, and show on it the horizontal and vertical components of the outer forces. This sketch need not be to scale but should not be materially distorted.

2. Indicate on the sketch by arrows and by the letters H and V, the assumed components of the reactions, using letters R and L as suffixes of H and V to indicate right and left reactions.

The direction of the components of the reactions that are unknown in direction may be assumed at random, *e.g.*, the horizontal component may be assumed as acting either to the right or to the left and the vertical component either up or down, but the components of the reaction the direction of which is known must be so assumed as to be consistent with this known direction.

3. Determine the unknown H and V components by the solution of the equations $\Sigma H = O$, $\Sigma V = O$, and $\Sigma M = O$, considering forces acting upward or to the right, and clockwise moments, as positive.

A positive result shows that the component in question acts in the direction *originally assumed*, and not necessarily that it acts up or to the right. With the magnitude of all components known, the magnitude of either reaction may be obtained by computing the square root of the sums of the squares of its two components. Its direction is determined by the direction of the components. The beginner is more likely to make errors by omitting some of the forces than in any other way. Particular attention may well be called to the fact that *horizontal forces may produce vertical reactions, and vice versa.*

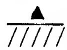
If the load, or any portion of it, is distributed over a considerable distance instead of being applied at a point, the resultant of this portion of the load may ordinarily be used in the computations as a concentrated load. This method, however, should be used only in reaction computations; it would in general be incorrect for the determination of shears, moments, and truss stresses. It is also incorrect for the determination of reactions in three-hinged arches.

It is always desirable to obtain a check by twice applying the equation $\Sigma M = O$, once about each point of support. This gives an independent check for at least one of the reaction com-

ponents, which in the case of a simple beam with vertical loads is sufficient and conclusive.

23. Reaction Conventions.—Hereafter, in both text and problems, structures supported at one end upon a set of rollers or by a tie rod will be considered as having the reaction at that point fixed in direction. The reasons for this in the case of rollers is stated in Art. 21. For the tie rod, it is sufficient to recall that such a rod is little better than a stiff rope and is incapable of carrying bending or compression; hence, the reaction that it carries must act along its axis and produce tension in the rod.

The conventional symbol  is used to indicate that the reaction at the point where the symbol is shown is to be considered as perpendicular to the supporting surface, whether the surface is horizontal, inclined, or vertical, and may be either toward it or away from it.

When the point of application of a reaction is fixed in position but not in direction, the symbol  will be used. This is not intended to represent a knife-edge bearing since the reaction may act in any direction, *i.e.*, up, down, horizontal, or inclined. If this symbol is combined with the previous symbol, then both point of application and direction of reaction are to be considered as fixed. If the reaction is carried by a tie rod, the rod will be so marked; in this case the point of application should be taken at the point where the rod is fastened to the structure.

24. Point of Application of Loads and Reactions.—In practice, it is seldom that the point of application of load or reaction is definitely fixed; it is, however, in many cases fixed within such small limits that no error arises in considering it as located at a definite point. This is the case when the structure is supported on steel pins, as in most bridges of considerable size; the reaction in such a case passes through the pin, which is generally but a few inches in diameter, and its resultant will pass through the pin center, or nearly so, unless the pin is badly turned or the bearing surface upon which it rests imperfect. With wheel loads the load acts at the point of tangency of wheel and bearing surface, which is practically a point; but as the wheel does not rest directly on the structure but has its load distributed by rails and ties, or by the floor if a highway bridge, it is not applied to the

structure itself at a point, though it is generally so considered as the error thus arising is small and on the safe side.

For ordinary beams that rest at the ends upon steel bearing plates inserted to distribute the load over the masonry supports, the assumption that the reaction is applied at the center of bearing is by no means an exact one. The actual distribution of the reaction in such a case is a function of the relative elasticity of the beam and support. If both beam and support were to be absolutely rigid—an impossible case—the reaction would pass through the center of bearing; if the support alone were to be rigid, the reaction would pass through the edge of the bearing plate; in the actual case where both beam and bearing surface yield to some extent, the reaction is distributed over the entire surface and its intensity varies uniformly or nearly so, as shown in Fig. 19. It will be noticed that the resultant pressure acts at a point between the center of bearing and inner edge of the masonry. The common assumption for such cases is to assume the reaction as applied at the center of the bearing. This assumption is on the *safe* side in designing the beam as a whole, but on the *unsafe* side in proportioning the area of bearing. However, the error for short beams, which deflect but little, is not serious. For long girders, which deflect considerably, the end bearing is usually made by a pin, supported upon a shoe, which in turn rests upon rollers, or by a rocker, a uniform distribution of the reaction being thus ensured.

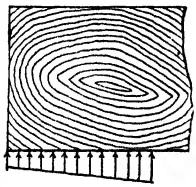


FIG. 19.

25. Solution of Reaction Problems.—The application of the methods of Art. 22 is illustrated in the problems of this article.

Problem: Compute horizontal and vertical components of reactions on beam shown in Fig. 20. Neglect weight of beam itself.

Solution: First apply $\Sigma H = 0$. This gives the equation $H_L = 0$, since the applied loads are all vertical and in consequence have no horizontal components.

Now apply $\Sigma M = 0$, taking for origin of moments the point of application of either reaction, thus eliminating one unknown. The equation that follows is derived by taking moments about the right end.

$$-10{,}000 \times 26 + 16V_L - 5{,}000 \times 12 - 10{,}000 \times 2 = 0$$

Therefore,

$$16V_L = 340{,}000 \text{ ft.-lb.} \qquad \text{and} \qquad V_L = +21{,}250 \text{ lb.}$$

Since the value of V_L is positive, the reaction acts in the direction assumed in the figure.

The application of $\Sigma V = 0$, using the value of V_L just obtained, gives the value of V_R and completes the solution of the problem. The equation follows:

$$-10{,}000 + 21{,}250 - 5{,}000 - 10{,}000 + V_R = 0$$

Therefore,

$$V_R = +3{,}750 \text{ lb.}$$

and acts upward as shown.

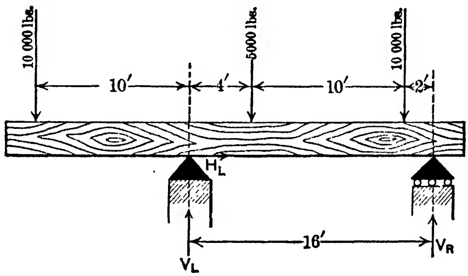


FIG. 20.

To check this value, apply $\Sigma M = 0$, using the left point of support for the origin of moments. The expression thus obtained is

$$-10{,}000 \times 10 + 5{,}000 \times 4 + 10{,}000 \times 14 - 16V_R = 0$$

Therefore,

$$V_R = +3{,}750 \text{ lb.}$$

which checks the value obtained by the application of $\Sigma V = 0$ and hence checks the value of V_L since this was used in the original determination of V_R.

Problem: Compute horizontal and vertical components of reactions on beam, shown in Fig. 21, neglecting weight of beam.

Solution: In this problem, V_L and H_L are independent of each other in magnitude and direction and each may be assumed as acting in either direction. V_R and H_R are, however, mutually related both in direction and magnitude since their resultant must act at right angles to the supporting surface, and hence make an angle of 60° with the horizontal. To fulfill this condition if V_R is assumed as upward, H_R must be assumed to the left. The ratio of their magnitude equals the ratio of the sides of a 30° triangle, as indicated by Fig. 22; hence, $V_R = H_R \cot 30° = 1.73H_R$.

To solve this problem, apply the equation $\Sigma M = 0$, taking moments about the point of application of the right-hand reaction. The following equation results:

$$-10 \times 25 + 20V_L - 10 \times 16 - 14.14 \times 10 = 0$$

The solution of this gives $V_L = +27.57$ tons; therefore, V_L acts upward as assumed.

It will be noticed that the lever arms about the origin of moments of all the horizontal forces are zero; hence, these terms do not appear in the equation. Had the inclined force been resolved at the *top* instead of the *bottom* of the beam, this condition would not have existed, but the value of the reaction would not have been changed since the moment of the horizontal

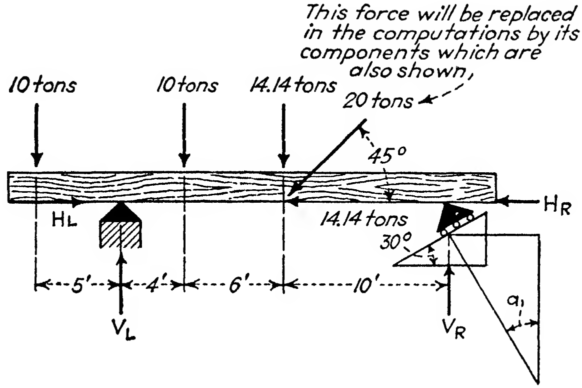


FIG. 21.

component would have been neutralized by the change in the moment of the vertical component due to its altered lever arm.[1]

The equation $\Sigma V = 0$ may now be used. This gives the following expression:

$$-10 + 27.57 - 10 - 14.14 + V_R = 0$$

Hence, $V_R = +6.57$ tons and acts as shown.

From Fig. 22, it is evident that $H_R = V_R \tan 30° = 0.577 V_R$; therefore, $H_R = 0.577 \times 6.57 = 3.79$ tons.

The application of $\Sigma H = 0$ completes the solution by giving the value of H_L. The equation is $H_L - 14.14 - 3.79 = 0$; hence, $H_L = 17.93$ tons and acts to the right.

To check the value of V_R, take moments about the left point of support. This gives the following expression:

$$-10 \times 5 + 10 \times 4 + 14.14 \times 10 - 20V_R = 0$$

[1] The device of resolving a force into its components at a point where the lever arm of one of the components is zero is a very useful one and frequently saves considerable labor. Its correctness is evident since the effect of a force upon a body as a whole always equals that of its components, no matter at what point the force is resolved or what may be the direction or length of the lever arms of the components; hence, if the lever arm of one of the components is zero the moment of the force equals the moment of the other component.

whence $V_R = +6.57$ tons, the value previously obtained, and in consequence the value of V_R is checked.

As an independent check of H_R and H_L cannot readily be made, a second computation of their value should be carried through, or the original computations carefully reviewed, the former being the safer method.

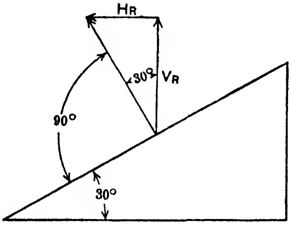


FIG. 22.

Problem: Compute horizontal and vertical components of the reactions for the truss shown in Fig. 23 for an assumed wind pressure of 30 lb. per square foot on a vertical surface.

Solution: Since the slope of the roof surface in this problem is about 30°, it will be assumed that the normal intensity of the wind pressure is 20 lb. per square foot (see table in Art. 13, Hutton's formula). The roof trusses are 20 ft. between centers; hence, the portion of the area of the windward side of the building supported by one truss has a length of 20 ft. for intermediate trusses and 10 ft. for end trusses. The reactions upon an intermediate truss will be computed.

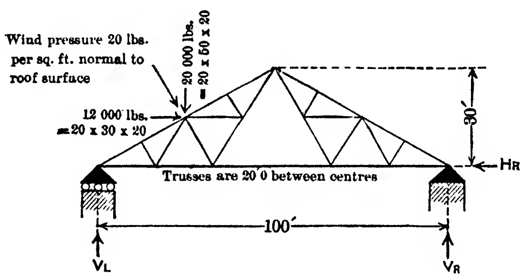


FIG. 23.

By the method of Art. 13, the horizontal and vertical components of the total wind pressure on the windward side are found to be as follows:

P_h = intensity of normal pressure multiplied by the vertical projection of roof surface $= 20 \times 30 \times 20 = 12{,}000$ lb.

P_v = intensity of normal pressure multiplied by the horizontal projection of surface $= 20 \times 50 \times 20 = 20{,}000$ lb.

The truss may now be considered as loaded with the two forces of 20,000 lb. and 12,000 lb. acting at center of windward surface, and the reactions due to these forces computed in the following way:

Applying $\Sigma M = 0$ about right end gives $100V_L + 12{,}000 \times 15 - 20{,}000 \times 75 = 0$, whence $V_L = +13{,}200$ lb., acting up as assumed.

Applying $\Sigma H = 0$ gives $12{,}000 - H_R = 0$, whence $H_R = +12{,}000$ lb., acting to left as assumed.

Applying $\Sigma V = 0$ gives $13{,}200 - 20{,}000 + V_R = 0$, whence, $V_R = +6{,}800$ lb., acting up as assumed.

Applying $\Sigma M = 0$ about left end as a check gives $-100V_R + 20{,}000 \times 25 + 12{,}000 \times 15 = 0$, whence $V_R = +6{,}800$ lb., acting up as assumed and agreeing with value previously obtained.

Problem: Compute horizontal and vertical components of the reactions for crane shown in Fig. 24. Neglect weight of structure itself.

Solution: The direction of the reaction at the top of the crane is fixed by the tie rod; hence, V_L and H_L cannot be assumed to act at random but must be so chosen that their resultant will act along the tie rod. Their magnitude will, of course, be equal since the tie rod makes an angle of 45° with the horizontal.

Applying $\Sigma M = 0$ about the bottom gives $-35H_L + 5{,}000 \times (20 + 30) + 20{,}000 \times 40 = 0$, hence, $H_L = +30{,}000$ lb., acting as assumed. Since the two components of the tie rod stress are equal, $V_L = 30{,}000$ lb. also acting as assumed.

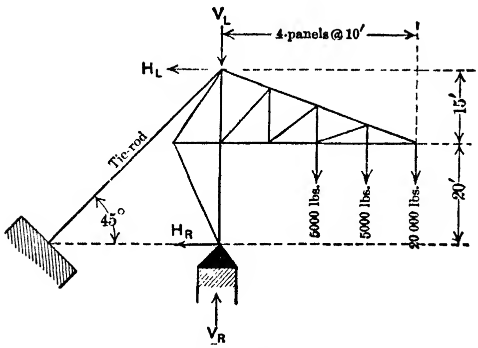


FIG. 24.

Applying $\Sigma H = O$, with the value previously found for H_L, gives $-30{,}000 - H_R = 0$; hence, $H_R = -30{,}000$ lb., acting to the *right* and *not* as assumed.

Applying $\Sigma V = O$, with the value previously found for V_L, gives $-30{,}000 - 5{,}000 - 5{,}000 - 20{,}000 + V_R = 0$; hence, $V_R = 60{,}000$ lb., acting up as assumed.

Applying $\Sigma M = O$, about the top as a check, gives $+35H_R + 5{,}000 \times 20 + 5{,}000 \times 30 + 20{,}000 \times 40 = 0$; hence, $H_R = -30{,}000$ lb., checking the value previously obtained.

It is always advisable to assume the reactions as acting in their probable directions to avoid complications. The opposite assumption was made for H_R in the foregoing problem in order to illustrate the solution with an incorrect assumption. The results will be found to agree in any case, provided that the work is correctly done, but it is confusing to have the reaction incorrectly indicated on the sketch. Sometimes, however, it is impossible to foretell the actual direction of a reaction.

In this problem the actual value of the reaction at the top should be found, since this gives the tension in the tie rod.

This value = 30,000/sin 45° = 42,430 lb. approximately.

This should equal $\sqrt{V_L^2 + H_L^2}$, which may be used as a check.

26. Shear and Bending Moment Defined.—*Shearing force* or *shear* at any section of a body is that force which tends to produce slipping along the given section.

The *bending moment* at any section of a body due to a set of coplanar forces is the *resultant moment, about an axis passing through the centroid of the section, of all the forces on either side of the section,* it being understood that the section and the axis are perpendicular to the plane of the forces.

Fractures due to shear are due either to transverse fracture of the grains or fibers or to the slipping of the fibers upon each other. Of the ordinary structural materials, wood is the only one of a fibrous character and shearing failures in this material ordinarily occur by longitudinal slipping of the fibers.

Fractures due to bending are caused by longitudinal failure of the fibers, either by tension or crushing.

27. Method of Computation, Shear, and Bending Moment.—The magnitude of the shear upon a given section due to a set of coplanar forces may be readily computed as follows: Resolve each force into two components, parallel and perpendicular, respectively, to the given section. *The algebraic sum of the components parallel to the section of all the forces upon either side of the section equals the shear.* That either side of the section may be considered is evident from the fact that for structures in equilibrium $\Sigma Y = 0$; hence, the algebraic sum of the forces on one side of the section and parallel to the Y-axis must be equal in magnitude and opposite in direction to the corresponding term for the other side of the section.

The magnitude of the bending moment upon a given section due to a set of coplanar forces may be computed by resolving the forces into horizontal and vertical components. For this case, however, it is necessary to include the moments of both sets of components, though again it is immaterial which side of the section is considered in computing the moment.

28. Signs for Shear and Bending Moment.—The signs for shears and bending moments must be used with care or errors will occur. Any reasonable convention may be adopted, but it

should be carefully observed that positive shear may represent forces acting in exactly opposite directions and that positive bending moment may represent either clockwise or counter-clockwise moment, depending in both cases upon the side of the section considered in making the computation. The distinction between the moment of forces in general as used, for example, in determining reactions, and the moment upon a cross section of a beam should be carefully observed. In the former case, clockwise moments should always be taken as of the same sign, since the effect of such moments upon the body as a whole is the same no matter upon what part of the body they may act. In a beam, however, clockwise moment upon the left of a given section produces the same effect upon the fibers as does counterclockwise moment upon the right. In both cases, compression is produced in the fibers of the upper portion of the section, and tension in those of the lower portion.

29. Shear and Moment, Common Cases.—In ordinary practice, it is seldom necessary to compute shears or moments, except for vertical sections of horizontal beams and trusses carrying *vertical* loads. For such cases the following conventions may be adopted.

Shear.—The shear upon a vertical section of a beam or truss equals the algebraic sum of all the outer forces (including reactions) upon either side of the section. It is positive when the resultant is *upward* on the *left* of the section or *downward* on the *right.*

Moment.—The moment upon a vertical section of a beam or truss equals the algebraic sum of the moments of all the outer forces (including reactions) upon either side of the section, about the neutral axis of that section. It is positive when the moment of the forces on the left of the section is *clockwise* or when the moment of the forces on the *right* of the section is *counterclockwise.*

30. Curves of Shear and Moment Defined and Illustrated.—A curve of shears or of moments is a curve the ordinate to which at any section represents the shear or moment at that section due to the applied loads. If the load is uniformly distributed, the curve may be a continuous smooth curve, a series of smooth curves, or a series of straight lines. If the loading consists of a series of concentrated loads, the curve will always be com-

posed of a series of straight lines. If the loading is a combination of concentrated and distributed loads, the curves may be composed of a combination of straight and curved lines.

It should be thoroughly understood that these curves represent the effect of loads that are *fixed* in *magnitude* and *position.* The shear and moment due to a set of moving loads constantly vary and hence cannot be represented by such curves except for a certain definite position of the loads. The effect of moving loads is shown more clearly by influence lines which are explained later. Typical curves of shears and moments are shown in Figs. 25 to 28.

It should be noticed that in all cases the ordinate to the curve of shears at any section equals the algebraic sum of the forces acting on either side of the section and that the curve of moments reaches its maximum positive and maximum negative values at points where the curve of shear crosses the axis.

This latter relation always exists and is demonstrated in Art. 33.

The computation of the values of the ordinates to the curve of moments at points a, b, c, and d of Fig. 25 are given below for illustration.

$$\begin{aligned}
\text{At } a,\ & 7.62 \times 5 && = +38.10 \text{ ft.-tons} \\
b,\ & 7.62 \times 13 - 5 \times 8 && = +59.06 \text{ ft.-tons} \\
c,\ & 7.62 \times 19 - 5 \times 14 - 10 \times 6 && = +14.78 \text{ ft.-tons} \\
d,\ & -5 \times 8 && = -40.00 \text{ ft.-tons}
\end{aligned}$$

Note that a point of maximum or minimum moment occurs in all cases where the curve of shears crosses the axis and that the moment curve is a straight line over any portion of beam where shear is constant.

31. Shear and Moment. Distributed Load.—In determining reactions a distributed load may usually be replaced by its resultant and the latter used as a concentrated load. This method is *incorrect* for shear and moment and should never be used for such cases unless the distributed load lies wholly on *one side* of the section under consideration. The reason for this may readily be seen. Both shear and moment are functions of the forces on *one side* only of a section, and all such forces must be included in the determination of either of these quantities. It is evident that if the structure is loaded with a

distributed load its resultant may act on either side of a given section, say on the right, while a considerable portion of the actual load may be on the left. If the shear or moment is computed for the forces on the left of the section with the distributed load replaced by its resultant, the serious error of neglecting a considerable portion of the loads will be made. For

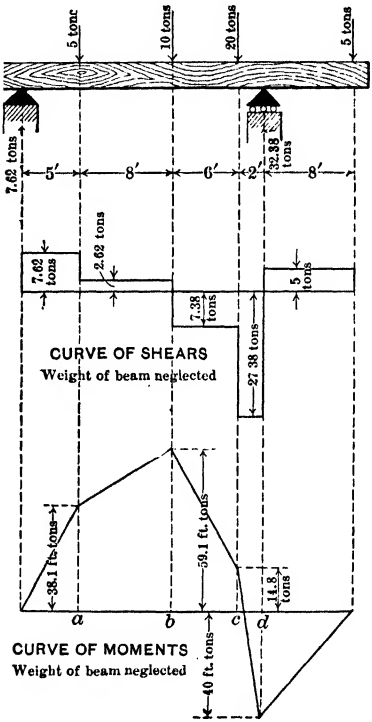


FIG. 25.

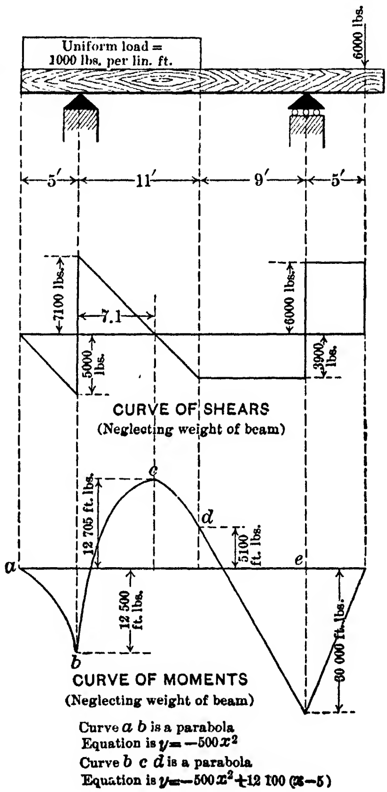


FIG. 26.

(Read Art. 31 before studying this figure.)

reactions, on the other hand, it is the influence of the load as a whole that is to be considered; hence, the resultant may properly be used except in special cases. To visualize the difference between the correct curves of shear and moment for the case of a beam carrying a uniformly distributed load and the same curves if drawn in accordance with the erroneous assumption that the

load may be replaced in magnitude and position by a concentrated load, see Fig. 27.

32. Shear and Moment. Uniformly Varying Load.—It is frequently necessary to determine shears and moments for a beam or girder loaded with a uniformly varying load. Such a condition may occur with a vertical member subjected to hydrostatic

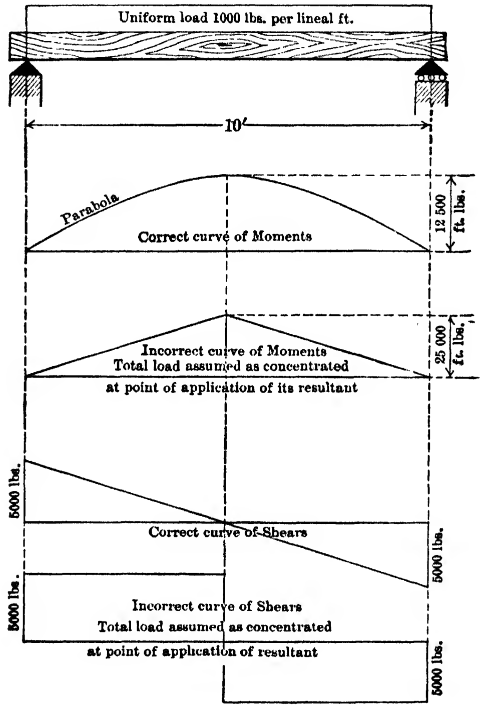


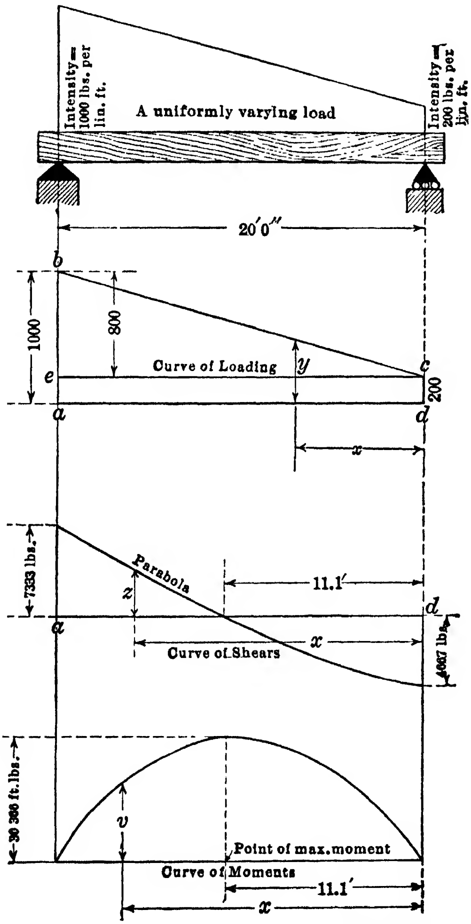
FIG. 27.

pressure, as in a canal lock or in a diagonal floor girder in a building.

The curves of shear and moment for such a girder are shown in Fig. 28 and the necessary computations follow.

Let the load be represented in intensity by the trapezoid *abcd*, the area of which represents the total load on the beam. If the trapezoid is divided into two parts by a line *ce* parallel to the axis *ad*, the effect of each portion may be treated separately and the problem simplified.

Magnitude of force represented by triangle *bce* =
$\frac{800}{2} \times 20 =$ 8,000 lb.
Magnitude of force represented by rectangle *adce* =
$200 \times 20 =$ 4,000 lb.
Total load = 12,000 lb.
= area of trapezoid *abcd*



$$y = 200 + \frac{800}{20}x$$

Note that the curve of shears corresponds to integral of the loading curve, hence

$$z = \int y\,dx = \int (200 + 40x)dx$$
$$= 200x + 20x^2 + C_1$$
$$-C_1 = 4{,}667 = \text{value of } z \text{ when } x = o$$
$$\therefore z = 200x + 20x^2 - 4{,}667$$

Note that the curve of moments corresponds numerically to integral of curve of shears, hence

$$v = \int z\,dx = 4{,}667x - 100x^2 - \frac{20}{3}x^3$$

FIG. 28.

Reaction.—The computation of the reactions should be divided into two operations: the determination of the reactions due to the load represented by the rectangle *aecd* and the deter-

mination of the reactions due to the load represented by the triangle *bec*.

$$\text{For the first case each reaction} = \frac{200 \times 20}{2} = 2{,}000 \text{ lb.} = V.$$

To determine the reactions due to the load represented by triangle *bce*, it is advisable to determine the position of the resultant of this load. This passes through the centroid of the triangle and hence is $^{20}\!/_3$ ft. from the line *ab* and $^{40}\!/_3$ ft. from *cd*. The left reaction V_L'' due to this load may now be determined by applying $\Sigma M = 0$ about the right-hand end of the beam. The following expression results: $-8{,}000 \times {}^{40}\!/_3 + V_L'' \times 20 = 0$; hence, $V_L'' = 5{,}333$ lb. The total left reaction V_L therefore equals $V_L'' + V = 7{,}333$ lb.

To obtain the right reaction, apply $\Sigma V = 0$. This gives $7{,}333 - 12{,}000 + V_R = 0$; hence, $V_R = 4{,}667$ lb., which may be checked by applying $\Sigma M = 0$ about the left end of the beam.

The curve of shears may now be drawn. Its equation referred to rectangular axes passing through point *d* with x positive to the left and z positive upward is $z = -4{,}667 + 200x + 20x^2$, in which the term $200x$ equals the area of a rectangle of height *cd* and length x, and the term $20x^2$ equals the area of that portion of the triangle *bce* comprehended between its vertex *c* and a vertical line drawn at a distance x from the vertex. This curve cuts the axis at a point 11.1 ft. from the right end, as may be seen by placing $z = 0$ and solving for x.

The curve of moments may be obtained in a similar manner. Its equation referred to the same origin is $v = 4{,}667x - \dfrac{200x^2}{2} - \dfrac{20x^3}{3}$. This equation may be written directly from the shear equation by multiplying each term in the latter, which represents a force, by the distance of the particular force from the section. Thus, 4,667 equals the right reaction and hence should be multiplied by x; $200x$ equals that portion of the load represented by a rectangle extending a distance x from the right reaction and hence should be multiplied by $x/2$; $20x^2$ equals that portion of the load represented by a triangle of length x, and with its vertex at the right reaction, and hence should be multiplied by $x/3$.

33. Location of Section of Maximum Moment.—It is a well-established principle of mechanics that the first derivative of the moment on a beam equals the shear; hence, the moment must have either a minimum or a maximum value at every section where the curve of shears crosses the axis of the beam. The following rule may therefore be stated: The maximum moment on a beam always occurs at a section where the curve of shears crosses the axis of the beam, *i.e.*, where the shear equals zero.

This rule may also be proved by the use of the theorem of Art. 34, since it is evident that the moment M_a begins to diminish when S_a changes from positive to negative, *i.e.*, passes through zero.

The reader will observe that if the equation for the curve of moments in Art. 32 is differentiated with respect to x the

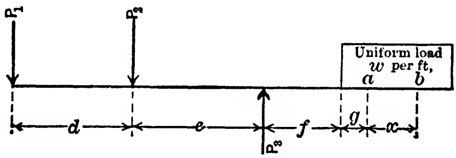


FIG. 29.

numerical value in the equation for the curve of shears will be obtained. In the light of what has just been stated, this is correct, and such a result should always be found.

The converse of this is also true, *viz.:*

That the moment curve is the integral of the shear curve with respect to x. It follows that the ordinate to the curve of moments at any section equals numerically the area between the shear curve and the beam axis extending between the end of the beam and the section. An inspection of the numerous shear and moment diagrams on the following pages will show that this relation occurs in every case. The student in testing this by integration must not forget the constant of integration.

34. Theorem for Computing Moments.—In computing moments at a number of consecutive points, as is often necessary in dealing with concentrated loads, the following theorem may be used to great advantage:

The moment at any section b of a structure loaded with parallel forces, either concentrated or distributed, is equal to the moment at any other section a, at a distance x from b, plus

(algebraically) the shear at a multiplied by x, plus (algebraically) the moment about b of the loads between a and b. This may be expressed as follows:

Let S_a = shear at section a.
M_a = moment at section a.
M_b = moment at section b.
x = distance between section a and section b measured at right angles to the line of action of the forces.
M_x = moment about b of forces between a and b.

Then,

$$M_b = M_a \pm S_a x \pm M_x$$

This may be proved in the following manner:

$$M_a = -P_1(d + e + f + g) - P_2(e + f + g) + P_3(f + g) - \frac{wg^2}{2}$$

$$M_b = -P_1(d + e + f + g + x) - P_2(e + f + g + x) + P_3(f + g + x) - w\frac{(g + x)^2}{2}$$

Therefore, by subtraction,

$$M_b - M_a = -P_1x - P_2x + P_3x - \frac{w}{2}(2gx + x^2)$$

But

$$P_3 - P_1 - P_2 - wg = S_a \qquad \text{and} \qquad \frac{wx^2}{2} = M_x$$

Therefore,

$$M_b = M_a + S_a x - M_x \tag{9}$$

This solution is perfectly general since no restrictions were imposed upon character or position of the loads.

35. Beams Fixed at Ends.—The beams hitherto dealt with have been supported at two points and have been statically determined. Sometimes, however, beams are used that are fixed at both ends by being built into the masonry or by other methods and hence, are statically undetermined. Complete treatment of such beams may be found in standard books on mechanics and will not be repeated here, although attention should be called to the fact that such beams are much stronger than beams of the same size that are merely supported at the ends.

A beam fixed at one end is also indeterminate with respect to the reactions, but the moment and shear at any section of the projecting end can be computed without difficulty.

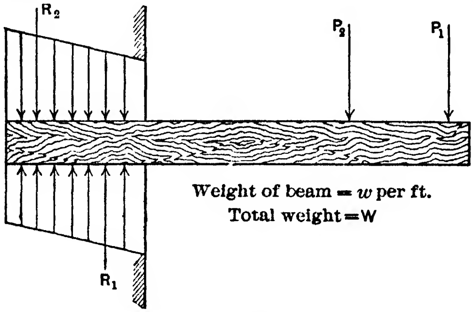


FIG. 30.

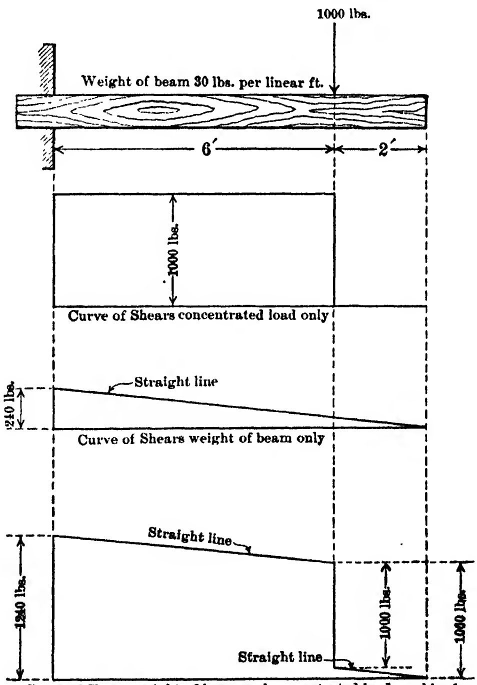


FIG. 31.

Such a beam is shown in Fig. 30, in which an assumed distribution of the reactions is indicated, *viz.*, a uniformly varying

downward reaction, the resultant of which is R_2, and another uniformly varying upward reaction, the resultant of which equals R_1. It is evident that $R_1 = R_2 + P_1 + P_2 + W$ and that the moment of R_2 about the point of application of R_1

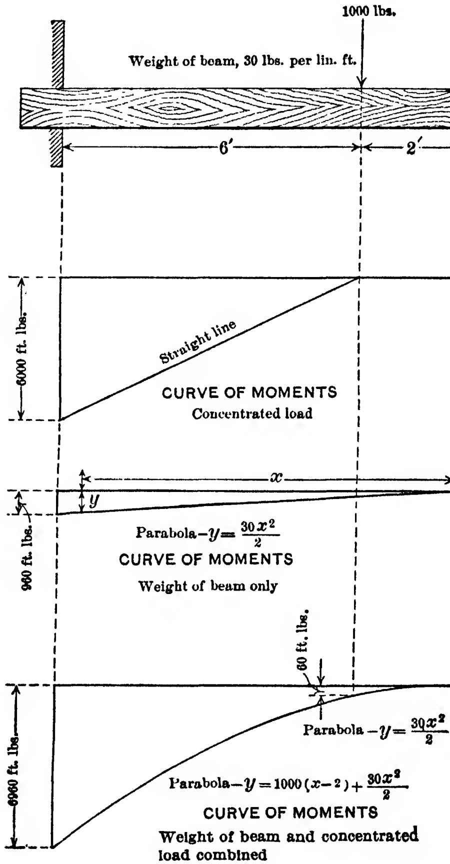


FIG. 32.

must equal the moment about the same point of P_1, P_2, and W. The actual distribution of the reaction depends upon the relative elasticity of beam and masonry and will not be discussed. The maximum bending moment and shear occur at, or very near, the edge of the masonry and can be computed with

no greater error than for ordinary beams resting on masonry abutments; hence, a beam of this sort can be designed with comparative certainty, provided that reasonable provision is made at

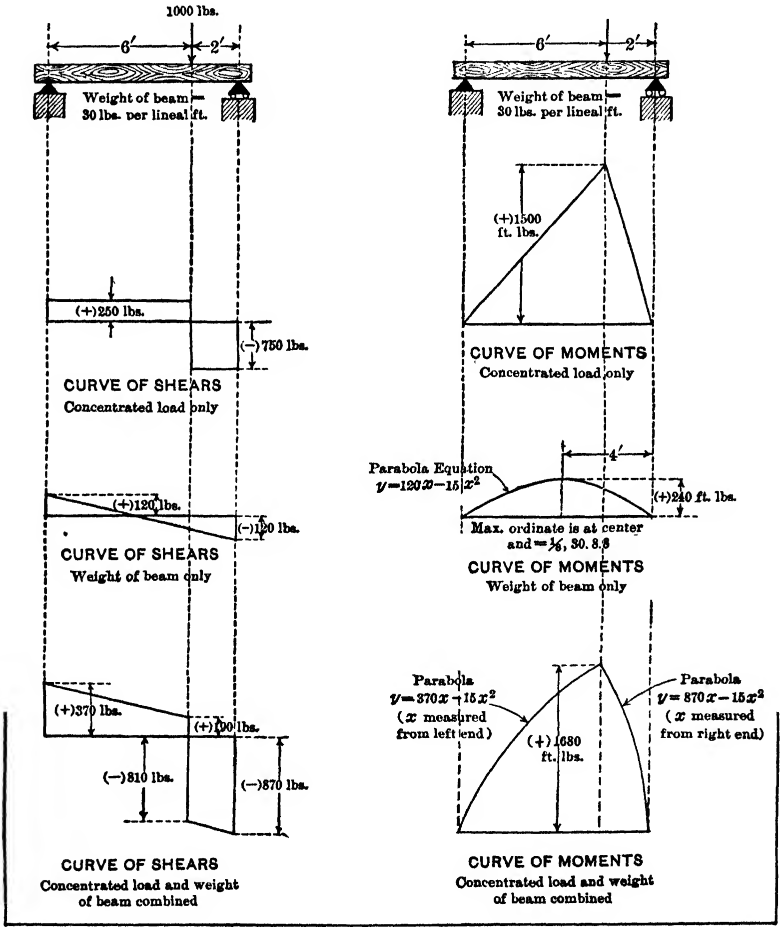


FIG. 33.

the ends for carrying the reactions and the moments at fixed ends.

36. Effect of Floor Beams.—Reactions, moments, and shears upon a structure *as a whole* are uninfluenced by the internal con-

struction. For example, the reactions at the ends of a structure due to a given loading are the same whether the structure is a simple beam or is made up of trusses, floor beams, and stringers. This immunity, however, does not extend to the individual members of the structure, which are influenced to a marked degree by the

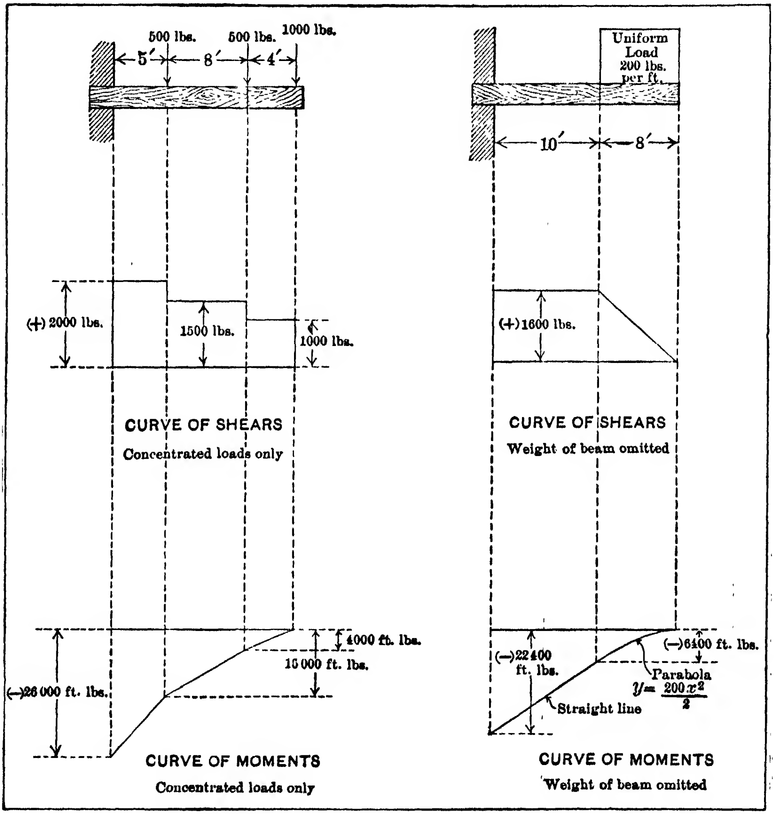


FIG. 34.

construction adopted. In the case of an ordinary bridge composed of trusses or girders, floor beams, and stringers, the shears and moments on the trusses vary from those which would exist if there were no floor beams, and this applies also to the reactions if floor beams are not used at the ends.

The effect of floor beams is to load the main girders or trusses with loads at definite points. This is clearly shown by the

figures accompanying Art. 1. The load reaches the stringers through the floor, is carried by them to the floor beams, and thence goes to the main girders. In consequence a girder carries only concentrated loads except for its own weight, and

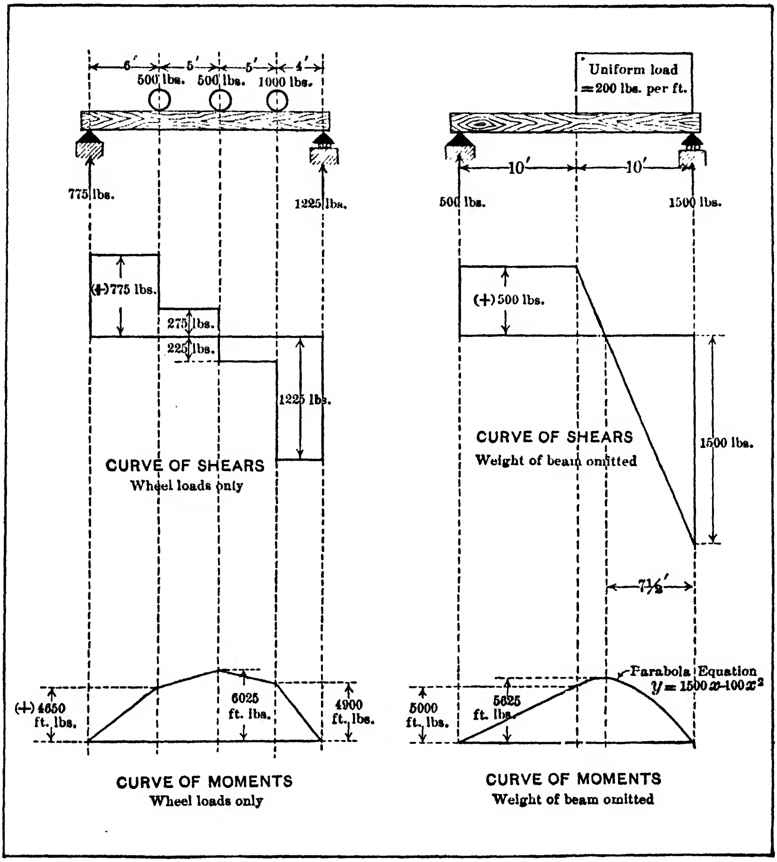


FIG. 35.

the curves of shear and moment for the applied loads are composed of straight lines.

37. Typical Curves of Shear and Moment.—A few curves of shear and moment have already been drawn to illustrate the text. In the figures that follow, the attempt has been made to cover a wide range of cases. The beginner should draw curves

for similar cases, changing the data to avoid copying, until he understands the subject thoroughly.

38. Influence Lines and Tables Defined.—In the determination of maximum shears, moments, reactions, and other functions

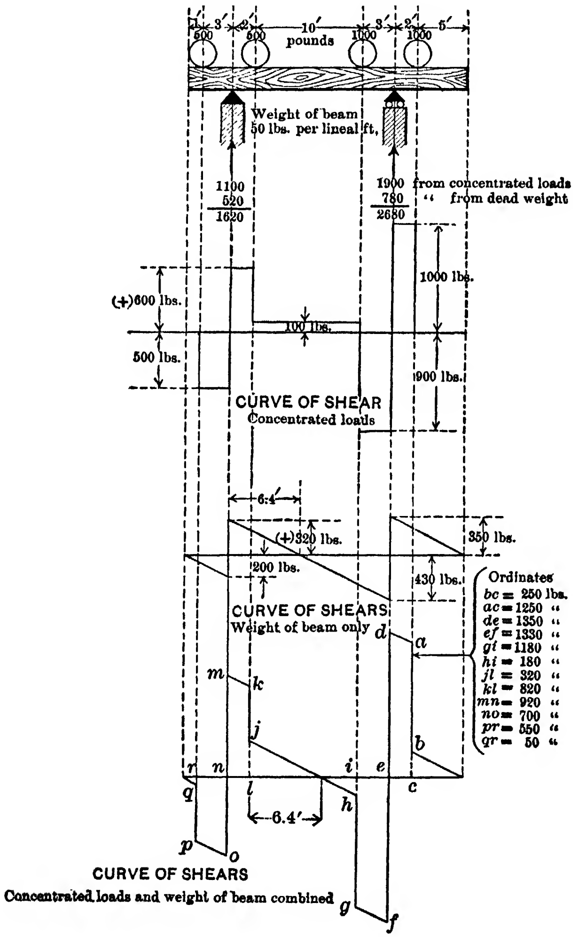
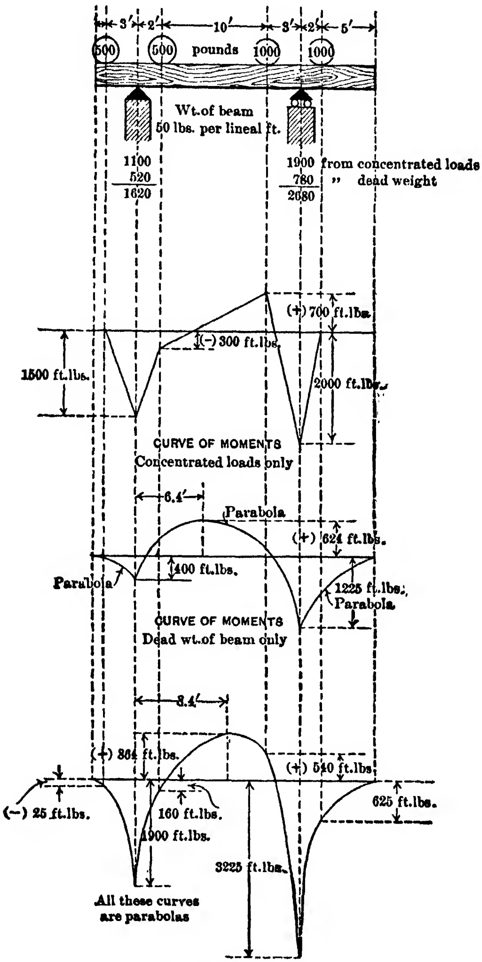


FIG. 36.

due to moving loads, it is frequently useful to study the effect of a load of unity as it moves along the structure. This may be done graphically by plotting a line called an influence line or

analytically by preparing an influence table in which are set down the values of the function under consideration when the



CURVE OF MOMENTS

Concentrated loads and weight of beam combined.

Fig. 37.

load is at various governing points, such as the panel points of a truss bridge. The simple illustrations on page 63 show clearly the character of the line and the table.

The influence line in this case is the locus of the values in the second column of the influence table and is merely the graphical representation of the equation for the shear at a due to a load of unity passing along the beam. If x is the distance of the load

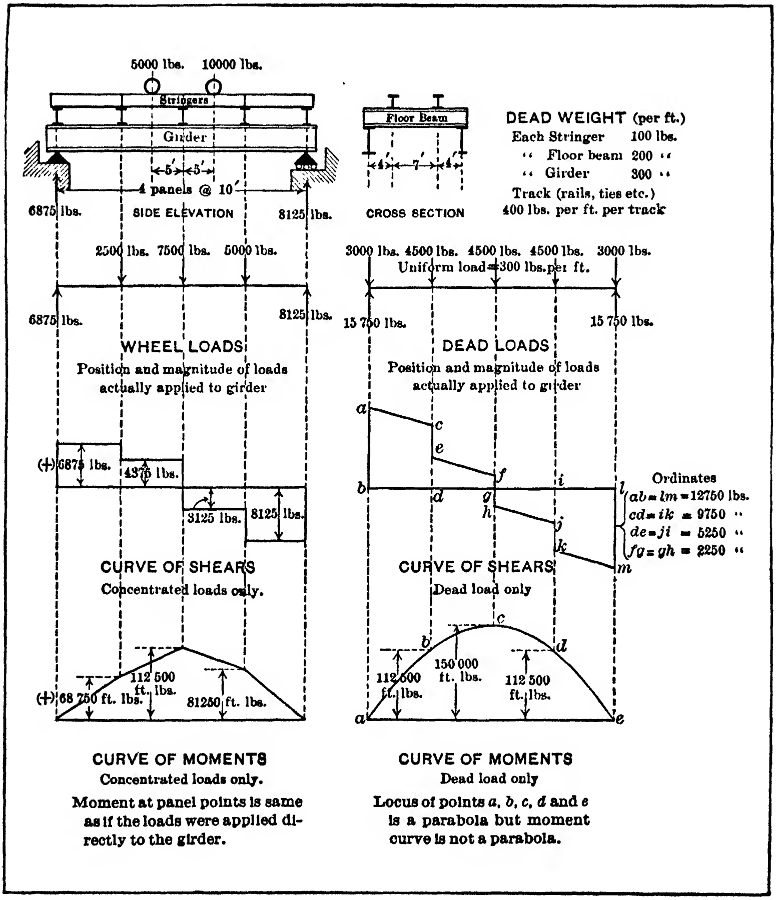


FIG. 38.

Note.—Floor beams are ordinarily riveted to sides of girders. Above construction is adopted here for sake of clearness.

from the left reaction and y the ordinate, the equations of the influence line will be as follows:

$$y = -\frac{x}{6}, \; x \text{ varying between 0 and } 2'$$

and

$$y = \frac{6 - x}{6}, \ x \text{ varying between } 2' \text{ and } 6'$$

The difference between an influence line and the curves given in the preceding articles should be carefully observed. A curve of shears, or moments, is a curve the ordinate to which at any point shows the shear, or moment, at that point caused by a set of loads, fixed in magnitude and position. The ordinate to the influence line shows instead the shear or moment at the section for which the influence line is drawn, due to a load of unity

INFLUENCE TABLE FOR SHEAR AT *a* OF BEAM SHOWN IN FIG. 39

Distance of Load from Right Reaction, Feet	Shear at *a*
1	$+\frac{1}{6}$
2	$+\frac{2}{6}$
3	$+\frac{3}{6}$
3.9	$+\frac{39}{60}$
4.1	$-\frac{19}{60}$
5	$-\frac{1}{6}$

Influence table for shear at *a* of beam shown in Fig. 39.

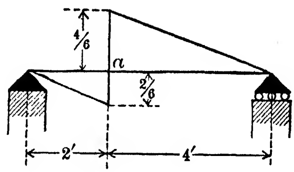


FIG. 39.—Influence line for simple beam. Shear at *a*.

acting at the point where the ordinate is measured. The examples in Art. 39 serve to illustrate influence lines for the more common cases of simple beams and girders.

The actual employment of influence lines and tables in practice seldom occurs except for complicated structures, where they are frequently almost indispensable. In this book the influence line will, however, be used with freedom, partly for purposes of illustration and demonstration and partly that the student may better familiarize himself with the behavior of various structures under moving loads.

39. Examples of Influence Lines. *a. Simple Beams and Girders (Figs.* 40 *to* 45).

b. Girders with Loads Applied through Floor Beams, as in Fig. 46 *(Figs.* 47 *to* 54).—The usual form of construction for such bridges is that in which the floor beams are riveted to the girder

webs and the stringers to the floor-beam webs. The type shown in Fig. 46 is chosen here for clearness in presentation. The influence lines, moments, shears, etc., would be identical in the two cases.

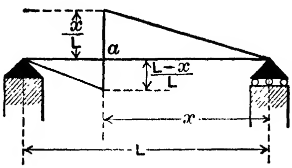


FIG. 40.—Influence line for shear at section a.

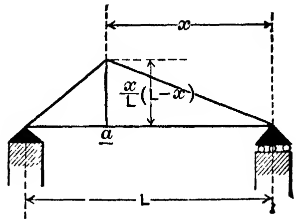


FIG. 41.—Influence line for moment at section a.

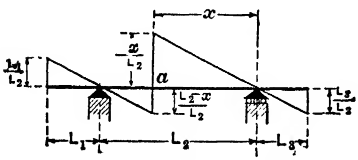


FIG. 42.—Influence line for shear at section a.

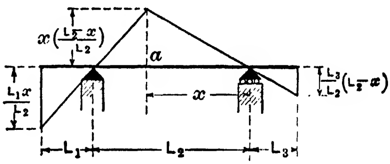


FIG. 43.—Influence line for moment at section a.

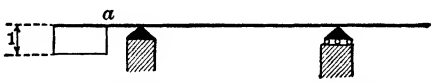


FIG. 44.—Influence line for shear at section a.

FIG. 45.—Influence line for moment at section a.

40. Properties of the Influence Line.—The following theorems may often be used to advantage:

1. The value of a given function due to a single load in a fixed position equals the product of the magnitude of the load and the ordinate to the influence line measured at the point where the load is placed. This needs no proof but follows directly from the definition of the influence line.

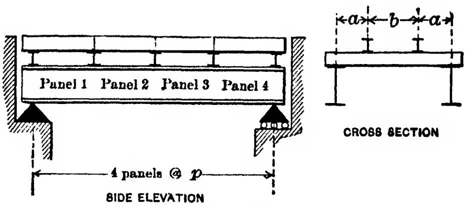


FIG. 46.

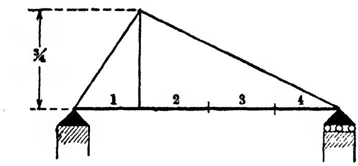


FIG. 47.—Influence line for shear in panel 1.

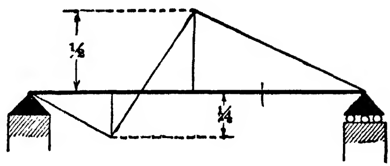


FIG. 48.—Influence line for shear in panel 2.

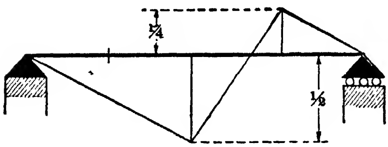


FIG. 49.—Influence line for shear in panel 3.

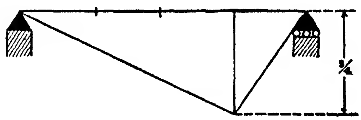


FIG. 50.—Influence line for shear in panel 4.

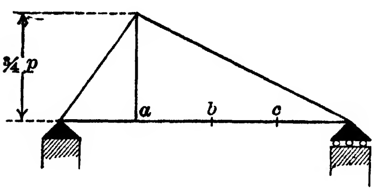


FIG. 51.—Influence line for moment at panel point *a*.

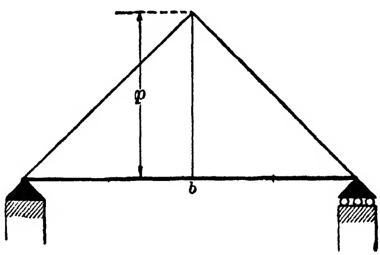


FIG. 52.—Influence line for moment at panel point *b*.

2. The value of a given function due to a uniformly distributed load equals the product of the *intensity* of the load and the *area* bounded by the axis of the beam, the influence line, and the ordinates drawn through the limits of the load. If this area is partly positive and partly negative, the algebraic sum of the two should be used.

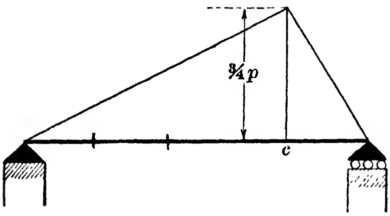


FIG. 53.—Influence line for moment at panel point c.

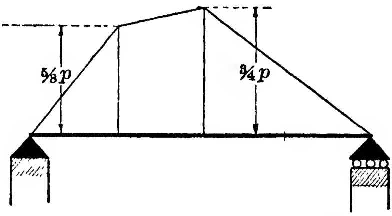


FIG. 54.—Influence line for moment at center of panel 2.

This theorem may be proved as follows:

Let bec represent an influence line for a portion of a given structure of length L. Let w equal the intensity of a uniformly distributed load.

Then the total load on a section of length $dx = wdx$ and the effect of this portion of the load upon the given function $= wydx$. Integrating between the limits 0 and L gives

$$w\int_0^L ydx$$

for the effect of a load covering the entire distance L. But ydx

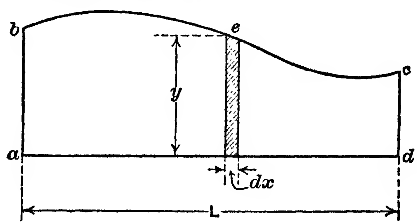


FIG. 55.

is the area of the infinitesimal strip subtended by dx, and $\int_0^L ydx$ is the area $abecd$; hence, $w\int_0^L ydx = w \times$ area $abecd$.

3. The value of a given function due to a set of concentrated loads equals the algebraic sum of the product of each load and

its corresponding ordinate to the influence line. This is a corollary of 1.

41. Neutral Point.—The influence lines shown in Figs. 47 to 50 cross the axis of the beam in each case except that for shear in the end panels. The point of intersection is called the *neutral point* since a single load placed at this point produces no shear in the panel where the intersection occurs.

Similarly, a neutral point may be found for moment, or for any other function for which an influence line may be drawn, provided that the influence line crosses the beam axis.

42. Position of Loads for Maximum Shear and Moment at a Definite Section.—The following important laws may be deduced from the influence lines given in Art. 39.

1. For a simple beam supported at the ends a single concentrated load causes maximum shear at a section when placed an infinitesimal distance on one or the other side of the section and maximum moment when placed at the section. A uniformly distributed live load produces maximum shear at a section when applied over the entire distance between the section and one or the other end of the beam and maximum moment when applied over the entire length of the beam.

2. In an end-supported girder or truss loaded by means of floor beams, a single concentrated load produces maximum shear in a panel when placed at the end of the panel adjoining the more distant reaction and maximum moment at a panel point when placed at that point. A uniformly distributed live load produces maximum shear in a panel when applied over the entire distance between the neutral point of that panel and the more distant reaction and maximum moment at any point when applied over the entire length of the structure.

43. Maximum Moments and Shears. Structures Supported at Ends.—In the preceding article, moments and shears at particular sections have alone been considered, and no attention has been given to the maximum values of these functions. These maximum values must, however, be computed before the structure can be designed. For single concentrated loads and for uniform live load the value of these quantities can be easily determined as follows, for beams supported at ends:

Case 1. Maximum shear, single concentrated load, beam without floor beams. The influence line shows that the maximum

value of the ordinate occurs either when $x = L$ or when $L - x = L$ and equals unity; hence, the maximum shear due to a load P occurs with the load at either end of the beam. Its value equals P.

Case 2. Maximum moment on beam under same conditions as Case 1. Here the ordinate to the influence line is a maximum at the load and equals $x/L(L - x)$. This can be easily shown to be a maximum when $x = L - x$; hence, the maximum moment due to a load P occurs when the load is at the center of the beam. Its value is $PL/4$.

Case 3. Maximum shear on same beam due to a uniform live load of intensity w. It is evident that the area between the influence line and the axis will be a maximum if section a is at either end; hence, the maximum shear equals $wL/2$.

Case 4. Same as Case 3, but maximum moment instead of shear. The maximum moment occurs for a load over the entire beam and occurs at the section where the ordinate is a maximum, which has already been shown in Case 2 to be at the center. The moment at the center equals $\frac{1}{8}wL^2$.

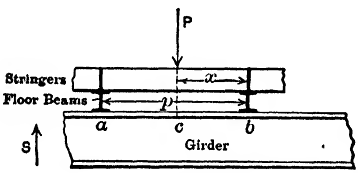


FIG. 56.

Case 5. Maximum shear, single concentrated load, girder with floor beams and equal panels. The maximum evidently occurs in the end panel, its value depending upon the number of panels. If n equals number of panels and P the load, the maximum shear $= P(n - 1)/n$.

Case 6. Same as Case 5, but for uniform load w per foot instead of concentrated load. Maximum shear occurs in end panels and with a load over the entire structure. Its value is $\frac{wp}{2}(n - 1)$ where p = panel length.

Case 7. Same as Case 5, but maximum moment instead of maximum shear. Place load at panel point nearest center. Maximum moment occurs at this panel point and equals $\left(\frac{P}{2}\right)\left(\frac{pn}{2}\right)$ if number of panels is even and $\frac{Pp}{4n}(n^2 - 1)$ if number of panels is odd.

Case 8. Same as Case 6, but maximum moment instead of maximum shear. Maximum moment occurs at panel point

nearest center with load over entire span. Its value is $\frac{1}{8}wL^2$ when number of panels is even and $\frac{1}{8}wL^2\left(1 - \frac{1}{n^2}\right)$ when number is odd.

In deriving these two quantities the following theorem may be used:

The moment at a panel point of a girder with floor beams equals that at the corresponding point of a simple beam under the same load.

The proof of the theorem is as follows:

Let Fig. 56 represent a portion of a girder carrying floor beams.

Let M_b = moment at panel point b.

M_a = moment at panel point a.

S = shear in panel to left of given panel.

Then, in accordance with rule given in Art. 34,

$$M_b = M_a + Sp - P\left(\frac{x}{p}\right)p = M_a + Sp - Px$$

This is also the value of the moment at b with the load P applied directly to the girder at the point c.

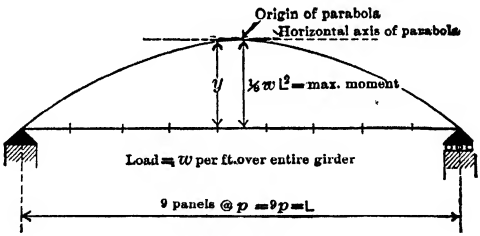


FIG. 57.

Of the formulas in this article the student is advised to memorize that for maximum moment at the center due to a uniform load, *viz.*,

$$M = \frac{1}{8}wL^2 \tag{10}$$

This formula is applicable not only to simple beams, but also to girders with floor beams provided that number of panels is even.

Since the moment at a panel point equals that at the corresponding point of a simple beam under the same load, the locus of the moments at the panel points for a uniform load over the entire beam is a parabola, with center ordinate equal to $\frac{1}{8}wL^2$;

hence, the ordinate at any panel point of a girder with an odd number of panels may be deduced from this value by applying the equation of a parabola. This is illustrated by Fig. 57 in which the ordinate y equals the maximum moment on the girder and is given by the equation

$$M = \frac{1}{8}WL^2\left[1 - \frac{(\frac{1}{2}p)^2}{(4\frac{1}{2}p)^2}\right] \tag{11}$$

44. Approximate Method for Maximum Shear.—In practice, it is common to determine the maximum shear produced by a uniform load on an *end-supported girder with floor beams* by the following approximate but safe method:

Compute the maximum positive shear in a panel as if all panel points to right were loaded with *full* panel loads and

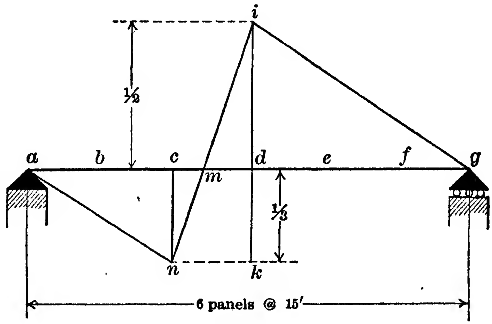


FIG. 58.

panel points at left with no load; for maximum negative shear, reverse this process.

This method is illustrated by the following example: Let the problem be the determination of the maximum positive shear in panel *cd* of the girder shown in Fig. 58 due to a uniform live load of 3,000 lb. per foot.

By the approximate method the shear should be computed for full panel loads at d, e, and f and for no loads at b and c and will therefore equal $\left(\frac{1 + 2 + 3}{6}\right)45{,}000 \text{ lb.} = 45{,}000 \text{ lb.}$

By the exact method the girder should be loaded from the right end up to the neutral point m in panel *cd*.

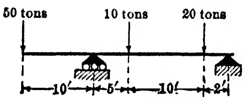
From the similar triangle of Fig. 58, it is evident that

$$\frac{md}{nk} = \frac{id}{ik} = \frac{1/2}{1/2 + 1/3} = \frac{3}{5}$$

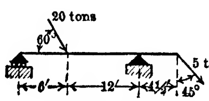
But $nk = cd = 15$ ft. Therefore, $md = 9$ ft.; hence, the area of the triangle $mig = 54 \times 1/4 = 13.5$ sq. ft. Since the maximum shear equals the area of the triangle mig multiplied by the intensity of the load per linear foot, its value is $3{,}000 \times 13.5 = 40{,}500$ lb., or considerably less than the value obtained by the approximate method, a relation that will always occur for the intermediate panels of an end-supported girder. For the end panels the neutral point occurs at the end of the panel; hence, for such panels the exact and approximate methods give identical results.

Problems

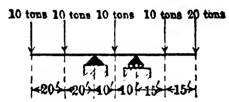
In Probs. 6 to 22, compute the horizontal and vertical components of each reaction.



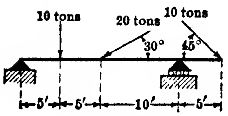
Prob. 6.



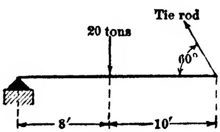
Prob. 7.



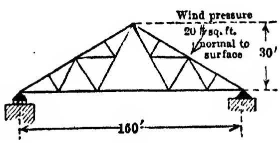
Prob. 8.



Prob. 9.

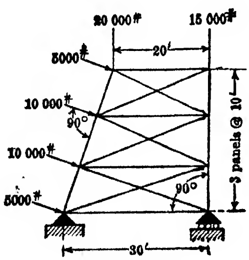


Prob. 10.

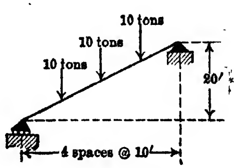


Prob. 11.

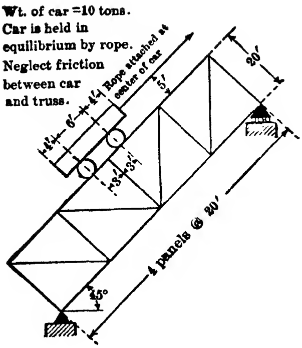
This truss is an intermediate truss of a series. Trusses spaced 20′ between centres.



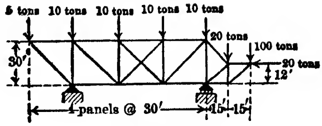
Prob. 12.



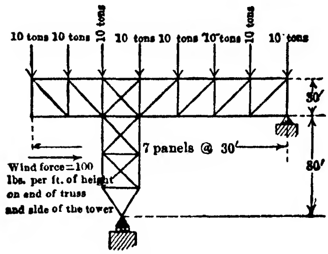
Prob. 13.



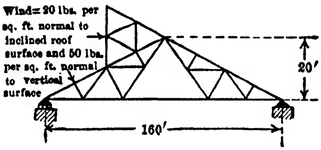
PROB. 14.



PROB. 15.

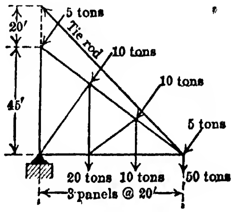


PROB. 16.

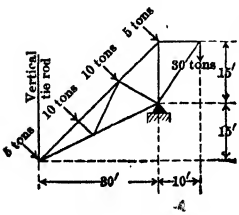


PROB. 17.

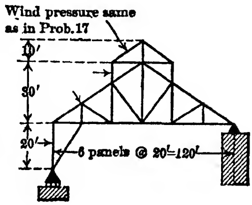
This truss is one of the end trusses of a series. Distance apart of trusses equals 20′ centre to centre.



PROB. 18.

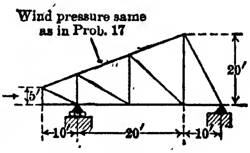


PROB. 19.



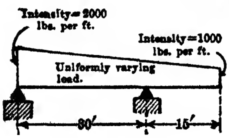
PROB. 20.

This truss is one of the intermediate trusses of a series. Distance apart of trusses equals 30′ centre to centre.

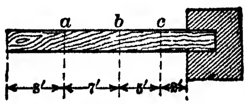


PROB. 21.

This truss is an end truss of a series. Distance apart of trusses equals 20′ centre to centre.



PROB. 22.

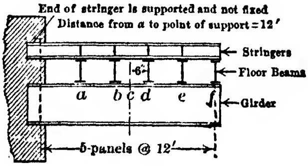


PROB. 23.

23. *a.* What is the magnitude of the shear at sections *a* and *c* with a concentrated load of 10,000 lb. at *b*?

b. What is the magnitude of the shear at sections *a*, *b*, and *c* with a uniform load of 1,000 lb. per linear foot over the entire beam?

24. *a.* Where should a single concentrated load be placed to cause maximum shear in panel *de*? In panel *ab*?



PROB. 24.
Stringers are directly over girders.

b. What is the magnitude of the shear at section *c* of the girder with a single concentrated load of 20,000 lb. applied to the stringer at the center of panel *bd*?

25. (In the following problems, relating to curves of moments and shears and to influence lines, positive values should be plotted above the axis and numerical values given for ordinates at all points where the curves change direction.)

Plot the curve of shears for beam shown in Prob. 23 with a uniformly varying load extending over the entire beam. Intensity of load at free end of beam, 2,000 lb. per foot; at fixed end, 1,000 lb. per foot.

26. (See Prob. 24 for figure for this problem.)

a. Plot the curves of shears and moments for a uniform live load of 1,000 lb. per foot per girder extending from the free end to the center of panel *ab* and applied to the stringers.

b. Compare the moment at each floor beam for the loading stated in *a* with that which would exist if there were no floor beams and the same load were applied directly to the girder (*i.e.*, a uniform load of 1,000 lb. per foot, extending 42 ft. from the free end of the girder).

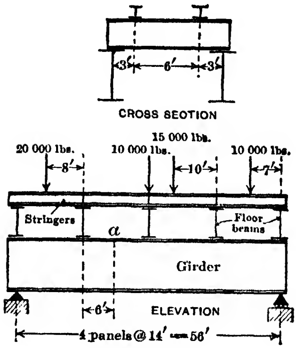
27. *a.* Draw curves of shear and moment for one girder. Loads are axial loads.

b. Draw similar curves for a total uniform load of 3,000 lb. per foot applied to the stringers and extending over entire span, and compare the moments at the floor beams with those which would occur at similar points if the load were applied directly to the girder.

c. Determine position of a single concentrated load for maximum shear at section *a*. For maximum moment at same section. Load to be applied at the stringers.

d. Draw the curves of dead moment and shear for following assumed weights:

Stringers, 300 lb. per foot per stringer (this includes weight of bridge floor). Floor beams, 100 lb. per lineal foot of floor beam. Girders, 200 lb. per lineal foot per girder.



PROB. 27.

28. (See Prob. 23 for figure for this problem.)

a. Plot the influence lines for shear in panel *ab* at sections *a* and *b*.

b. Plot the influence lines for moment at sections *a* and *b*.

29. (See Prob. 24 for figure for this problem.)

a. Plot the influence lines for shear in panel *ab* and in panel *ef* of girder. Assume girder to be directly under a stringer and load to be applied at the stringer.

b. Plot the influence lines for moment at sections *a* and *d*.

c. From an inspection of the influence line, determine over what portion of the beam a uniform load should extend in order to produce maximum shear in panel *ab*, and compute the magnitude of this shear, assuming the uniform load to equal 1,000 lb. per linear foot and to be applied at the stringers.

d. Same as *c*, except substitute moment at section *a* for shear in panel *ab*.

30. *a.* Plot the influence lines for shear at sections *a*, *b*, and *c*.

b. Plot the influence lines for moment at sections *a*, *b*, and *c*.

c. From an inspection of the influence lines, determine where a single load should lie to give maximum shear at section *c*. To give maximum moment at section *a*.

d. From an inspection of the influence lines, determine what portions of the beam should be loaded with a uniform load per foot to give maximum shear at section *c*. To give maximum moment at section *a*.

e. Compute the maximum shears at sections *a* and *c* due to a uniform live load of 2,000 lb. per foot, and state in each case whether the shear is positive or negative.

f. Compute the maximum moments at sections *a* and *b* due to the load given in *e*, and state whether positive or negative.

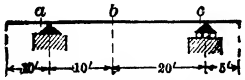
31. *a.* Plot influence lines for shear in panels 0–1 and 1–2. Make same assumption as to relative position of stringers and girders as in Prob. 29, and assume loads to be applied at stringers.

b. Plot influence lines for moment at sections 1 and 2.

c. From an inspection of the influence line, determine where a single concentrated load should lie to cause maximum positive shear in panel 1–2 and maximum positive moment at section 2.

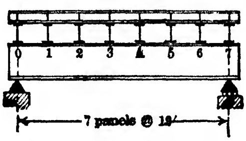
d. Compute by the "influence-line method" the exact maximum positive shear produced in panel 1–2 by a uniform live load of 2,000 lb. per foot, and check this result by computing the shear analytically by using panel loads.

PROB. 30.

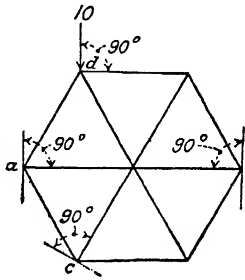


Sections *a* and *c* are to be assumed at an infinitesimal distance from the adjoining point of support.

PROB. 31.



PROB. 32.



e. Compute the maximum positive live shear in panel 1–2 by the approximate method given in Art. 44.

32. The reactions at a, b, and c of this structure act along the lines shown. Compute their values resulting from the application of the load shown at d. Structure is a regular hexagon.

CHAPTER III

CONCENTRATED LOAD SYSTEMS

45. Shear at a Fixed Section. Girder without Floor Beams.— To determine the position of loads that will produce maximum shear at a given section of a simple end-supported beam or deck girder a method of trial may be employed. Stated briefly, this consists of moving the loads along the beam by intervals

Loads are in thousands of lbs. per wh

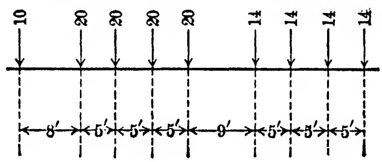


Fig. 59.

corresponding to the distance between wheels and writing expressions for the change in shear thus produced. This process is repeated until the shear is found to decrease.

This method is based upon the fact that the maximum shear at a given section of a simple beam carrying concentrated loads

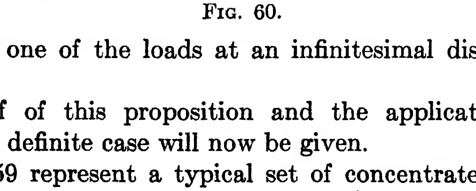


Fig. 60.

occurs with one of the loads at an infinitesimal distance from the section.

The proof of this proposition and the application of the method to a definite case will now be given.

Let Fig. 59 represent a typical set of concentrated loads, in this case a single consolidation locomotive, and let it be desired to compute the maximum shear at section a, for the beam shown in Fig. 60.

The influence line for the section is shown in Fig. 60 and shows clearly that for maximum positive shear at section *a* most of the heavy loads must be the right of *a*.

To prove that one of the loads should lie an infinitesimal distance to the right of the section, or practically *at* the section, proceed as follows: Suppose the loads to be on the beam as shown in Fig. 61.

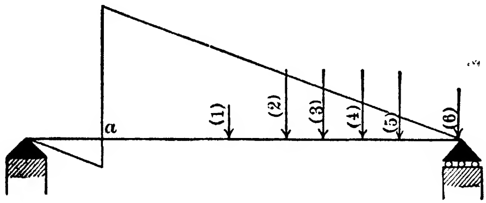


FIG. 61.

As the shear due to a set of concentrated loads in any position equals the summation of the product of the loads and their ordinates, it is evident that starting with loads in the position shown in Fig. 61 the shear at *a* will be increased by moving the loads to the left until load 1 reaches the section. If the loads are moved still further until load 1 passes to the left of the section, there will be a sudden decrease in the shear due to load 1 crossing the section. The new position is shown by Fig. 62, from which it is again evident that if the loads are moved still further to the left there will be an increase in shear until load 2 comes to the section and that the result of load 2 crossing the section will be another sudden decrease in shear, after which the shear will again increase till another load reaches the section, and so on. It is also clear that the effect of a load coming on the span at the right or going off at the left during

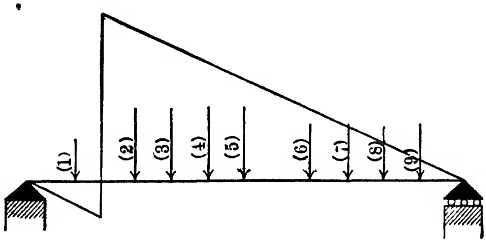


FIG. 62.

the process of moving up the loads will not affect the above conclusions.

Figure 63 is a graphical illustration of the changes in shear at *a* of the beam shown in Fig. 60 as the loads move to the left. The ordinates represent this shear with load 1 at the point where the ordinate is shown. In consequence the line 1–2 shows the increase in shear at *a* due to moving load 1 on the span until load 2 reaches the right end; line 2–3 shows the increase due to moving to the left the first two loads until load 3 reaches the right abutment; and so on, up to line 9–10, which shows the effect

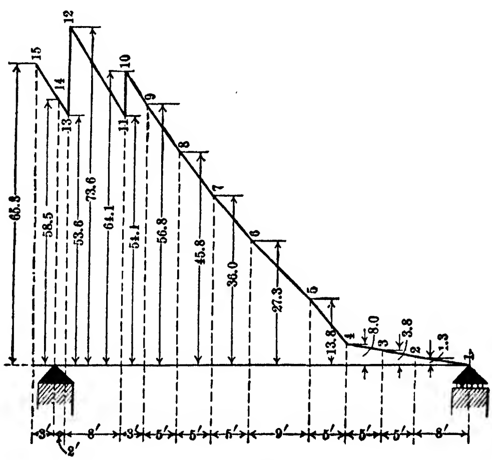


FIG. 63.

of moving the first 9 loads, *i.e.*, all the loads, until the first load reaches the section *a*. When the first load crosses the section, the shear drops suddenly by 10,000 lb. and then increases again, as shown by line 11–12, until the second load reaches section *a*. As this load crosses the section, the shear is diminished by 20,000 lb. and then increases, as shown by line 13–14, until the first load passes off the span. This does not produce a sudden change in shear but changes the slope of the line, as shown by 14–15.

From the preceding discussion, it is evident that the following method may be used to determine the location of locomotive loads for maximum positive shear at any section of a simple beam:

Starting with all the loads to the right of the section and with load 1 at the section, write an expression for the change in shear due to moving load 2 to the section. If this expression shows a decrease, it is evident that load 1 at the section gives the maximum shear. If, on the other hand, the expression shows an increase, it will be necessary to write another expression for the change due to moving up load 3, and so on, until a decrease is finally obtained.

In practice the operation is simple, as is shown by the following example for the beam and loads of this article. It will be noticed that instead of writing an equation for the change in shear the method used is to write an inequality, one side of which shows the increase in the left-hand reaction due to moving up those loads *which are on the span to begin with and remain on or which come on during the process of moving*, and the other side of which shows the effect of a load crossing the section or going off the span to the left.

The numerical expressions for the case in question will now be given.

With (1) at section, move up (2).

$$146 \times 8/60 > 10$$

As the left-hand quantity is greater than the right, it is evident that the shear is increased by moving up load 2.

With (2) at section, move up (3).

$$(146 - 10)5/60 + \delta + (10)2/60 < 20$$

Since in this case the left-hand side of the inequality is less than the right-hand, it is evident that there is no further increase and that the maximum shear will occur with load 2 at section *a*.

As the left-hand side of the foregoing expression may not be entirely clear, a few words of explanation may be added. The first term shows the increase in the left-hand reaction due to moving up those loads which are on the span to begin with and which remain on the span. The second term δ represents the slight increase in the shear at the section due to loads that may have come on the span at the right end of the bridge during the process of moving up the loads. This term is always small and may often be ignored. Its value in the present case is 0. The third term gives the shear caused by load 1 when load 2 is

at section a. This shear being negative and disappearing during the movement on account of the load going off the span, an increase in shear is obtained which is, therefore, placed on the left-hand side of the inequality.

The position of the loads for maximum shear having been determined in the foregoing fashion, it remains simply to compute this shear in the ordinary manner by figuring the left-hand reaction and subtracting therefrom the loads between it and the section.

46. Moment at a Fixed Section.—The method of determining the position of loads for maximum moment differs somewhat

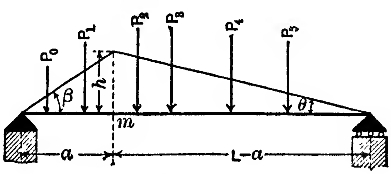


FIG. 64.

from that used in determining the position for maximum shear and is as follows:

Let the original position of the loads be as shown in Fig. 64.

Let ΔM = increase in moment at m due to moving all the loads a short distance d to the left.

Then, since the change in the moment at m caused by the movement of the load system equals the summation of the product of each load by the *change* in length of the influence-line ordinate corresponding to that load, the following expression for the increase in moment may be written:

$$\begin{aligned}\Delta M &= (P_2 + P_3 + P_4 + P_5)d \tan \theta - (P_0 + P_1)d \tan \beta \\ &= (P_2 + P_3 + P_4 + P_5)d\frac{h}{L - a} - (P_0 + P_1)d\frac{h}{a}\end{aligned}$$

Therefore,

$$\frac{\Delta M}{hd} = \frac{P_2 + P_3 + P_4 + P_5}{L - a} - \frac{P_0 + P_1}{a}$$

This equation shows that the moment at m will be increased by moving the loads to the left if the *average load per foot* on the *right* of m is greater than the *average load per foot* on the

left. The converse of this proposition is also true. It should be noted that if the average load per foot on the right equals the average load per foot on the left there will be no change in moment caused by moving the loads.

The foregoing equations are true, provided that the relative position of the loads does not change, *i.e.*, if no load comes on from the right, or goes off to the left, or passes the section. It may be readily seen, however, that if the average load per foot on the right exceeds that on the left a movement to the left sufficient to bring another load on from the right or to cause a load to go off to the left serves to increase the moment more rapidly and hence does not vitiate the conclusion that the loads should be moved to the left. It is also evident that the movement to the left should be continued until P_2 reaches the section. Hence, we have the following theorem:

For maximum moment at any section of an end-supported girder, one load must lie at the section, and the loads must be so located that with that load just to the right of the section the average load per foot on the right is greater than that on the left, while with that load just to the left of the section the average load per foot on the left is greater than that on the right.

A special case of the foregoing is when the average load per foot on one side equals the average load per foot on the other side. In this case a load does not have to lie at the section; but if it does lie at the section the moment will be equal to the maximum, and hence the theorem applies for this case, also.

It should be noticed that the proof of this theorem would be equally applicable to any case where the influence line is composed of two straight lines and that in consequence the theorem is very useful for many cases other than that of moment on a simple beam; *e.g.*, in finding position of loads for maximum stress in truss members.

The application of this theorem is simple, but it sometimes happens that several loads of the same system will be found to satisfy the above criterion. This is explained by the fact that a different set of loads may be on the span for each position, and consequently there may be several maxima. In such cases, it is usually necessary to compute either the value of each maximum or else the change in moment due to moving the loads from one maximum position to another.

A numerical example of the determination of the position for maximum moment will now be given.

Let the loads and span be as in Art. 45, and let the problem be to find the position giving maximum moment at *a*.

	Average load per ft. on left		Average load per ft. on right
First try load 2.			
Load 2 to right of section,	$^{10}/_{10}$	$<$	$^{136}/_{50}$
Load 2 to left of section,	$^{30}/_{10}$	$>$	$^{116}/_{50}$
Load 2 gives a maximum.			

	Average load per ft. on left		Average load per ft. on right
Try load 3.			
Load 3 to right of section,	$^{20}/_{10}$	$<$	$^{116}/_{50}$
Load 3 to left of section,	$^{40}/_{10}$	$>$	$^{96}/_{50}$

Load 3 also gives a maximum.

It is seen by inspection that in this case it is unnecessary to try load 4 and that loads 2 and 3 are the only ones giving maximum moments. To determine which of these is the larger, it is advisable to compute both independently and then check the results by computing the change in moment produced by starting with load 2 at *a* and moving load 3 to *a*.

That the maximum moment at a given section due to a set of concentrated loads always occurs with a load at the section is apparent from the fact that the maximum moment for a given position of loads occurs where the shear curve crosses the axis, *i.e.*, where the shear equals or passes through zero, and that this can never happen except at one of the loads.

47. Shear. Girder with Floor Beams.—For such girders the maximum shear in every panel must be computed. The method of determining the position of the loads differs in detail from that given in Art. 45, although the same general method may be used.

To illustrate this case the bridge shown in Fig. 65 is chosen. Here again, for greater clearness, the stringers and floor beams are shown above the girders, though, as already explained, such

construction is uncommon. End floor beams are also used, but this makes no difference in the method or its application.

Consider first the position of loads for shear in panel 1. In this case, it is clear that the maximum shear occurs with the same condition that would produce maximum moment at panel point a, since the proof given in Art. 46 applies equally well here. In consequence, one of the loads must lie at a. To determine which, either the method of moving up the loads explained in Art. 45 may be used, or that of obtaining the position of the loads for maximum moment at a. If the latter plan is adopted, it may happen that more than one position will be found to give a maximum, and hence an extra computation will be needed. This latter is, however, useful as a check and is not a conclusive argument against the method since an approximate check computation with another load at the section should invariably be made.

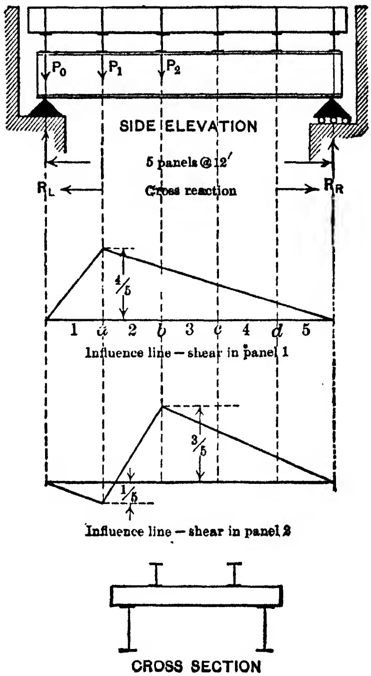


FIG. 65.

The fact that the maximum shear in the end panel and the maximum moment at the first panel point occur simultaneously is important. It follows that, since none of the live loads can be applied to the girder between panel points, the maximum live moment at the first panel point equals the product of the maximum live shear in the end panel and the length of that panel.

For intermediate panels the latter method can not be used since it is incorrect, except for cases where the influence line is composed of two straight lines forming a triangle with the axis of the beam, a condition that does not occur for intermediate panels. For such panels, therefore, the method of Art. 45 will be adopted. Examination of the influence line shown in Fig. 65 for

the shear in the second panel, which is typical of the influence lines for all intermediate panels, shows that the loads when brought on from the right must at least be moved to the left until the first load reaches b. Further movement to the left will cause a decrease in the shear due to the first load, but an increase due to the loads on the right. If the result is a net increase, the loads should be moved until load 2 reaches b. This conclusion is uninfluenced by the action of other loads that may come on the span from the right or by the fact that load 1 may pass a. Further movement to the left produces an additional increase in shear due to loads to the right of b, but a decrease due to load 2, and either an increase or a decrease due to load 1. If the expression for the change is positive, it will remain so until load 3 reaches the section unless load 1 passes off the bridge, which would lower the rate of increase and perhaps cause a decrease. This condition is, however, not likely to occur and may be neglected.

In view of the foregoing, it may be stated that for maximum positive shear in either end or intermediate panels, one of the loads must lie at the panel point to the right.

Before proceeding to a numerical illustration of these principles, the student should observe that the increase in shear in panel 1 equals the increase in R_L minus the increase in P_O; that the increase in shear in panel 2 equals the increase in R_L minus the combined increase in P_O and P_1; and similarly for other panels.

A load that passes off the span in the process of moving up should always be considered by itself. It should be noted that the change of shear, or of any other function, due to removing a load from a structure, is equal to the shear or other function caused by the load when on the structure. Hence, to find the change in shear due to a load passing off the span, compute the shear due to it in its original position before the loads are moved.

The application of these principles to the structure shown in Fig. 65 will now be given for the locomotive shown in Fig. 59.

Shear in End Panel. Method of Moving Up the Loads.

	Increase in R_L		Increase in P_O
Start with load 1 at panel point a.			
Move load 2 to a.	$146 \times 8/60 + \delta$	$>$	$10 \times 8/12$

Therefore, shear is increased. $\delta = 0$ in this case
Move load 3 to a.

$$136 \times \tfrac{5}{60} + \delta > 20 \times \tfrac{5}{12} + 10 \times \tfrac{4}{12} \times \tfrac{4}{5}\text{*}$$

Therefore, shear is increased. $\delta = 0$ in this case

The fact that the increase in moving up load 3 is very slight and that the next step of moving up load 4 would materially increase the change in P_0 without increasing that in P_1 makes it evident that load 3 at the section gives the maximum shear.

The value of the maximum shear in the end panel may now be computed. The expression for it is

$$\tfrac{20}{60}(53 + 48 + 43 + 38) + \tfrac{14}{60}(29 + 24 + 19 + 14) - 20 \times \tfrac{5}{12} = 73 \text{ (1,000 lb. units)}$$

To show that the conclusions above are correct the shear with load 2 at a will be computed.

$$\tfrac{20}{60}(48 + 43 + 38 + 33) + \tfrac{14}{60}(24 + 19 + 14 + 9) + 10 \times \tfrac{4}{12} \times \tfrac{4}{5} = 72.07 \text{ (1,000-lb. units)}$$

The value of this is less than that for load 3 at the section and is therefore in accordance with the conclusions of the previous method.

Shear in End Panel. Average-load Method.

	Average load per panel on left		Average load per panel on right	
Load 2 to right of panel point a,	$\tfrac{10}{1}$	$<$	$\tfrac{136}{4}$	Therefore, load 2 gives a maximum.
Load 2 to left of panel point a,	$\tfrac{30}{1}$	$>$	$\tfrac{116}{4}$	
Load 3 to right of panel point a,	$\tfrac{20}{1}$	$<$	$\tfrac{116}{4}$	Therefore, load 3 gives a maximum.
Load 3 to left of panel point a,	$\tfrac{40}{1}$	$=$	$\tfrac{96}{4}$	

* The last term in the above expression gives the shear due to load 1 when load 2 is at a. Its value is obtained by computing the floor-beam reaction P_1 and the shear due to it. The reaction P_O may be ignored since it produces no shear in the girder. The same result should be obtained by the usual method of computing R_L and subtracting P_O from it; this gives $(\tfrac{56}{60} - \tfrac{8}{12})10$, which equals the value already found.

Load 4 to right of panel point a, $^{40}/_{1} > {}^{96}/_{4}$ Therefore, load 4 does not give a maximum.

From these expressions, it is seen that by the application of the average-load criterion, loads 2 and 3 are found to give maxima and that it is necessary to calculate both to determine the greater.

It should be noted that in the application of the average-load method the average load per panel instead of the average load per foot has been used. This is simpler and gives the same result when the panels are of equal length, as in the bridge under consideration. If the panels are of unequal length, *this method would be incorrect.*

Shear in Second Panel. Method of Moving Up the Loads.

	Increase in R_L	Increase in $(P_0 + P_1)$	
Start with load 1 at panel point b.			
Move load 2 to b.	$104 \times {}^{8}/_{60} + \delta >$	$10 \times {}^{8}/_{12}$	Therefore shear is increased.

$\delta = 14 \times {}^{9}/_{60}$, but it is evident that this value need not be computed.

	Increase in R_L	Increase in $(P_0 + P_1)$	
Move load 3 to b.			
	$132 \times {}^{5}/_{60} + \delta <$	$10 \times {}^{4}/_{12} + 20 \times {}^{5}/_{12}$	Therefore, load 2 gives a maximum.

$\delta = 14 \times {}^{2}/_{60}$ (necessary to compute in this case since otherwise results would be doubtful).

The right-hand side of the inequality above may require some explanation.

With load 2 at b,

$$P_0 = 0 \qquad \text{and} \qquad P_1 = {}^{8}/_{12}10$$

With load 3 at b,

$$P_0 = {}^{1}/_{12}10 \qquad \text{and} \qquad P_1 = {}^{11}/_{12}10 + {}^{5}/_{12}20$$

Therefore,

$$\text{Increase in } (P_0 + P_1) = 10 \times ({}^{12}/_{12} - {}^{8}/_{12}) + {}^{5}/_{12}20$$

That is, the increase in $(P_0 + P_1)$ when load 1 is moved from the second into the first panel equals the reaction on the floor beam at b due to load 1 when load 2 is at b, plus the increase in the reaction on the floor beam at a due to moving the second load into the second panel.

The student will observe that in all cases where no load goes off or comes on the span or goes out of the panel the distance which the loads are moved appears on both sides of the inequality and may be omitted. Moreover, the denominator of the left-hand term equals the span length and that of the right-hand term the panel length. Hence, we may say that for such conditions the shear will be increased by moving up the loads whenever the average load per foot on the entire span exceeds that on the given panel.

The value of the maximum shear in the second panel equals

$$10 \times 44/60 \times 20/60(36 + 31 + 26 + 21) + 14/60(12 + 7 + 2) - \frac{10 \times 8}{12} = 43.56 \text{ (1,000-lb. units)}$$

As an approximate check the corresponding shear with load 3 at b has been computed and found to be 43.37. If the increase in shear as determined from the expression for the increase due to moving up load 3 is added to this, the result should equal the shear with load 2 at b, thus giving an exact check.

48. Formula for Position of Loads for Maximum Shear for Intermediate Panels. *Girder with Floor Beams.*—The method of moving up the loads as shown in the preceding article is simple and very general. It is applicable not only to the determination of the position of loads for maximum shear but to the determination of position for many other functions. The student should understand it thoroughly and apply it to many different cases until he thoroughly comprehends the influence of such load systems upon the various portions of the girder.

For the practitioner who may wish a definite formula for determining the position, the following may be of use for *intermediate panels* of end-supported girders, the loads being assumed to come on from the right.

The maximum shear in a given panel will occur with that load at the right-hand end of the panel which, if the loads are

moved to the left until the next load reaches the right-hand end of the panel, will first satisfy the following expression:

$$\sum \frac{Pa}{L} \leqq \sum \frac{P_1 a_1}{p}$$

in which P = any load which may be on the span or which may come on during the moving up of the loads from one position to another

a = distance which that load is moved on the span

P_1 = any load which may be at any time in the panel under consideration during the process of moving the loads

a_1 = the distance which P_1 moves in the panel

L = total length of span

p = length of panel under consideration

If no load comes on or goes off the span and if no load passes out of the panel, $a = a_1$. It follows that for this case the first load of the system should lie at the panel point unless the average load per foot on the entire span with that load just to the left of the panel joint is greater than that on the given panel, in which case the second load should be tried at the panel point, and so on, until the position for maximum shear is determined.

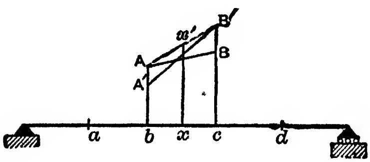


FIG. 66.

49. Maximum Moment. *Girder with Floor Beams.*—For girders with floor beams, it is customary to compute maximum moments at panel points only. If, for any reason, the maximum moment between panel points is desired, it may be obtained with sufficient accuracy by interpolation.

For uniform live loads and for concentrated loads that are fixed in position, interpolation gives exact results since the curve of moments for such loads consists of a series of straight lines. The same is also true for moments due to the weight of the floor system but is slightly in error for the weight of the girder itself. For a system of moving loads, this method is somewhat inaccurate but is on the safe side and hence may be used with security. This is shown by the following demonstration, which refers to Fig. 66.

Let the ordinate bA represent the maximum live moment at any panel point b, due to a concentrated-load system. For the position producing this maximum the moment curve for the portion of the girder between b and c will be the line AB, Bc representing the moment at c for the position of the loads giving maximum moment at b. If the loads are now moved so as to give a maximum moment at c, we shall have cB' and bA' as the ordinates for moments at c and b, respectively, for this new position, and $B'A'$ will be the moment curve between b and c. It is evident from the figure that interpolation between the maximum moment at b and that at c will give a safe value for the maximum moment at any point in the panel, since the line AB can never rise above AB' nor the line $B'A'$ above $B'A$; therefore, the ordinate xx' for the moment at x can never be less than the actual maximum moment at x. It will readily be seen by drawing an influence line for the moment at x that for maximum moment some load should lie at either panel point b or panel point c; that the moment at c with the loads in the position necessary for maximum moment at b can never exceed the maximum moment at c and will almost invariably be less than that; and that this principle holds good for the condition when the moment at b is a maximum. This proof is perfectly general and applicable to any panel.

50. Moment and Shear at the Critical Section.—The cases already treated have been for shear and moment at stated sections of simple beams and for panels and panel points of girders with floor beams. For the latter, it is necessary and sufficient to compute the maximum shear in each panel and the maximum moment at each panel point, since thereby the maximum values of these functions will be obtained. For beams or girders that are not loaded by means of floor beams, it is also necessary to compute maximum values of shears and moments, and in addition, for long girders, the values at intermediate sections should be taken with sufficient frequently to ensure a good design.

To determine the maximum values, it is necessary to locate the sections at which they occur, *i.e.*, the *critical sections*.

For shear the critical section is an infinitesimal distance from one of the points of support. This needs no demonstration, as an inspection of influence lines for various sections, including one at the end, furnishes sufficient proof.

For the moment with uniform load, it has already been shown that the maximum moment in an end-supported beam occurs at the center and equals $\frac{1}{8}wL^2$, when w equals the load per foot and L the span.

With a system of concentrated loads the maximum moment may not occur at the center, though the critical section will be very near the center. To treat this case, it is necessary to make use of the already established principle that for maximum moment at any section of a beam under a system of concentrated loads one of the loads must lie at the section. If, therefore, it is possible to determine the location of the system of loads as they cross the span such that the moment at any

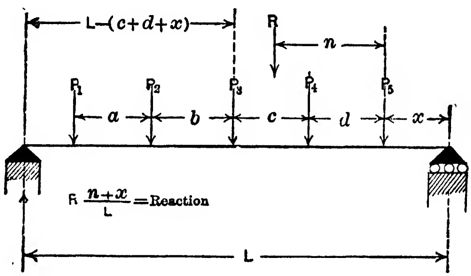


FIG. 67.

one load is a maximum, the problem can be solved by trying a sufficient number of loads and computing the different maxima. As will appear later the critical section in an end-supported beam is always near the center of the span; hence, as a rule, only loads need be tried that are found to give a maximum moment at the center.

Consider the set of loads shown in Fig. 67, and let the problem be the determination of the position of these loads in order that the moment at P_3 may be a maximum. Let R be the resultant of the loads P_1 to P_5 and n its distance from the last load P_5.

Let x be the distance from P_5 to the right support when the loads lie in the proper position for maximum moment at P_3.

Then the moment at P_3 is given by the equation

$$M_3 = R\frac{(n + x)}{L}[L - (c + d + x)] - P_1(a + b) - P_2b$$

For maximum value of M_3, differentiate with respect to x and put the first derivative equal to 0. This gives

$$\frac{dM_3}{dx} = \frac{R}{L}[-n + L - c - d - 2x] = 0$$

Therefore, in order to find the maximum moment at P_3 as the loads cross the span, P_3 must be so located that

$$-n + L - c - d - 2x = 0$$

or

$$L - (c + d + x) = n + x$$

That is, the resultant of the loads on the span when the maximum moment at P_3 occurs must lie as far from the right support as the load itself lies from the left support, or, in other words, the *center* of the *span* must lie *halfway* between the *resultant* and the *load.*

The following examples serve to illustrate the application of this principle:

Problem: Compute the absolute maximum moment on a simple beam of 12-ft. span due to two wheel loads of 10,000 lb. each spaced 6 ft. between centers.

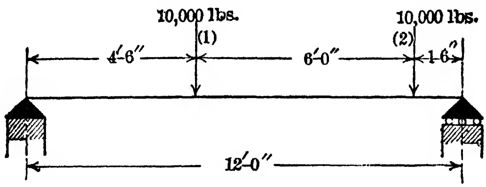


FIG. 68.

Solution: In this case, there are two equal loads; hence, it is immaterial which load is considered. For maximum moment at load 1 the loads should lie as shown in Fig. 68, the center of the span being halfway between load 1 and the resultant of the two loads. The moment at the first load will then equal

$$20{,}000\frac{(6 - 1\frac{1}{2})^2}{12} = 33{,}750 \text{ ft.-lb.}$$

The maximum moment at the center for this beam would be 30,000 ft.-lb.; hence, the absolute maximum moment exceeds the maximum center moment by over 10 per cent.

It should be particularly noted that the demonstration that has been given serves to fix the position for maximum moment

at a given load only with certain assumed loads on the span and that if a different set of loads is on the span the position will be different. To illustrate this, consider the same loads as in the previous examples and a span of 10 ft. There are then two positions of the first load that give maximum moment. First, assume only the first load to be on the span; in this case, it should be placed at the center and the moment would be 25,000 ft.-lb. Second, assume two loads on the span; in this case, the center of the span should be halfway between the resultant of the two loads and the first load, and the maximum moment at the first load will equal

$$20{,}000\frac{(5 - 1\tfrac{1}{2})^2}{10} = 24{,}500 \text{ ft.-lb.}$$

which is somewhat less than with one load at the center. In such a case the length of span can easily be determined for which one load at the center gives a moment at the load equal to that with two loads on the span. In general, it is necessary to consider both cases when dealing with two loads.

The absolute maximum moment on spans above 25 or 30 ft. in length does not materially differ from the maximum center moment, and in practice the latter is generally used.

For the loads previously considered with a 30-ft. span the absolute maximum moment equals

$$20{,}000\frac{(15 - 1\tfrac{1}{2})^2}{30} = 121{,}500 \text{ ft.-lb.}$$

and maximum center moment equals

$$20{,}000(\tfrac{12}{30})15 = 120{,}000 \text{ ft.-lb.}$$

The difference is about 1 per cent.

The following example serves to show the application of this principle for a locomotive loading:

Problem: Determine the maximum moment on a simple beam of 21-ft. span due to the locomotive given in Art. 45.

Solution: First determine which load or loads give maximum moment at the center, as it is probable that one of these loads will give the absolute maximum moment. By applying the criterion for maximum moment, loads 3 and 4 are found to give maxima, but it is clear that the center moment with load 3 at the center will equal the center moment with load 4 at the

center and that it makes no difference whether we use one or the other load. Let the maximum moment therefore be determined at load 3, it being assumed that loads 2 to 5 are on the span. The position for maximum moment will then be as shown in Fig. 69, and the moment at load 3 will equal

$$80{,}000\frac{(10\frac{1}{2} - 1\frac{1}{4})^2}{21} - 20{,}000 \times 5 = 225{,}950 \text{ ft.-lb.}$$

In this case, it is impossible to get more than four loads on the span at once. If three loads are on the span, the resultant coincides with load 3; hence, for a maximum for this assumption, load 3 should lie at the center. But this is inconsistent with three loads being on the span; hence, a maximum at load 3 with only three loads on the span cannot be obtained, and the case considered gives the absolute maximum moment.

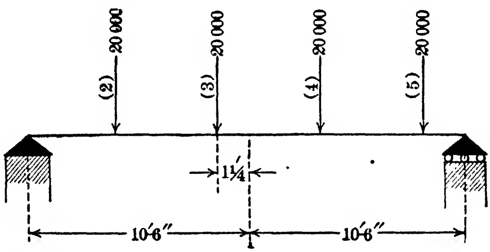


FIG. 69.

The maximum center moment for these loads occurs with either load 3 or load 4 at the center and equals

$$80{,}000 \times \tfrac{8}{21} \times 10\tfrac{1}{2} - 20{,}000 \times 5 = 220{,}000 \text{ ft.-lb.}$$

so that in this case the difference is 2.7 per cent.

51. Moments and Shears. *Floor Beams and Transverse Girders.*—As a preliminary step in the examination of this case the influence lines shown in Figs. 70 and 71 have been drawn. These are influence lines for stringer reactions on floor beams. Since the stringers are simple beams of length equal to one panel and are supported at the ends upon the floor beams, it is evident that a load moving along the bridge causes no reaction on a floor beam unless it is on one of the stringers in one of the panels adjoining the floor beams in question. Figure 70 represents the stringer reactions on an intermediate floor beam and Fig. 71 on an end floor beam.

It will be noticed that the influence line shown in Fig. 70 has the same characteristics as the influence line for moment

at any section of a simple end-supported beam; hence, the demonstration of Art. 46 is applicable. The conclusion may therefore at once be drawn that for maximum reaction on an intermediate floor beam one load must lie at the beam and that that load must be one which, when placed just to the right of the given floor beam, makes the average load per foot on the stringers in the right-hand panel greater than on those in the left panel and, when placed just to the left of the floor beam, reverses this condition. For the end floor beam, the maximum reaction occurs for the loading giving maximum stringer reaction and equals that reaction.

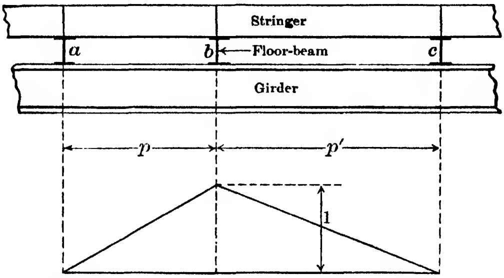


FIG. 70.—Influence line for stringer reaction on floor beam at b.

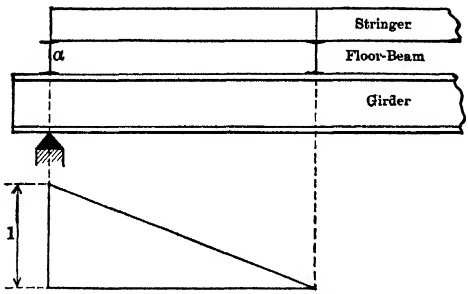


FIG. 71.—Influence line for stringer reaction on floor beam at a.

It remains to consider the actual moments and shears on the floor beams. Curves of moments and shears for a floor beam due to stringer reactions are shown in Fig. 72.

Both shear and moment are direct functions of the stringer reactions. The maximum moment must occur at one of the stringers, since the floor beam is in the condition of a girder loaded with concentrated loads and the curve of shears can

cross the axis only at a load. The case illustrated is not the usual one, since the stringers are unsymmetrically placed with respect to the center of the floor beam. Were the floor beam symmetrical, the maximum moment would occur at both stringers and at all points between. Since, in the actual design, the dead load of the floor beam would also have to be considered, the maximum combined live and dead moments for the ordinary symmetrical floor beam occurs at the center.

For floor beams where the stringers in one of the adjoining panels are not of equal length, *i.e.*, where the panel is a skew panel, special treatment is necessary. It is usually advisable

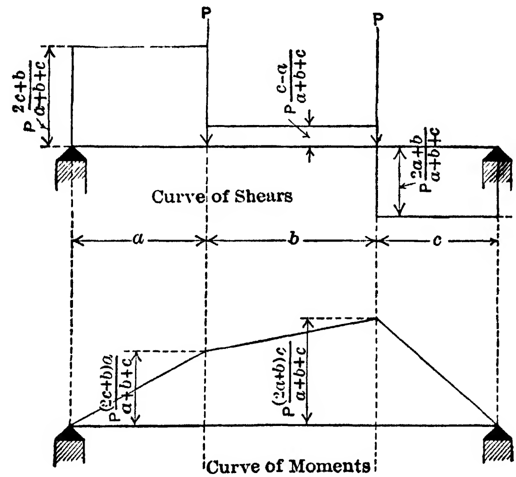


FIG. 72.

to treat this case by means of influence lines without attempting to apply special rules.

The application of the methods just given to the determination of the maximum moment and shear on a floor beam (or a transverse girder, such as a cross girder in an elevated railroad structure) will now be illustrated.

Problem: Determine maximum moment and shear on floor beam *b* of Fig. 73 for loads shown in Fig. 59, Art. 45.

It may be easily seen that a given load when equidistant from the floor beam *b* produces a greater reaction if on the longer stringer; hence, it is probable that the maximum reaction in this case will occur with the greater number of loads on the 15-ft. panel. Let the loads, therefore, be brought on from the left.

	Average load per ft. on left		Average load per ft. on right	
Load 2 to left,	$^{60}/_{15}$	$>$	$^{10}/_{10}$	Load 2 does not give a maximum.
Load 2 to right,	$^{60}/_{15}$	$>$	$^{30}/_{10}$	
Load 3 to left,	$^{60}/_{15}$	$>$	$^{20}/_{10}$	Load 3 gives a maximum.
Load 3 to right,	$^{40}/_{15}$	$<$	$^{40}/_{10}$	
Load 4 to left,	$^{54}/_{15}$	$<$	$^{40}/_{10}$	Load 4 does not give a maximum.

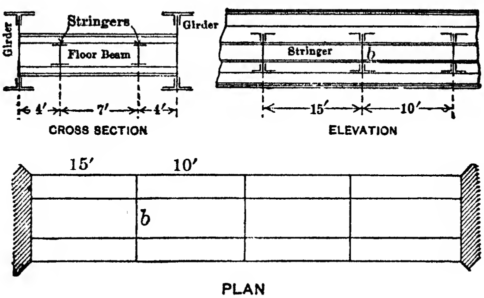


FIG. 73.

Load 3 at b with loads coming on from left evidently gives the maximum floor-beam reaction. Its value is given by the expression,

$$20 \times \frac{5 + 10}{10} + 20\frac{10 + 5}{15} = 50$$

That is, the reaction on the floor beam at each stringer connection equals 50,000 lb.

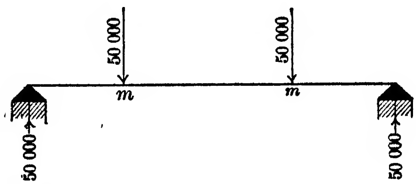


Diagram showing loads on floor beam.

FIG. 74.

The floor beam is then in the condition shown by Fig. 74.

The maximum shear = 50,000 lb., and the maximum moment = 200,000 ft.-lb.

Before concluding this article the beginner should be cautioned to avoid the mistake that is frequently made of adding the maximum live reactions on two adjoining stringers to determine the floor-beam load at a point such as m in Fig. 74. The fact that the

MOMENT DIAGRAM—COOPER'S E_{40} LOCOMOTIVE

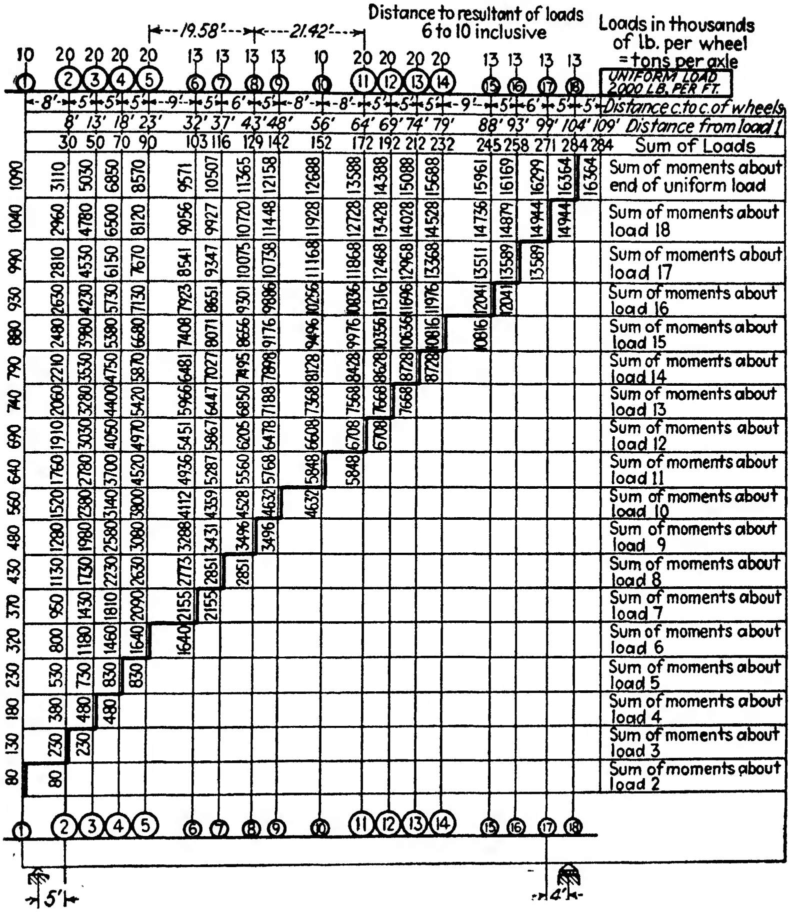


Fig. 75.

Moments are in thousands of foot-pounds per rail and are the summations of the moments of all the loads to the left of and including that under which they appear about the load indicated in right-hand columns; *e.g.*, the value 10256 under load 10 and opposite load 16 in right-hand column is the moment about load 16 of the loads 1 to 10.

Moments to right of zigzag line are moments about load under which they appear of all loads to left thereof.

Application of Diagram. To find moment about a given load, say load 14, of a certain number of loads to left, say 6 to 13, proceed as follows: Find, in diagram, moment about 14 of all loads to left; subtract from it moment about 14 of loads 1 to 5. The result will be the desired moment. The expression thus obtained = 8728 − 5870 = 2858. The moment of the same loads about a point between 14 and 15 and distant x ft. from 14 can be found by adding to quantity just obtained the product of the sum of the loads 6 to 13 and the distance x (see Art. 34).

maximum reaction on a stringer occurs when one of the heavy loads lies at the end of the stringer is sufficient to show that the same condition cannot exist on the adjoining stringer in the next panel, because such a condition would necessitate two wheel loads occupying practically the same place at the same time.

52. Moment Diagram.—To save repetition of computations for a given set of concentrated loads when used for varying spans, it is customary to use a moment diagram upon which certain quantities frequently required and unaffected by the length of spans are placed once for all. Upon this diagram the loads are plotted to a convenient scale at top and bottom of sheet for convenience in reading, and the quantities desired are placed between. The diagram is used in connection with another sheet upon which the span is drawn to the scale used in plotting the loads. The diagram shown in Fig. 75 is of a convenient form and is self-explanatory.

The use of the diagram can be readily understood from the simple example that follows. Let the problem be the computation of the moment at the second panel point from the left of a span having five panels at 20 ft. when load 8 is at the given panel point. Place the plotted span on a separate sheet so that load 8 is over panel point 2; the ends of the span will then be in the position shown at bottom of Fig. 75. Since the desired moment equals the moment of the left reaction due to loads 2 to 17 about load 8, minus the moment of loads 2 to 8 about the same point, it is necessary to compute these two quantities. The moment of the left reaction equals the moment of the loads on the span about the right reaction divided by five panel lengths and multiplied by two panel lengths.

This equals

$$\tfrac{2}{5}[13{,}589 - 990 + (271 - 10) \times 4] = \tfrac{2}{5}13{,}643 = 5{,}457.2$$

The moment of loads 2 to 7 about panel point 2 equals

$$2{,}851 - 10 \times 43 = 2{,}421$$

Hence, the moment desired $= 5{,}457.2 - 2{,}421 = 3{,}036.2$, expressed in units of thousands of pounds per rail or tons per track.

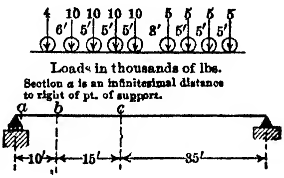
This method involves the application of the principle stated in Art. 34, which should be thoroughly understood.

Problems

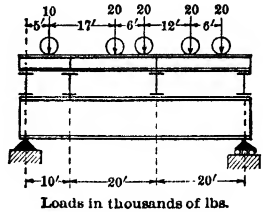
33. *a.* Compute the maximum positive shear in thousands of pounds at sections *a*, *b*, and *c* for the system of moving loads shown.

b. Compute maximum moment in foot-pounds at *b* and *c* for the system of moving loads.

c. Compute uniform live load per foot that would give a maximum moment at section *b* equal to that found for the system of moving loads.



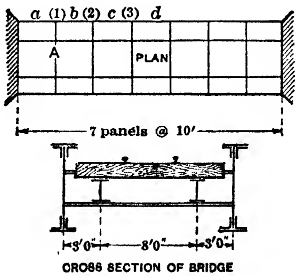
Prob. 33.



Prob. 34.

34. Draw curves of shear and moment for one girder for the concentrated loads shown in the figure (bridge has two girders located symmetrically with respect to the loads).

35. *a.* Draw influence lines for shear in panels *a* and *b* of one girder. Assume two loads of unity to pass over the structure—one on each rail.



Prob. 35.

b. Compute maximum dead shear in panels *a*, *b*, and *c* of one girder and dead moment at panel points 1, 2, and 3 for the following dead loads:

Stringers, 100 lb. per foot per stringer.
Track (rails, ties, etc.), 400 lb. per foot per track.
Floor beams, 100 lb. per foot per floor beam.
Girders, 300 lb. per foot per girder.

c. Draw curve of dead moments for floor beam *A*, using above dead loads.

36. *a.* Determine the position for maximum positive shear in panels *a* and *b* of bridge in Prob. 35:

1. For the system of concentrated loads shown in Fig. 75 coming on from right.
2. For the same loads with train running in opposite direction, *i.e.*, coming on from left.
3. In panel *a* for one of the locomotives shown in Fig. 75 followed, at a distance of 5 ft., by the uniform load, the loads coming on from right.

b. Compute the live shear in panels *a* and *b* for each of the positions previously determined.

c. Determine the position of the concentrated-load system for maximum moment at panel point 2, considering only the first and third cases given under *a*. Try driving wheels only.

d. Compute the live moment at panel point 2 for each of the positions previously determined.

e. Compute the maximum live moment at panel point 1 for the system of concentrated loads previously used, trying both one and two locomotives with uniform load.

f. Compute the minimum uniform live load per foot that will give a live shear in panel *b* equal to 93,750 lb.

37. Compute for the bridge of Prob. 35 the maximum live shear and moment on floor beam *A*, using same loads as in Prob. 36.

CHAPTER IV

BEAM DESIGN

53. Formulas.—In order to determine the proper size of beams required to carry given external bending moments and shears, it is necessary to make use of formulas expressing the relation between the outer and inner forces. Such formulas are deduced in all standard books on mechanics and are as follows for beams of homogeneous material and of ordinary proportions:

$$M = \frac{fI}{y} \tag{12}$$

$$v = \frac{VQ}{bI} \tag{13}$$

The terms in these formulas are as follows for any cross section of the beam:

M = external bending moment at section, in.-lb.
I = moment of inertia, in.^{4}, about the neutral axis of the section
y = distance, in., from neutral axis to any fiber
f = direct fiber stress at distance y from neutral axis
Q = statical moment,* in.^{3}, about the neutral axis of the cross section of that portion of the section lying either above or below an axis parallel to the neutral axis and at distance y from it
v = intensity of the longitudinal shear per square inch along any plane parallel to the neutral plane of the beam and at distance y from it
V = external shear, lb., at the section
b = width of beam at distance y from neutral axis

The application of these formulas to actual problems of design requires the selection of beams such that at no section shall My/I exceed the maximum allowable value of f, or VQ/bI exceed the allowable value of v.

* Statical moment of a given area about any axis equals the area multiplied by the distance from its centroid to that axis.

It is evident that under all conditions f will attain a maximum value for any given cross section at the fiber farthest removed from the neutral axis, since y will then be a maximum. For beams of uniform width the largest value of v for any given section will occur at the neutral axis since the statical moment Q has its maximum value about the neutral axis and b is constant. The absolute maximum values of f and v for a beam are functions of the external moment and shear and of the cross section of the beam. If the beam is of constant cross section throughout, then these maximum values will occur at the section where M and V, respectively, have maximum values.

In a beam of variable section, f and v may attain maximum values at several points. For greatest economy of material the maximum values of f and v for the different cross sections of the beam should be constant throughout its length, but it is seldom or never attempted to obtain this condition, since the additional labor cost would far exceed the saving due to economy of material.

Substituting limiting values in formula (12), the following working formula is obtained:

$$M = \frac{sI}{c} \tag{14}$$

in which s = maximum allowable working value of f, lb. per square inch.

c = distance, in., from neutral axis to the extreme fiber of any given section.

M = maximum allowable bending moment on the beam, in.-lb.

For beams of rectangular cross section having a height h and a width b formula (14) becomes

$$M = \frac{s(\frac{1}{12}bh^3)}{h/2} = \frac{1}{6}sbh^2 \tag{15}$$

The maximum value of v for beams of rectangular cross section is given by formula (16) in which A = area of the cross section.

$$v = V\frac{\left(b\frac{h}{2}\right)\frac{h}{4}}{b(\frac{1}{12}bh^3)} = \frac{3V}{2bh} = \frac{3V}{2A} \tag{16}$$

54. Method of Design.—Frequently the design of beams requires merely the application of formula (14), and the shearing strength need not be considered. In the case of comparatively short beams, however, the shearing strength is important and should be investigated. In wooden beams, this is especially important, since the resistance of wood to longitudinal shear is small and such beams may fail by splitting longitudinally. The design of reinforced concrete beams also requires the application of formula (13).

55. Wooden Beams.—In selecting wooden beams, care should be taken to use commercial sizes only. The following table gives such sizes, in inches, available along the Atlantic seaboard in 1939.

Spruce:

2×3, 2×4, 2×5, 2×6, 2×7, 2×8, 2×9, 2×10
3×4, 3×6, 3×8, 3×10, 3×12
4×4, 4×6, 4×8, 4×10
6×6, 6×8, 6×10

8 to 16 ft. are ordinary lengths.
16 to 22 ft. are less common.
Longer lengths are obtained with difficulty.

Yellow Pine: Sizes about the same as for spruce, also:

8×8,	4×12,	2×14,	2×16
	6×12,	3×14,	3×16
	8×12,	4×14,	4×16
		6×14,	6×16
		8×14,	8×16
		10×14,	10×16
		12×14,	12×16
		14×14,	14×16
		16×16	

Lengths of yellow pine sticks are longer than for spruce and run up to 40 ft., and it is usually possible to obtain even 50-ft. lengths except for the largest sizes.

Douglas fir is available in all the sizes above and up to 18 in. by 18 in., with lengths up to 60 ft.

The cost of wooden beams depends upon the price of lumber per board foot. This is subject to considerable variation; if a close estimate is desired, a dealer should be consulted.

56. Steel Beams.—Such beams are usually made with a cross section of the shape of the letter H or I in order to obtain a large moment of inertia from a comparatively small amount of material. They are rolled from solid pieces of steel in varying heights and thicknesses. In selecting such beams the manufacturers' handbooks should be consulted, and sections marked "standard" chosen since the selection of a "special" section is likely to cause delay in filling the order. These handbooks give all the properties of the beams, such as area, weight, moment of inertia, etc., and may be relied upon as accurate.

The cost of steel beams is dependent upon the weight of the beam and upon the amount of punching, riveting and other work that has to be done. The price is usually figured on a "cent per pound" basis, the price of the rolled beam being taken as the base price and the other prices added thereto. Other things being equal, the lightest beam having the requisite strength and stiffness is the most economical. The base price is published from time to time in such engineering papers as the *Engineering News-Record*, *Iron Age*, etc.; *e.g.*, in the *Engineering News-Record* of Dec. 1, 1938, the price at New York of carbon-steel beams was quoted at $3.75 per 100 lb.

Cost of erection per ton depends upon distance of project from railway or waterfront, and other local conditions.

57. Examples of Beam Design.

Problem: Design wooden and steel beams for a 12-ft. span. Beams to be supported at ends and to be loaded with a total uniform load (live and dead) of 1,000 lb. per foot. Allowable unit stresses for timber to be 1,300 lb. per square inch in bending, 120 lb. per square inch in longitudinal shear, and 380 lb. per square inch in bearing across the grain. For steel, allow 18,000 lb. per square inch fiber stress in bending and 11,000 lb. per square inch shear on gross section of web. Fifty per cent of total load to be added in the case of the steel beam to allow for impact.

Solution: Maximum moment is at center of beam and equals

$$\tfrac{1}{8}1{,}000 \times 12 \times 12 = 18{,}000 \text{ ft.-lb.}$$

Maximum shear is at end of beam and equals 6,000 lb.

For wooden beam,

$$M = \tfrac{1}{6}sbh^2$$

Therefore,

$$18{,}000 \times 12 = \tfrac{1}{6}1{,}300bh^2$$

Therefore,

$$bh^2 = 997$$

An 8- by 12-in., a 6- by 14-in., or a 4- by 16-in. beam each has a value of bh^2 greater than that required and may be used.

The area of cross section needed to carry shear may be determined by Eq. (16) and is given by the following expression:

$$A = \frac{18{,}000}{240} = 75 \text{ sq. in.}$$

Evidently the 4- by 16-in. beam is too small, and one of the other beams should be selected. The 6- by 14-in. is the cheapest and should be chosen if conditions permit its use. The longer side should always be placed parallel to the plane of the loads, *i.e.*, vertical if the loads are vertical. This was the position assumed in solving for bh^2, and none of the beams selected would be strong enough if not so placed.

The bearing area on the abutment should also be determined. If the reactions were uniformly distributed over the bearing surface, there would be needed 6,000/380 = 16 sq. in. To allow for unequal distribution, 50 per cent will be added to this, giving 24.0 sq. in. The 6- by 14-in. beam would therefore need to extend 24 in./6, or 4 in., over the abutment.

For the steel beam the moment after allowance for impact is made = 27,000 ft.-lb.
Therefore,

$$\frac{I}{c} = \frac{27{,}000 \times 12}{18{,}000} = 18.0$$

The term I/c is known as the section modulus. Values of this for various beams are given in the handbooks issued by steel makers, and the lightest beam having a modulus equal to or greater than the figure above should be selected. A 10-in. I beam, weighing 25.4 lb. per foot has sufficient strength in bending and will be chosen.

The distribution of shear over an I beam is more complicated than in a rectangular beam. It will, however, be shown later that the shear is practically all carried by the web over which it is distributed almost uniformly. Making this assumption the area required in the web would be 9,000/11,000 sq. in., and as the actual area in the beam selected is far in excess of this the beam is evidently strong enough to carry the shear and hence may be used with safety.

The sizes selected were based upon the assumption that the beam would have no rivet or bolt holes and no other reductions in the cross-section area. If such reductions occur, the value of I should be corrected to allow for the reduction in section, and the value of c also changed if the position of the neutral axis is shifted by the change in area. Methods of making such corrections will be given in full in Chap. V.

Another important element to be considered in selecting beams is that of vertical and horizontal stiffness. This will be considered in Art. 59, it being assumed for the present that the beams designed in this article are supported laterally where necessary and that their vertical deflections are not excessive.

Problem: Design wooden and steel beams for a single-track electric-railroad bridge of 12-ft. span carrying the electric car shown in Fig. 76. Assume track (ties, rails, etc.) to weigh 400 lb. per lineal foot (200 lb. per foot per rail) and each beam to weigh 40 lb. per lineal foot. Allowance *for impact to be 25 per cent. Unit stresses as in previous problem.*

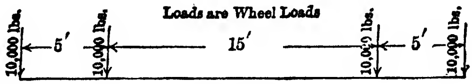


FIG. 76.

Solution:

Maximum Moment

$$\text{Dead} = \tfrac{1}{8}240 \times 12 \times 12 = 4{,}300 \text{ ft.-lb.}$$

$$\text{Live} = 20{,}000\frac{(6 - 1.25)^2}{12} = 37{,}600 \text{ ft.-lb.}$$

$$\text{Impact} = \underline{9{,}400} \text{ ft.-lb.}$$

$$\text{Total moment} = 51{,}300 \text{ ft.-lb.}$$

Maximum Shear

$$\text{Dead} = 240 \times 6 = 1{,}440 \text{ lb.}$$

$$\text{Live} = 10{,}000 \times \frac{19}{12} = 15{,}830 \text{ lb.}$$

$$\text{Impact} = \underline{3{,}960} \text{ lb.}$$

$$\text{Total shear} = 21{,}230 \text{ lb.}$$

Steel Beam.—For a steel beam, on the assumption of no reduction due to bolt holes,

$$\frac{I}{c} = \frac{51{,}300 \times 12}{18{,}000} = 34.2$$

$$\text{Web area needed for shear} = \frac{21{,}230}{11{,}000} = 1.93 \text{ sq. in.}$$

A 12-in. I beam 31.8 lb. is large enough for bending, and as it has a web area of 4.20 sq. in. its strength in shear is far greater than necessary.

As the actual weight of the beam is slightly less than originally assumed, no recomputation is necessary. A considerable error might, however, have been made in the original assumption without requiring a recomputation since the moment and shear due to the weight of the beam are small percentages of the total moment or shear.

A 10-in. wide flange (CB) beam weighing 33 lb. per foot could also be used if depth of beam must be limited but would be somewhat more expensive.

Wooden Beam.—For wooden beam, impact being neglected

$$41{,}900 \times 12 = \tfrac{1}{6}\, 1{,}300\, bh^2$$

Therefore,

$$bh^2 = 2{,}310$$

$$\text{Area needed for shear} = \frac{3}{2}\,\frac{17{,}270}{120} = 216 \text{ sq. in.}$$

One beam 14 by 16 in. with 16-in. side vertical fulfills both requirements and will be chosen. Its weight is somewhat in excess of that assumed, but as its strength is also in excess of the requirements no revision need be made.

58. Composite Beams.—The cases just treated are of simple beams only, but it sometimes happens that a composite beam is used, as, for example, a so-called *flitch beam* consisting of two wooden beams and a steel plate bolted together and used as one beam. Another example is that of two beams of unequal size laid side by side. For both of these cases the load carried by each member is in proportion to the product of its moment of inertia and modulus of elasticity and can be easily computed. Still another case is that of one beam laid on the top of another, but not fastened to it. Such a beam is of slightly greater strength than two beams laid side by side; the additional strength is due, however, to friction between the beams and should be neglected in design. If the beams are fastened together by rivets or, in the case of wooden beams, by keys, the combination may be figured as one beam with a cross section corresponding to that of the two beams, provided that the rivets, or keys, are of sufficient strength to carry the longitudinal shear which would exist at the plane of contact. Reinforced-concrete beams form the most important class of composite beams, but these will not be considered in this book. For a full discussion of these beams the student is referred to standard treatises on reinforced concrete.

59. Stiffness.—Beams are seldom used for bridge spans exceeding 60 ft. in length, since above that span the ratio of length to depth is so great that the deepest beam made, a 36-in. CB beam, lacks sufficient stiffness. It is common to specify that railroad-bridge beams exposed to bending shall, where possible, have a depth not less than one-twelfth of the span and that highway-bridge beams shall have a depth not less than one twenty-fifth of the span. For buildings, longer beams are admissible, but the length should generally be restricted to twenty times the depth for floors and less than that if the floor is to be subjected to vibration and shock. In roofs, somewhat longer beams may be used. For long-span bridges the beams should be cambered, *i.e.*, bent upward so that when loaded with dead and live load they will be practically horizontal. Manufacturers are equipped to furnish such curved beams.

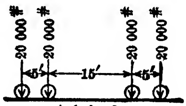
When the beams are not supported laterally, the ratio of length to width of the compression flange should be considered and the allowable unit stresses reduced accordingly. The rule in common use for railroad and highway bridges is to reduce the allowable unit stress by $5(l/b)^2$ when l = unsupported length and b = width of flange. For spans of considerable length, it is usually more economical to use lateral bracing between the top flanges of the beams than to use the heavier beams that would otherwise be necessary. It is assumed in the problem of Art. 57 that such bracing is used.

Problems

38. *a*. Design a steel I-beam stringer for bridge and loads given in Probs. 35 and 36, using unit values of Art. 18, and allowing 75 per cent for impact.

b. Design a yellow-pine stringer for the same bridge, using unit values of Art. 18 and neglecting impact. Proportion stringer for bending only, but compute the maximum intensity of longitudinal shear in the stringer selected.

39. *a*. Design a steel I-beam stringer for an electric-railroad bridge, the bridge to be a single-track bridge with 15-ft. panels and stringers



PROB. 39*a*.

PROB. 39*b*.

located symmetrically with respect to the rails. Assume total dead weight of track and stringers to be 600 lb. per lineal foot of bridge. Use live loads shown, and allow 25 per cent for impact. Unit stresses as given in Art. 18.

b. Design steel I-beam stringers for the bridge for the electric locomotive shown. All other conditions to be the same as in the previous problem.

CHAPTER V

PLATE-GIRDER DESIGN

60. Plate Girders Defined.—A plate girder is essentially an I beam made, not out of one solid piece of metal, but out of a number of pieces riveted together. Figure 3 shows a plate-girder bridge, and Fig. 77 shows the cross section of a typical plate girder.

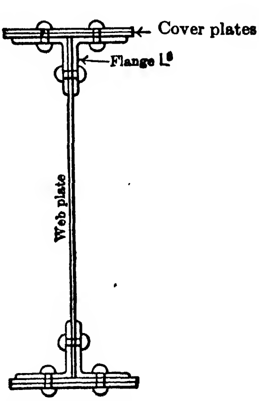


Fig. 77.—Cross section of a plate girder.

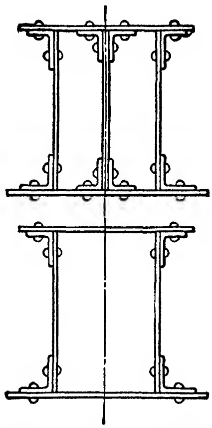


Fig. 78.—Cross sections of two box girders.

Plate girders are rarely made of greater depth than 10 ft. 6 in. owing to difficulties in transportation by rail, and a length of 100 ft. is seldom exceeded for the same reason, although girders 125 ft. in length have been made and shipped in one piece. Occasionally plate girders are made in sections and spliced in the field, but this expedient is not common and should not be adopted except to meet some unusual condition, such, for example, as one in which either economy or appearance requires a girder continuous over several spans.

A plate girder with more than one web is called a *box girder*. It is used in situations where great strength with limited depth is required.

61. Plate-girder-web. Theory.—Plate girder webs are usually proportioned on the assumption that all the transverse shear is uniformly distributed over the gross area of the web and that the allowable unit stresses in standard bridge specifications are determined on this basis.

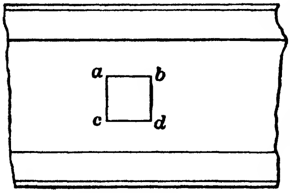


FIG. 79.

That this method is essentially correct is shown by the following demonstration:

Let Fig. 80 represent the square prism *abcd* from the web of the plate girder shown in Fig. 79, and let it be assumed that there are shearing forces acting on all four surfaces, and direct stresses on the two vertical surfaces.

Let t = its thickness = thickness of web.

s_h = intensity of the shearing force on the surface *ab*.

s_h' = intensity of the shearing force on surface *cd*.

s_v = intensity of shearing force on surface *bd*.

s_v' = intensity of shearing force on surface *ac*.

f = intensity of the direct stress on surface *bd* (assumed compression for convenience).

f' = intensity of the direct stress on surface *ac* (also compression).

Application of the equations of equilibrium give the following results:

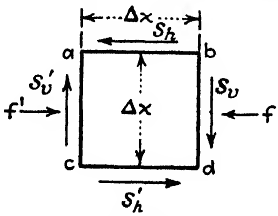


FIG. 80.

$$s_v t\Delta x = s_v' t\Delta x$$

Therefore,

$$s'_v = s_v$$

$$s_h t\Delta x + ft\Delta x = s_h' t\Delta x + f' t\Delta x$$

Therefore,

$$s_h = s_h' + (f' - f)$$

$$(s_h t\Delta x)\Delta x - (s_v t\Delta x)\Delta x + [(f - f')t\Delta x]\frac{\Delta x}{2} = 0$$

Therefore,

$$s_h = s_v + \frac{(f' - f)}{2}$$

As the distance Δx becomes infinitesimal, $\frac{f' - f}{2}$ approaches zero; hence, at the limit, $s_h = s_h' = s_v$.

It therefore follows that the intensity of the horizontal shear in the web of a plate girder at any point on a vertical plane equals the intensity at the same point on a horizontal plane.

Since the intensities of the vertical and horizontal shears are equal, it is evident that the distribution of the vertical shear can be determined by the application of formula (13), from which it at once follows that the vertical shear is distributed over the web with approximate uniformity since Q is the only term in the equation affected by the distance from the neutral axis and its value changes much more slowly than does the distance from the axis. The numerical examples given in Art. 64 show the degree of approximation of this assumption for certain typical girders.

In many girders the thickness of the web is determined by imposing restrictions upon its minimum thickness to prevent undue corrosion. For railroad bridges, it is common to specify that the web shall be not less than 3/8 in. thick.

62. Plate-girder-flange. Theory.—Formula (14) applies to girders as well as to beams. It is, however, in inconvenient form for use and may be replaced in practice for symmetrical plate girders by a less accurate but more easily applied formula. The formula recommended for plate-girder flanges is as follows:

$$A = \frac{M}{sh} - \frac{1}{12}th_1{}^* \tag{17}$$

The derivation of this formula is as follows:

Let A = net area, sq. in., of tension flange (through rivet holes).
h = distance, in., between centers of gravity of the two flanges.
s = allowable unit stress in bending.
t = thickness of web, in.
h_1 = depth of web, in.
M = maximum bending moment on given section, in.-lb.

* This formula should not be used for unsymmetrical girders or for very shallow girders with heavy flanges, where the distance between centers of gravity of flanges is much less than the total depth of the girder, or for other abnormal cases.

I = total moment of inertia of gross cross section about neutral axis.

A_1 = gross area of each flange.

h_2 = depth out-to-out of flanges.

$I_{c.g.}$ = moment of inertia of each flange about its own center of gravity.

The following equation for I may now be written:

$$I = 2I_{c.g.} + 2A_1\left(\frac{h}{2}\right)^2 + \frac{1}{12}th_1^3$$

The term $2I_{c.g.}$ is small in comparison with the other terms and may be omitted without serious error, this being on the safe side. In consequence the value of I/c for a symmetrical cross section may be written thus:

$$\frac{I}{c} = \frac{\frac{2A_1h^2}{4} + \frac{1}{12}th_1^3}{\frac{h_2}{2}} = \frac{A_1h^2}{h_2} + \frac{1}{6}\frac{th_1^3}{h_2}$$

But

$$\frac{I}{c} = \frac{M}{s}$$

Therefore,

$$\frac{M}{s} = \frac{A_1h^2}{h_2} + \frac{1}{6}\frac{th_1^3}{h_2}$$

Hence,

$$A_1 = \frac{M}{s}\frac{h_2}{h^2} - \frac{1}{6}\frac{th_1^3}{h_2}\frac{h_2}{h^2} = \frac{M}{s}\frac{h_2}{h^2} - \frac{1}{6}\frac{th_1^3}{h^2}$$
$$= \frac{M}{sh}\left(\frac{h_2}{h}\right) - \frac{1}{6}th_1\left(\frac{h_1}{h}\right)^2$$

For girders with ordinary depths, say one-sixth to one-twelfth of the span, the value of h is seldom larger than that of h_1 and is usually smaller.[1] If h_1/h is therefore assumed as unity, the last term in the equation above will ordinarily be less than its true value; and since this term is small compared with the term involving M, the slight change in its value by the approximation above

[1] It is good practice to proportion girders so that h is less than h_1.

will affect the value of A but little, and that on the safe side, since it reduces the value of the negative term. This approximation will therefore be made.

The assumption that h_2/h = unity will also be made. This is on the unsafe side, since h is usually less than h_2, and to assume it equal gives a smaller value for A_1 than is required. The error in making this assumption is largest in shallow girders having large flange angles, as may be seen in the numerical examples given later.

By making these approximations the formula becomes

$$A_1 = \frac{M}{sh} - \frac{1}{6}th_1$$

For columns in direct stress, it is customary to make no deduction whatever for rivet holes since it is assumed that the rivets, which are driven while hot and ordinarily under high pressure, fill the holes so completely as to become an integral portion of the material. This is open to some doubt in the case of thick material or hand-driven rivets or shop rivets with long grips and may be vitiated at any section by a loose rivet, but for most cases this assumption is probably a reasonable one. For sections in tension, full allowance for rivet holes must be made, since under no circumstances can tension be transmitted through a rivet hole.

The last term in formula (17) represents the bending resistance of the web. As there are usually vertical rows of rivets in the web for floor-beam connections, stiffener angles, etc., and as these may occur at the section carrying maximum moment, they must be considered.

To allow for such holes, it may be assumed that a vertical row of holes 1 in. in diameter and $2\frac{3}{4}$ in. apart may occur in the tension half of the web. This would decrease the moment of inertia by two-elevenths approximately, thus making the last term in the equation $\frac{9}{66}th_1$ or, say $\frac{1}{8}th_1$.

Allowance for rivet holes in the tension flange must also be made. This may be done by substituting A for A_1, which is in reality equivalent to providing for rivet holes in both flanges. This may seem excessive, but some excess is necessary since the section has been considered as solid and with its neutral axis at mid-height, whereas in reality the influence of the rivet holes in the tension portion is to shift the neutral axis from the

center, thus diminishing the moment of inertia and increasing the distance from neutral axis to extreme fiber. The substitution of A for A_1 is, however, more than sufficient for this purpose and helps to diminish the error made in placing h_2/h = unity.

This modification gives the following formula, which is adopted by many engineers:

$$A = \frac{M}{sh} - \frac{1}{8}th_1$$

The last term in this equation represents the resistance of the web to bending. As it is sometimes difficult to splice the web completely, the last term in the foregoing equation was changed in Eq. (17) by using $\frac{1}{12}th_1$.

The assumptions made in deriving formula (17) are of such a character as to make the formula inaccurate for girders having great depth in proportion to their length. Such girders are not common in bridges but are sometimes used in architectural work and should be solved by the direct application of formula (14).

It should be stated, furthermore, that experimental knowledge of the distribution of stress in plate girders is insufficient to permit a confirmation of the accuracy of formulas of the type of (17). Formulas of this character have, however, been in use for many years with satisfactory results and may well be considered as safe working formulas. Formula (17) is more conservative than that usually employed.[1]

63. Degree of Approximation of Flange Formula.—In order to show the degree of approximation of formula (17), in comparison with formula (14), the problems that follow have been inserted.

Problem: Compute allowable bending moment M for the girder shown in Fig. 81. Assume no intermediate web stiffeners, and hence only one rivet hole (flange rivet) in tension half of girder. Allowable unit stress = s.

Allowable Moment by Moment-of-inertia Method.

	Area, square inches
Top angles, two 6 in. × 4 in. × ½ in., at 4.75	9.5 gross
Bottom angles, two 6 in. × 4 in. × ½ in., at (4.75 − 0.5)	8.5 net
Web, 29 in. × ½ in.	14.5 net
Total effective area of cross section	32.5 sq. in.

[1] The 1938 specifications for railroad bridges of the American Railway Engineering Association require plate girders to be designed by the moment-

Distance of center of gravity of cross section above axis xy

$$= \frac{1 \times 1\frac{1}{2} \times 13\frac{1}{8}}{32.5} = 0.6 \text{ in.}$$

$$h = 30.25 \text{ in.} - 3.98 \text{ in.} \doteq 26.27 \text{ in.}$$

Let Ixy = moment of inertia of *gross* section about axis xy and $I_{c.g.}$ = moment of inertia of any piece about an axis parallel to xy and passing through the center of gravity of the piece.

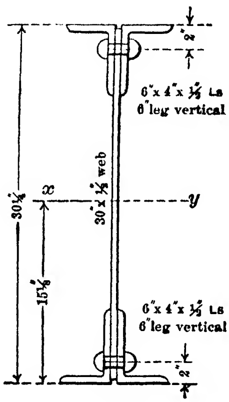


FIG. 81.

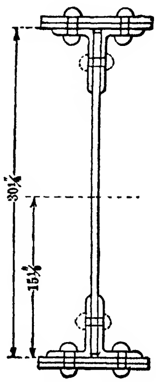


FIG. 82.

I_{xy} of webs $= \frac{1}{12} \times \frac{1}{2} \times 30 \times 30 \times 30$	$= 1{,}125$
I_{xy} of angles $= I_{c.g.}$ of angles $+ \frac{A_1h^2}{2} = 4 \times 17.4 + 4 \times 4.75 \times (13.13)^2$	$= 3{,}345$
Total I_{xy}	$= 4{,}470$
Deduct for flange rivet hole $1\frac{1}{2} \times 1 \times (13.13)^2$	$= 259$
I of net section about axis xy	$= 4{,}211$
To obtain I for net section about axis passing through center of gravity, deduct 32.5×0.6^2	$= 12$
I of net section about neutral axis	$= 4{,}199$

$$\frac{I}{c} = \frac{4{,}199}{(15\frac{1}{8} + 0.6) \text{ in.}} = 267$$

Therefore,

$$M = s\frac{I}{c} = 267s$$

Allowable Moment by Formula (17).—By transformation of terms in the formula, M is found to be given by the expression

$$M = (A + \tfrac{1}{12}th_1)sh = (8.5 + 1.25)(s)(26.27) = 256s$$

---

of-inertia method, on the assumption that the neutral axis is to be located at centroid of gross section.

Since the allowable moment as computed by formula (17) is less than that given by the moment-of-inertia method, the formula for this girder is on the safe side. The approximation is about 4 per cent.

Problem: Compute allowable bending moment for the same girder, assuming a row of rivet holes in the web at the point of maximum bending moment in addition to the flange rivet holes.

Solution: For practical reasons the web rivets nearest the flanges should be located not less than $1\frac{1}{2}$ in. from the edge of the flange angles, or for this girder $7\frac{1}{2}$ in. from the backs of the angles. In order to get even rivet spacing, let this distance be made $7\frac{5}{8}$ in., thus giving 15 in. for the distance between these rivets and permitting the use of five spaces at 3 in. for the horizontal rivets between flanges.

Assume that only the rivet holes in that portion of girder below the neutral axis, *i.e.*, the tension half, need be deducted.

Net area of cross section $= 32.5 - 3 \times \frac{1}{2} = 31.0$.

The position of the center of gravity of the cross section above the axis

$$xy = \frac{1\frac{1}{2} \times 13\frac{1}{8} + \frac{1}{2}(7\frac{1}{2} + 4\frac{1}{2} + 1\frac{1}{2})}{31.0} = 0.85 \text{ in.}$$

The deduction from I_{xy} to allow for rivet holes will be

$$1 \times 1\frac{1}{2} \times (13.13)^2 + \frac{1}{2} \times (7.5^2 + 4.5^2 + 1.5^2) = 298$$

Therefore,

$$I \text{ of net section about axis } xy = 4{,}470 - 298 = 4{,}172$$

and about neutral axis

$$= 4{,}172 - 31.0 \times 0.85^2 = 4{,}150$$

Therefore,

$$\frac{I}{c} = \frac{4{,}150}{15.12 + 0.85} = 260$$

and

$$M = \frac{sI}{c} = 260s$$

The approximation in this case is somewhat less than 2 per cent and is also on the safe side.

Problem: Compute allowable bending moment for the same girder, assuming two 10- by $\frac{1}{2}$-in. plates to be added to each flange and no web rivets at the critical section (see Fig. 82). (Note that the horizontal and vertical flange rivets are frequently staggered, and hence a section containing the vertical rivets may not contain horizontal rivets; also, that the material in the horizontal legs of the flange is thicker than that in the vertical leg, and that hence the reduction due to rivet holes is larger.)

Allowable Moment by Moment-of-inertia Method.

	Area, square inches
Top angles, two 6 in. $\times$ 4 in. $\times$ ½ in., at 4.75	= 9.5 gross
Bottom angles, two 6 in. $\times$ 4 in. $\times$ ½ in., at (4.75 − 0.5)	= 8.5 net
Top plates, two 10 in. $\times$ ½ in., at 5	= 10.0 gross
Bottom plates, two 10 in. $\times$ ½ in., at (5 − 1)	= 8.0 net
Web, 30 in. $\times$ ½ in.	= 15.0 gross
Total effective area of cross section	= 51.0 sq. in.

Computation of h,

Gross area of one flange = 19.5 sq. in.

From back of angles to center of gravity of angles = 1.99 in.

Hence, from center of gravity of angles to center of gravity of flange

$$= \frac{10 \times (1.99 + 0.5)}{19.5} = 1.27 \text{ in.}$$

Therefore,

$$\begin{aligned} h &= 30.25 \text{ in.} - 2 \times (1.99 \text{ in.} - 1.27 \text{ in.}) \\ &= 28.81 \text{ in.} \end{aligned}$$

Allowance for rivet holes $= 1 \times 1\frac{1}{2} \times 2 = 3$ sq. in., hence, center of gravity of cross section above axis xy

$$= \frac{3 \times (15\frac{1}{8} + \frac{1}{4})}{51} = 0.9 \text{ in.}$$

Allowable Moment by Moment-of-inertia Method.

I_{xy} of web		= 1,125
I_{xy} of angles		= 3,345
I_{xy} of plates $= 20 \times (15\frac{5}{8})^2$		= 4,883
Total I of gross section about axis xy		= 9,353
Deduct for rivet holes $3 \times (15\frac{3}{8})^2$	= 709	
Correction for I about neutral axis $= 51 \times 0.9^2$	= 41	
		750
I of net section about neutral axis		= 8,603

$$\frac{I}{c} = \frac{8{,}603}{16\frac{1}{8} + 0.9} = 505$$

Therefore,

$$M = 505s$$

Allowable Moment by Formula (17).

$$M = (A + \tfrac{1}{12}th_1)sh = (8.5 + 8.0 + 1.25)(s)(28.81) = 511.4s$$

Here the formula errs on the unsafe side, the approximation being slightly over 1 per cent.

Problem: Compute allowable bending moment for girder shown in Fig. 83. Assume a vertical row of rivets, 3 in. center to center, at the point of maximum moment and that all flange rivets shown occur in the same cross section as the web rivets.

Composition of girder
- 1 web, 90 in. × ½ in.
- 2 top angles, 6 in. × 6 in. × ¾ in.
- 3 top plates, 16 in. × ⅝ in.
- 2 bottom angles, 6 in. × 6 in. × ¾ in.
- 3 bottom plates, 16 in. × ⅝ in.

Allowable Moment by Moment-of-inertia Method.

		Area, square inches
Top angles, two 6 in. × 6 in. × ¾ in., at 8.44	=	16.88 gross
Bottom angles, two 6 in. × 6 in. × ¾ in., at (8.44 − 2 × ¾)	=	13.88 net
Top plates, three 16 in. × ⅝ in., at 10	=	30.00 gross
Bottom plates, three 16 in. × ⅝ in., at (10 − 1.25)	=	26.25 net
Web, 90 in. × ½ in. − 14 in. × ½ in. (deducting rivet holes in lower half)	=	38.00 net
Total effective area of cross section	=	125.01 sq. in.

Computation of h,

Back of angles to center of gravity of angles $= 1.78$ in.

$$\frac{\text{Moment of plates about center of gravity of angles}}{\text{Gross area of flange}} = \frac{30 \times 2.72}{46.88} = 1.74 \text{ in.}$$

$\overline{0.04}$

Therefore,

$$h = 90.5 \text{ in.} - 0.08 \text{ in.} = 90.42 \text{ in.}$$

Moment about xy of rivet holes in tension half of girder is as follows:

Web, ½(43.25 + 37.5 + 34.5 + · · · + 1.5) = ½ × 296.8	=	148.4
Flange, 1½ × 43¼ + 5.25 (45.81)	=	305.4
Total	=	453.8

$$\text{Distance of neutral axis above } xy = \frac{453.8}{125.0} = 3.63 \text{ in.}$$

Maximum value of c $= 50.75$ in.

I_{xy} of web	=	30,375 gross
I_{xy} of angles $= 112 + 2 \times 16.88 \times (43.47)^2$	=	63,906 gross
I_{xy} of plates $= 17 + 2 \times 30 \times (46.187)^2$	=	128,014 gross
I of gross section about axis xy	=	222,295
Deduct for rivet holes, $\frac{1}{2}[(43.25)^2 + (37.5)^2 + \cdots (1.5)^2] + 1\frac{1}{2}(43.25)^2 + 5.25(45.81)^2$	=	18,050
I of net section about axis xy	=	204,245
Correction for I about neutral axis	=	1,647
I of net section about neutral axis	=	202,598

$$\frac{I}{c} = \frac{202{,}598}{50.75} = 3{,}992$$

Therefore,

$$M = 3{,}992s$$

Allowable Moment by Formula (17).

$$M = (A + \tfrac{1}{12}th_1)sh = (13.88 + 26.25 + 3.75)90.42s = 3{,}967s$$

In this case the approximation equals about ½ per cent on the safe side.

The examples that have been given show that formula (17) gives for ordinary girders a very close approximation to the value obtained by the ordinary beam formula $M = sI/c$. It is much more convenient to use, since by it the required flange area can be directly computed, after the web is determined, by estimating the value h, which can be done by the experienced computer with little error. The actual application of the formula to the design of a girder will be shown later.

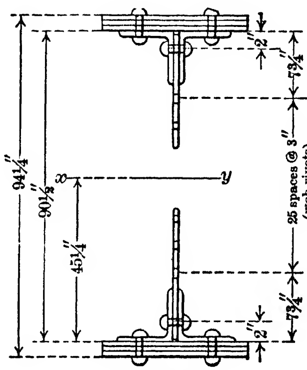


FIG. 83.

64. Degree of Approximation of Shear Formula.—To show the degree of approximation involved in the ordinary assumption that the vertical shear is distributed uniformly over the gross area of the web, the maximum intensity of shear will be computed for the girders in the last article. This occurs at the neutral axis and will be computed in each case, using for Q the statical moment of the portion of the gross area of the girder above this axis, although the same numerical result would be obtained by considering the portion below the axis since the statical moment of the entire section about the neutral axis equals zero.

Problem: Compute maximum intensity of shear in the web of the girder shown in Fig. 81, assuming a vertical row of web rivets to occur at section of maximum shear.

Solution:

$$Q \text{ for angles} = 9.5 \times (15.12 - 2.84) = 116.7$$

$$Q \text{ for web} = \tfrac{1}{2} \times (15.00 - 0.85) \times \frac{(15.00 - 0.85)}{2} = 50.0$$

$$\text{Total value of } Q = 166.7$$

From formula (13),

$$v = V\frac{166.7}{4{,}150 \times \frac{1}{2}} = 0.0803V$$

(See Art. 63 for value of I.)

By the assumption that the shear is uniformly distributed over the gross area of the web the following result is obtained:

$$v = \frac{V}{15} = 0.0667V$$

The value thus obtained seems to be on the unsafe side but is compensated for by using a low unit shearing value.

Problem: Compute maximum intensity of shear in the web of the girder shown in Fig. 83, assuming that no cover plates occur at section of maximum shear.

Solution:

Net area $= 68.76$* sq. in.

Center of gravity above xy =

$$\frac{148.4 + 1.5(43.25 + 44.875)}{68.76} = 4.08 \text{ in.}$$

$$\begin{array}{lr}
I_{xy} \text{ of gross area of web and angles} & = 94{,}281 \\
\text{Deduct for rivets } \tfrac{1}{2}[(43.75)^2 + (37.5)^2 + \cdots (1.5)^2] & \\
\quad + 1\tfrac{1}{2}[(43.25)^2 + (44.875)^2] & = \underline{10{,}052} \\
I \text{ of net section about } xy & 84{,}229 \\
\text{Correction for } I \text{ about neutral axis} = 68.76 \times 4.08^2 & = \underline{1{,}144} \\
I \text{ of net section about neutral axis} & = 83{,}085 \\
Q \text{ of angles} = 16.88 \times 39.39 & = 665 \\
Q \text{ of web} = (45 - 4.08)\left(\frac{1}{2}\right)\frac{(45 - 4.08)}{2} & = \underline{419} \\
 & 1{,}084
\end{array}$$

Therefore,

$$v = \frac{1{,}084V}{83{,}085 \times \frac{1}{2}} = 0.0261V$$

By the assumption that the shear is distributed over the gross area of web, the following value would be obtained:

$$v = \frac{V}{45.0} = 0.0222V$$

This result is again lower than the value obtained by the more exact method.

* In determining the area it is assumed that the maximum shear may occur at the point where the first cover plate begins; hence, vertical flange rivets may occur at the section but cover plate should be ignored.

For I beams the shear may be treated in somewhat the same manner. The values obtained for a 10-in. I beam, 25.4 lb. per foot, are as follows, assuming no rivet holes in cross section:
By formula (13),

$$v = \frac{14.02}{0.31 \times 122.1}V = 0.370V$$

By assumption that shear is uniformly distributed over gross area of web,

$$v = \frac{V}{3.10} = 0.322V$$

The error in this case is on the safe side.

For a 24-in. I beam 79.9 lb. per foot, the corresponding values would be $0.097V$ and $0.083V$, the error in this case being also on the safe side.

65. Allowance for Rivet Holes.—In the design of girders, it is necessary to make due allowance for the tension-flange rivet

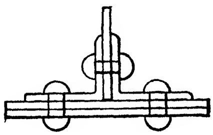


FIG. 84.

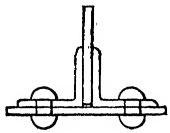


FIG. 85.

holes in advance of the completion of the detailed drawings. No accurate rule for doing this can be given since the actual reduction of strength by rivet holes needs more thorough experimental investigation than it has yet received. The following rules may, however, be used as a guide:

1. For flanges with cover plates, and angles with legs wider than 4 in., assume that both vertical and horizontal rivet holes may occur in the same section, these holes being as shown in Fig. 84.

2. For flanges with flange angles of 4 in. or less in width and with cover plates, deduct two holes from each section as shown in Fig. 85.

3. For flanges without cover plates, deduct one hole from each section as shown in Fig. 86.

In all cases the rivet hole should be assumed to have a diameter ⅛ in. more than the nominal diameter of the rivet. This

is necessary since the hole is usually punched with a diameter $\frac{1}{16}$ in. greater than that of the cold rivet, and the edges of the hole may be damaged somewhat in punching. The rivets commonly used in structural work are $\frac{7}{8}$ in. in diameter; hence, for these rivets the hole should be assumed as 1 in. in diameter. In light work, $\frac{3}{4}$- or $\frac{5}{8}$-in. rivets are occasionally used, and in very heavy girders 1-in. or larger rivets are sometimes employed.

It should be stated that, though it is seldom that more than three rivet holes actually occur in the same section of the tension flange, the fact that a zigzag section through holes not in the same right section may be the controlling section should be carefully considered; for example, the zigzag section *AB* in Fig. 87 may be the controlling section.

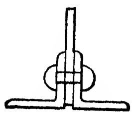


FIG. 86.

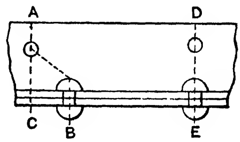


FIG. 87.

The 1935 specifications of the American Railway Engineering Association covering the allowance to be made for rivet holes in a tension member read as follows:

The net width for any chain of holes extending progressively across the part shall be obtained by deducting from the gross width the sum of the diameters of all the holes in the chain and adding, for each gage space in the chain, the quantity,

$$\frac{s^2}{4g}$$

s = pitch of any two successive holes in the chain.
g = gage of the same holes.

The net section of the part is obtained from that chain which gives the least net width.

For angles, the gross width shall be the sum of the widths of the legs less the thickness. The gage for holes in opposite legs shall be the sum of the gages from back of angle less the thickness.

For splice members, the thickness shall be only that part of the thickness of the member which has been developed by rivets beyond the section considered.

The diameter of the hole shall be taken as ⅛ inch greater than the nominal diameter of the rivet.[1]

The pitch is the distance between rivets along the axis of the member and the gage is the transverse distance between lines of rivets.

Attention should be called to the fact that, although the maximum moment on a girder ordinarily occurs at only one section and that at this section the rivet pitch may be a maximum, a maximum flange stress is developed wherever a cover plate ends, and the rivet pitch at such a point may be little if any larger than the minimum allowable pitch.

66. Example of Girder Design.—An example of the complete design of the cross section of a girder will now be given.

Problem: Determine cross section of a girder to carry a maximum bending moment of 1,250,000 ft.-lb. and a maximum shear of 160,000 lb. Depth of girder back to back of flange angles = 48½ in.

Allowable Unit Stresses.

Bending	= 18,000 lb. per sq. in.
Shear on gross web area	= 11,000 lb. per sq. in.

Solution:

Web, net area of cross section required = 160,000/11,000 = 14.55 sq. in.
Depth of web = 48 in.
Thickness of web = 14.55/48 = 0.31 in. or $\frac{5}{16}$ in.
Flange. Assume h to equal depth of web = 48 in.

$$\text{Trial } A = \frac{1{,}250{,}000 \times 12}{18{,}000 \times 48} - \frac{1}{12} \times \frac{5}{16} \times 48 = 17.36 - 1.25 = 16.11 \text{ sq. in.}$$

Trial section. Rivets in flange assumed as in Fig. 83.

2 angles, 6 in. × 4 in. × ½ in., at 4.75	= 9.50 − 2.00 =	7.50 sq. in.
2 plates, 14 in. × ⅜ in., at 5.25	= 10.50 − 1.50 =	9.00 sq. in.
	=	16.50

Computation of h for this section:

Center of gravity of angles from back of angles = 0.99 in. (from handbook).

$$\text{Correction to allow for cover plates} = \frac{10.5 \times 1.36}{10.5 + 9.50} = 0.71.$$

Therefore,

$$h = 48.5 \text{ in.} - 2 \times (0.99 \text{ in.} - 0.71 \text{ in.}) = 47.94 \text{ in.}$$

[1] The development of this method is discussed at length in an article by Chapin, *Proc. Am. R. R. Eng. Assoc.*, Vol. 36, 1935.

Hence, original assumption for value of h, though slightly too large, is sufficiently precise and the trial section may be used.

67. Flange Rivets and Riveted Joints.—The flange rivets form the only connection between the flanges and the web; hence, the determination of the proper size and distance apart of these rivets is an essential feature in girder design. The diameter of the rivet is ordinarily fixed by practical considerations, the common practice for structural work being to use $\frac{7}{8}$-in. rivets. The distance apart of the rivets has to be computed. The question of riveted connection between two members is also of great importance.

Thorough treatment of rivet spacing is found in treatises upon mechanics and will not be given here. The essential points with which the structural designer must be thoroughly conversant are as follows:

A riveted connection may fail in one of the following ways:

a. By the shearing of the rivets.

b. By the crushing of the rivets or of one of the pieces upon which they bear.

c. By the tearing of the rivets through one of the connected pieces.

Under *a* it should be noted that the allowable shearing value of the rivet may be found by multiplying its cross-section area by the allowable shearing stress per square inch of gross section and that the area of a $\frac{7}{8}$-in. rivet is 0.60 sq. in. and of a $\frac{3}{4}$-in. rivet 0.44 sq. in.

In designing rivets to resist shear, the plane upon which the maximum shear occurs must always be determined. If the maximum shear is equally distributed over two planes, the rivet is said to be in *double shear*.

The permissible bearing, or crushing, strength of a rivet against a given plate is determined by multiplying the allowable bearing strength per square inch by the diameter of the rivet and the thickness of the plate in question.

To satisfy the requirements stated in *c*, use the following empirical rule: Rivets may not be spaced closer than three times the diameter, and the distance of a rivet from the edge or end of a piece may not be less than $1\frac{1}{4}$ in. for a $\frac{7}{8}$-in. rivet, if the edge in question is rolled or planed, or $1\frac{1}{2}$ in. if it is sheared, though where possible this distance should be at least twice the diameter of the rivet. The distance from the center of a

rivet to the edge of a plate shall not exceed eight times the thickness of the plate. For other sizes of rivets, proportional allowances should be made.

The two following examples show the application of these rules to some simple cases:

Problem: Determine number of ⅞-in. rivets needed in row *a* to connect plates shown in Fig. 88.

Allowable shearing stress = 11,000 lb. per sq. in.
Allowable bearing stress = 27,000 lb. per sq. in.

Solution: The maximum shear on a plane through the rivets = 50,000 lb. As the rivets are ⅞ in. in diameter, one rivet will carry in shear 11,000 × 0.6 = 6,600 lb.; hence, if the strength of the joint is limited by the shearing strength of the rivets there are needed 50,000/6,600 = 8. These rivets,

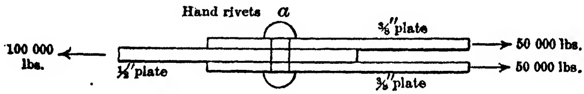


FIG. 88.

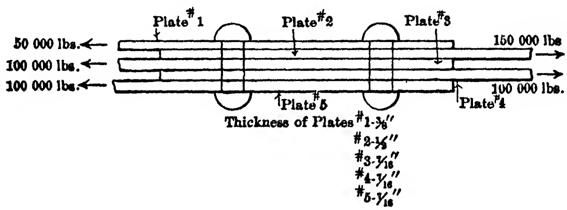


FIG. 89.

are limited in crushing strength by the ½-in. plate which carries 100,000 lb. The value of the rivet in bearing against this plate equals ⅞ × ½ × 27,000 = 11,800 lb., and the number required = 100,000/11,800 = 8 (+), or 9. As this is larger than the number needed to prevent shearing, nine rivets must be used.

Problem: Determine number of ⅞-in. rivets required to connect plates in joint shown by Fig. 89. Use same rivet values as in previous problem.

Solution: The maximum shear = 100,000 lb. and occurs between plates 2 and 3 or 4 and 5. The number of rivets needed to carry this shear = 100,000/6,600 = 15 (+), or, say 16.

The bearing strength is evidently limited by plate 2, which carries 150,000 lb. and has a thickness of ½ in., this producing a greater stress on the rivets than would be the case for the 7/16-in. plate carrying 100,000 lb. The number required for bearing = 150,000/11,800 = 12 (+), or 13; hence, for this joint the number of rivets is limited by shear.

The examples just given illustrate methods of computing connection rivets for plates carrying direct stress. Sometimes, however, it is necessary to transmit torsion as well as direct stress by means of rivets. Such a condition often occurs in steel-frame building construction where the connections of girders to columns must be given considerable rigidity to provide proper transmission of the wind stresses.

The condition commonly occurring in such a case is represented diagrammatically by Fig. 90, in which the load P is applied at a distance x from the center of the group of rivets, thus producing

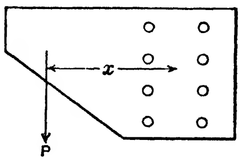


FIG. 90.

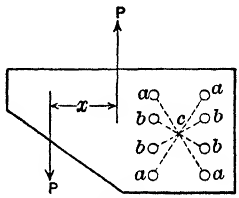


FIG. 91.

a torsion Px which must be carried by the rivets in addition to the direct load P.

If the torsion is produced by a couple, as in Fig. 91, then the vertical load upon the rivets will be zero and a rivet may be legitimately assumed to offer a resistance to torsion varying directly with its distance from the center of gravity of the group of rivets and acting at right angles to the line connecting it with the center of gravity. Upon this basis the rivets at a, Fig. 91, should each be computed as equally stressed up to the allowable working load, and the other rivets would carry such proportion of the working stress as the distance cb is to ca when c is the center of gravity of the group of rivets.

The resistance to torsion of such a group of rivets may therefore be expressed as follows:

Let r = the allowable working value of the most stressed rivet.

I = summation of the squares of the distances from the center of gravity of the group of rivets to each rivet.

d = distance from center of gravity of group of rivets to the most stressed rivet = ac, Fig. 91.

R = resistance to torsion of the group of rivets.

Then

$$R = \frac{r}{d} I \tag{18}$$

For the case shown in Fig. 90, this method must be modified to allow for the effect of the vertical load. To make this correction, it is only necessary to determine the allowable resistance to torsion consistent with the rivet carrying its share of the vertical load.

This may best be done graphically by the method indicated in Fig. 92. In this case the total vertical load is 20,000 lb., of which each rivet is assumed to carry 2,500 lb. With an allowable total working value of, say 7,500 lb. per rivet, the component at a perpendicular to the line ac is found graphically to be 5,800 lb. The corresponding allowable components of the stress in the rivets at d, b, and e are larger; hence, the rivets at a furnish the minimum resistance to torsion for the given working value and consequently give the limiting value of r in Eq. (18).

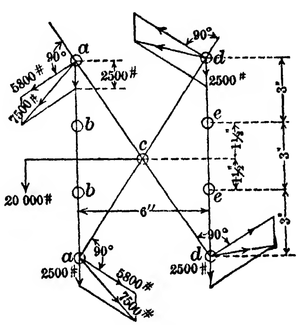


FIG. 92.

Tables giving the resistance to torsion of various groups of rivets have been prepared by E. A. Rexford and are published by the Engineering News Publishing Company.

68. Flange Rivets. Approximate Method of Computing Pitch. A plate-girder flange receives its stress through the rivets which connect it to the web; hence, the number of flange rivets needed in any given distance is directly proportional to the increase in flange stress in the same distance. If the girder is of constant height, the flange stress is proportional to the bending moment; hence, the rate of increase of the flange stress is proportional to the rate of increase of the bending moment, which at any given section equals the external shear at that section in accordance with the well-known principle of mechanics that the first differential coefficient for the bending moment equals the shear. It follows that for such a girder the required rivet pitch (the pitch is to be considered as the distance apart of the rivets measured along the flange) varies in inverse proportion to the shear.

A knowledge of the variation of the pitch is, however, insufficient; it is necessary to determine the pitch itself. The following method of doing this is obvious: Compute the total stress in the flange at any section, and compute also the total stress at a

section 1 in. from this section; the difference between the two stresses gives the increase in flange stress per longitudinal inch at that portion of the girder, and this increase must be carried into the flanges through the rivets. If one rivet can carry p lb. and if the increase in flange stress is x lb., the proper rivet pitch to use in that portion of the girder is $p \div x$.

If the portion of the bending moment carried by the web is neglected in rivet-pitch computations, this approximation being small and on the safe side, the rate of increase of the total flange stress at any section of an ordinary girder may be found by dividing the rate of increase of the moment at that section by the distance between the centers of gravity of the flanges. Since the rate of increase of the moment equals the shear, the increase in flange stress at any section equals the shear divided by the distance between the centers of gravity of the flanges.

The foregoing discussion refers entirely to the rivets needed to transmit to the flange the stress developed by bending. If the flange rivets must also transmit a vertical load, as in the case of the top flange of a deck girder supporting track ties, due allowance must be made for the effect of such a load by computing the resultant of the vertical load per lineal unit and the increase in flange stress in the same distance and using this resultant as the stress to be carried by the rivets per linear unit in accordance with the following formula:

$$p = \frac{R}{\sqrt{(V/h)^2 + m^2}} \tag{19}$$

in which p = maximum allowable pitch, in., at section under consideration.

R = allowable stress on rivet, lb.

h = distance between centers of gravity of flanges, in.

V = maximum external shear on given section, lb.

m = load per lineal in. supported directly by flange.

If there is no vertical load imposed directly upon the flange, $m = 0$.

R will be the smaller of the following two values: (*a*) The product of the allowable bearing value per square inch by the diameter of the rivet in inches by the thickness of the web plate. (*b*) Twice the product of the allowable shearing stress per square inch by the cross-section area of the rivet in square inches.

Owing to the fact that if the distance between rivets is too great the different pieces in a compression member may wrinkle, it is customary to specify a maximum pitch in the line of stress not greater than seven times the diameter of the rivet or twelve times the thickness of the thinnest plate connected. This restriction is frequently the controlling factor in determining the rivet pitch and is commonly applied to tension members as well as to compression pieces.

Equation (19) is applicable only to rivets through the vertical leg. These are the rivets that carry the flange stress into the web. The rivets through the horizontal leg serve to transmit a part of this flange stress into the cover plates and in consequence may have a larger pitch. It is customary, however, to use the same pitch for the vertical as for the horizontal rivets;[1] hence, the method given is, in general, all that is necessary.

69. Flange Rivets. Precise Method of Computation of Pitch. The method represented by Eq. (19) is the approximate method of figuring rivet pitch that is generally used in plate-girder design. In order to understand thoroughly the question of rivet pitch and to be able to figure properly the pitch in other cases that may arise, such as columns carrying bending, it is necessary to develop a more exact method. Such a method may also be well employed in investigating existing girders the strength of which may be in doubt. To obtain such a method the formula for horizontal shear may be used.

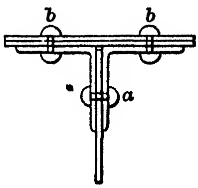


FIG. 93.

With reference to Fig. 93, it is evident that the function of the rivets at a is to prevent the flange angles from sliding along the web; *i.e.*, the rivets must resist the longitudinal shearing tendency of the angles. Hence, if this tendency can be computed, the rivet pitch necessary to withstand it can be determined. This value can be easily determined by multiplying the intensity of the longitudinal shear at the bottom of the angles by the thickness of the web, the statical moment about the neutral axis of the girder of the angles and cover plates combined being used for Q in the determination of the intensity, but not of the

[1] At the ends of the cover plates, it is customary to place the vertical rivets at a small pitch for a distance of 1 or 2 ft. to ensure that the stress may be properly carried into the plate.

portion of the web included between the flange angles, since the stress in this is carried by the web itself. This gives the shearing force per longitudinal inch in the flange, which equals the increase in flange stress, and this can be used as before in figuring the pitch. By the same method the correct pitch for the rivets at b can be determined by computing Q for the cover plates only.

70. Flange Rivets. Example in Computation of Pitch.—To illustrate the application of these methods the girder shown in Fig. 83 will be considered.

Problem: Determine the rivet pitch for ⅞-in. rivets required at a section where the shear is 300,000 lb., assuming that at this section all the cover plates are needed.

Solution: The rivets through the vertical legs of the angles will first be considered, it being assumed that the outer force is applied directly to the web and not through the flange. Let the bearing value per square inch of the rivets be taken as 27,000 lb. and the shearing value as 13,500 lb. The strength of the rivet will then be limited either by bearing on the ½-in. web, which equals $\frac{1}{2} \times \frac{7}{8} \times 27{,}000 = 11{,}800$ lb., or by double shear, which equals $0.60 \times 13{,}500 \times 2 = 16{,}200$ lb. As the bearing value is smaller, it should be used.

Increase in flange stress per linear inch.

Approximate method,

$$\frac{300{,}000}{90.4} = 3{,}320 \text{ lb.}$$

Exact method,

$$\frac{VQ}{I} = 300{,}000\frac{16.88(45.25 - 1.78) + 30.0(45.25 + 0.94)}{222{,}295} = 2{,}850 \text{ lb.}$$

Since one rivet can carry 11,800 lb., the required pitch by the approximate method is

$$\frac{11{,}800}{3{,}320} = 3.54 \text{ in., or, say } 3\frac{1}{2} \text{ in.}$$

and by the exact method

$$\frac{11{,}800}{2{,}850} = 4.15 \text{ in. or, say } 4\frac{3}{16} \text{ in.}$$

It is evident that the approximate method is decidedly on the safe side in this case.

Were the required pitch less than three diameters of the rivet, it would be necessary to locate the rivets in two rows as shown in Fig. 94, where a pitch of 2 in. is assumed.

To determine the pitch of the vertical rivets, the exact method should be used. The increase in flange stress per inch is

$$\frac{30 \times (45.25 + 0.94)}{222,295} \times 300,000 = 1,870 \text{ lb.}$$

The value of each rivet in this case is evidently its strength in *single* shear; but as there are two vertical rivets in each cross section, the pitch may be obtained by dividing the value of one rivet in *double* shear by the increase in flange stress per inch. This gives

$$\frac{16,200}{1,870} = 8.75 \text{ in.}$$

As this exceeds twelve times the thickness of the $\frac{5}{8}$-in. plates (see Art. 68), the pitch of these rivets should be made 6 in.

It will be noticed that the pitch is the distance between rivets measured along the axis of the angle. The vertical distance between the rows of rivets must be sufficient to make the distance d equal to or greater than three times the diameter of the rivet, or $2\frac{5}{8}$in. for a $\frac{7}{8}$-in. rivet. The distance e should not be less than $1\frac{1}{4}$ in., as already noted, and should preferably be $1\frac{1}{2}$ in. or $1\frac{3}{4}$ in. The distance c is determined by the amount of room needed for driving the rivet. The standard values for different angles are given in steel makers' handbooks.

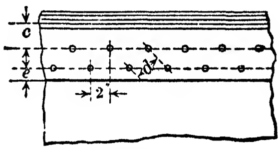


FIG. 94.

This example shows that the pitch of the vertical rivets may be considerably greater than that of the horizontal rivets. It is, however, usually made the same for practical considerations.

There remains one other case to be treated—that of a girder supporting a load on the upper flange. To illustrate the method required for this case, let the girder shown in Fig. 81 be considered, and let it be assumed that this girder is a railroad-bridge stringer with ties resting directly upon the top flange. Let it also be assumed that the maximum wheel load crossing the stringer is 24,000 lb.; that the maximum end shear including impact is 100,000 lb.; and that the pitch of the rivets at the end of the stringer is to be determined. With the allowable unit stresses previously used the limiting value of the rivet is $\frac{7}{8} \times \frac{1}{2} \times 27,000 = 11,800$ lb. Using the approximate method the increase in flange stress is found to be $100,000/26.27 = 3,800$ lb. per linear inch. This value must be combined with the vertical load carried by the rivets. Since the rail has considerable strength as a beam, it is evident that a wheel load will not be carried

entirely by one tie but will be distributed over several. The common assumption is that one wheel load is distributed over three ties. If the assumption is made, one wheel load will be distributed over the rivets in the space m, Fig. 95. If the ties are 8 in. wide and spaced 6 in. apart in the clear, three ties will occupy a total distance of 42 in.; hence, the vertical load per inch that the rivets must bear is 24,000/42 or 570 lb., the dead weight being so small compared with the live load as to be negligible. To allow for impact, this value should be doubled since these rivets are more directly affected by the shock of the locomotive than any other portion of the structure. To obtain the rivet pitch, it is therefore necessary to divide the value of one rivet by the resultant of 3,800 and 1,140. This resultant may be obtained quickly and with sufficient accuracy by the graphical method indicated in Fig. 95. Its value is found to be 3,970; hence, the proper pitch at the end is 11,800/3,970 = 2.98 in. or, say 3 in.

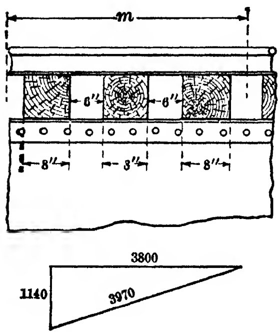


FIG. 95.

It frequently happens that the determination of the flange section of girders is materially influenced by the question of rivet pitch, and the experienced designer will always look into this before selecting flange angles.

71. Direct Web Stresses.—It has previously been shown that the intensity of the horizontal shear at any point in the web of a girder equals the intensity of the vertical shear and that these reach their maximum values at the neutral axis. Consider again an infinitesimal prism at the neutral axis. The shearing forces acting on this prism are shown in Fig. 96. These forces will develop internal forces of tension and compression, the value of which may be found as follows:

Let the thickness of the prism at right angles to the paper be unity, and let v equal the intensity of the shear. Then the total shearing force on each side $= vdx$. These forces being resolved into components, the value of each is found to be $vdx/\sqrt{2}$, acting as indicated in Fig. 97. The effect of these components is to produce on the diagonal plane bd a total tension $= 2vdx/\sqrt{2}$. Since

the length of $db = \sqrt{2}dx$, the intensity of the tension on it is $\dfrac{2vdx}{\sqrt{2}\sqrt{2}dx} = v.$

In the same manner a compressive force may be shown to act on ac the intensity of which is also v. It therefore follows that at the neutral axis there exist both a tension and a compression acting upon planes at right angles to each other and at 45° to that axis, and that the intensity of these forces is equal to that of the shear. If the prism in question had been taken above or below the neutral axis, these conditions would have been modified somewhat through the introduction of direct fiber stresses. The effect of the shear in producing direct stresses would not be changed; *i.e.*, the shearing forces would develop

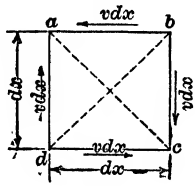


FIG. 96.

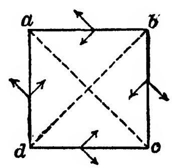


FIG. 97.

direct stresses as before, but the final value of the tension or compression upon any plane would have to be obtained by combining the direct stresses due to shear and the direct fiber stresses due to bending.

The expression for the maximum direct stress for this more general case is developed in books on mechanics and is as follows:

$$p' = \frac{p_x + p_y}{2} \pm \frac{1}{2}\sqrt{4v^2 + (p_x - p_y)^2}$$

In this equation, p' = intensity of the maximum direct stress occurring at a given point on any plane, p_x and p_y are the intensities of the direct stresses acting at the same point on two rectangular planes passing through the point, and v is the intensity of the shear on each of these two latter planes.

Figure 98 illustrates this condition. It is evident that if point a is at the neutral axis of a beam subjected to bending but not to direct stress, p_x and p_y are both zero and $p' = v$.

The expression for the angle θ between the x plane and the plane upon which the maximum intensity occurs is also derived

in mechanics and is as follows for a beam subjected to bending only: $\cot 2\theta = p_x/2v$.

At the neutral axis of such a beam or girder, $v = p'$; hence, $\theta = 45°$.

That tension and compression act as shown in Fig. 97 is also evident from the distortion produced by the shearing forces. It is plain that under the action of horizontal and vertical shear the prism which is rectangular when unstressed will take the shape shown, greatly distorted, in Fig. 99, and that hence the line *ac* will be lengthened and the line *bd* shortened. These changes can be produced only by tension and compression at 45° to the axis.

In plate girders the existence of this compression at 45° to the axis is of considerable importance since, if it is not recog-

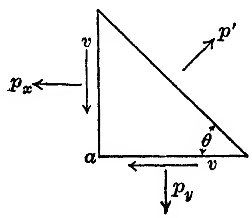


FIG. 98.

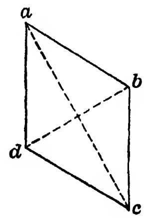


FIG. 99.

nized and proper means are not taken to provide for it, failure will occur through sidewise buckling of the web. To prevent such failure the web must be made of such thickness that there will be no danger of excessive compression, or else the buckling tendency must be restrained by other means. The latter is the common method and is accomplished by the use of stiffeners in the form of vertical angles riveted to the web and extending from top to bottom of the girder. Sometimes, however, it is more economical of material as well as of labor to increase the web thickness rather than to use stiffeners. In reinforced-concrete beams the diagonal tension is an important factor since concrete is very weak in tension and means must generally be taken to provide against failure by diagonal rupture at 45° to the axis, either by steel rods placed at approximately 45° to the axis or by vertical stirrups.

The combination of the direct stress due to shear with that due to bending gives a resultant compression in the web acting

in the direction indicated by the dotted line in Fig. 100. The shape of this line is dependent upon the relative value of the shear and the direct stress. Its ordinates are plotted from axis *mn*. At the end of a girder where the shear is a maximum and the bending moment a minimum, it would lie at about 45° to the axis throughout its entire length. At the section where the moment is a maximum the shear is zero; hence, at this section there is no direct stress in the web at the neutral axis, and the direct stress above or below this axis is parallel to it.

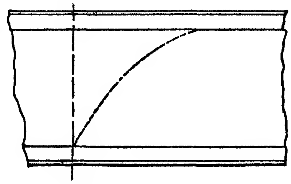


FIG. 100.

72. Web Stiffeners.—The web of a plate girder is a thin plate subjected to shear and direct stresses, which latter consist of both tension and compression acting at right angles to each other, as explained in the previous article. The compression forces tend to buckle the web which must either be thick enough to resist this buckling or be restrained laterally. The usual practice is to prevent such buckling by means of pairs of stiffener angles placed at right angles to the flanges and located near enough together to reduce the unsupported length of any diagonal strip of the web sufficiently to prevent buckling.

The determination of the required distance apart of the stiffeners for a given thickness of web involves a study of the elastic stability of such a plate and has been discussed by numerous writers in recent years,[1] resulting in the adoption of the following rules for intermediate stiffener spacing by both the American Railway Engineering Association and the American Association of State Highway Officials:

If the depth of the web between the flanges or side plates of a plate girder exceeds 60 times its thickness, it shall be stiffened by pairs of angles riveted to the web. The clear distance between stiffeners shall not exceed 72 inches nor that given by the formula:

$$d = \frac{255{,}000t}{S}\sqrt[3]{\frac{St}{a}} \qquad (20)$$

[1] See article entitled Elastic Stability of Plates Subjected to Compression and Shear, by Hovey, in *Proc. Am. R. R. Eng. Assoc.*, Vol. 36, 1935.

d = clear distance between stiffeners, in inches.
t = thickness of web, in inches.
a = clear depth of web between flanges or side plates, in inches.
S = unit shearing stress, gross section, in web at point considered.
The width of the outstanding leg of each angle shall be not more than 16 times its thickness and not less than two inches plus $\frac{1}{30}$th of the depth of the girder.

The following example illustrates the method of using this formula:

Problem: Determine the required spacing of web stiffeners in the following girder:

Depth, $60\frac{1}{2}$ in. back to back of angles.
Web, 60 in. $\times$ $\frac{3}{8}$ in.
Flange angles, 6 in. $\times$ 4 in. $\times$ $\frac{1}{2}$ in. with 4-in. leg vertical.
Maximum shear (live, dead, and impact) = 240,000 lb.

Solution:

$$d = \frac{255{,}000 \times \frac{3}{8}}{10{,}670}\sqrt[3]{\frac{10{,}670 \times \frac{3}{8}}{52\frac{1}{2}}} = 38 \text{ in.}$$

For that portion of a girder where the bending moment is large and the shear relatively small, the conditions in the web differ materially from those assumed in developing formula (20). For example, at the point of maximum bending moment the shear is zero, and in consequence no web compression exists in the half of the girder between the neutral axis and the tension flange, while the web compression in the other portion of the girder is parallel to the flange and increases in intensity as the distance from the neutral axis increases. Between the section of maximum shear and that of maximum moment the condition varies from that assumed in developing the formula to that just stated. To ensure that the web is sufficiently thick to prevent buckling under stresses parallel to the flanges, specifications commonly require that its minimum thickness for carbon steel shall be $\frac{1}{170}d$ and that it shall be somewhat greater for alloy steel. Some modification is, however, permissible if the maximum fiber stress due to bending is less than the permissible stress.

In addition to the angles required to stiffen the web against buckling, stiffener angles should be used at all points where concentrated loads of considerable magnitude are applied to the girder, in order to transmit these loads into the web without overstressing the flange rivets. The design of such stiffeners consists

in selecting angles of sufficient area in the outstanding legs to withstand the load without crushing and with sufficient total area to carry the applied load as a column, using the formula of Art. 18, and considering the unsupported length to be approximately one-half the depth of the girder. The outstanding legs should extend as close as possible to the edges of the flange angles.

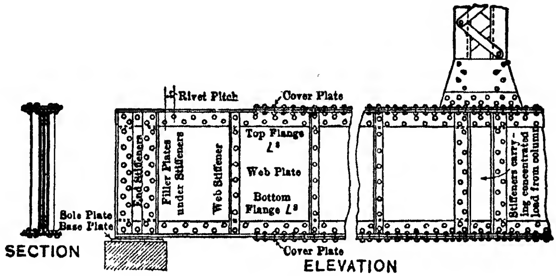


FIG. 101.

The number of rivets necessary to transmit the load into the web must also be determined, the value of the rivet being limited either by bearing on the web or by double shear. Both end and intermediate stiffeners are indicated in Fig. 101.

73. Flange Plates.—Flange plates are frequently used to increase the flange area and thereby give a variable and more economical flange. It is not considered good design to use many cover plates. In general the total area of cover plates should not exceed one-half the total flange area, unless the largest size

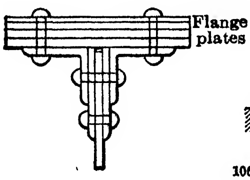


FIG. 102.

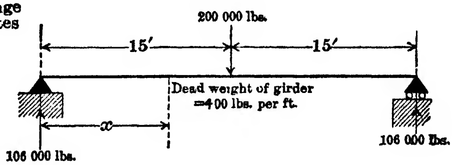


FIG. 103.

angles are used. As the length of rivets should be limited in order to ensure good results, the thickness of the metal in the flanges should not exceed 4½ times the diameter of the rivet unless the number of rivets is increased over the theoretical number or special tapered rivets are used. In case a larger flange area is required, vertical flange plates may be used, as shown in Fig. 102, or a box girder.

To determine the proper location of the ends of a cover plate, it is necessary to equate the bending moment that the girder can carry without the cover plate to the external bending moment at the end of the plate.

The following example serves to illustrate this method:

Problem: How far from the ends of the girder shown in Fig. 103 may the ends of the bottom cover plates be located? Girder to consist of a 48- by $\frac{9}{16}$-in. web, 6- by 6- by $\frac{5}{8}$-in. flange angles, and two 16- by $\frac{1}{2}$-in. cover plates on each flange. Allowable fiber stress = 16,000 lb. per square inch.

Solution:

Effective area of the tension flange members:

					Sq. in.
Two angles, 6 in. × 6 in. × $\frac{5}{8}$ in., at	7.11	= 14.22	− 2.50	=	11.72
Two plates, 16 in. × $\frac{1}{2}$ in., at	8.00	= 16.00	− 2.00	=	14.00
Web, $\frac{1}{12} \times 48 \times \frac{9}{16}$				=	2.25

To locate end of outside cover plate proceed thus:

Effective flange area after plate is cut = 11.72 + 7.0 + 2.25 = 20.97 sq. in.

Distance from back of angles to center of gravity of flange

$$= 1.73 \text{ in.} - \left(8 \text{ in.} \times \frac{1.73 + 0.25}{22.22}\right) = 1.0 \text{ in.}$$

$h = 48.5 \text{ in.} - 2.0 \text{ in.} = 46.5 \text{ in.}$

Bending moment that girder can carry with one cover plate on flange

$$= \frac{20.97 \times 16{,}000 \times 46.5}{12} \text{ ft.-lb.}$$

$$= 1{,}300{,}000 \text{ ft.-lb.}$$

Let x = distance, ft., from end of girder to point where plate should begin.

$$\text{Bending moment at } x = 106{,}000x - \frac{400x^2}{2}$$

Therefore,

$$106{,}000x - \frac{400x^2}{2} = \frac{20.97 \times 16{,}000 \times 46.5}{12} = 1{,}300{,}000$$

The value of x as determined from this equation is 12.5 ft.

The actual length of the cover plate should be somewhat longer than the theoretical length, in order that its stress may be properly carried into it. A foot is usually allowed at each end for this purpose. If this allowance is made, the cover plate in question would begin 11.5 ft. from the end of the girder and its length would be 7 ft.

The value of x for the cover plate nearest the flange is given by the following expression:

$$106{,}000x - \frac{400x^2}{2} = \frac{13.97 \times 16{,}000 \times 45}{12}$$

In the case of girders subjected to moving concentrated-load systems the following graphical method may be used to advantage:

Plot the external bending moment at each panel point or at correspondingly frequent intervals in the case of a deck girder without floor beams, and connect these points by a smooth curve, which will be the curve of bending moments. This curve is practically a series of straight lines and may be so used with safety if desired, the influence of the weight of the girder being offset by the fact that a straight-line curve for moving concentrated loads gives excess moments throughout except at panel points (see Art. 49). Compute the allowable moment, m,[1] by the application of formula (17) for the controlling conditions, *viz.:* no cover plate; one cover plate on each flange; two cover plates on each flange; and so on, up to the maximum number of cover plates used less one. Plot these moments to the same scale as the external bending moments. Since these moments for each case are constant throughout the length of the girder, each may be represented graphically by a straight line parallel to the girder axis, the points of intersection of which with the curve of bending moments locate the ends of the cover plate.

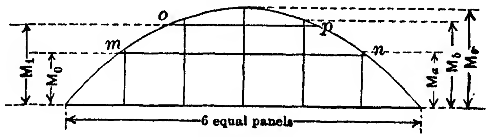


FIG. 104.

This method is shown for a girder with two cover plates by Fig. 104. M_a, M_b, and M_c are the external bending moments: M_0 is the allowable moment without cover plates; M_1 with one cover plate. (The moment with two cover plates need not be plotted.) The cover plate nearest the flange should extend from m to n, the outer cover plate from o to p. These are theoretical lengths, and the actual plate should be made somewhat longer, as previously stated.

It should be noted that the curves of moments for end-supported deck girders carrying uniform loads and of uniform weight per foot are parabolas, and the ends of the plates for such girders

[1] The allowable moment which the girder can carry is called the *moment of resistance.*

may be located by means of the following parabolic formula, which may also be used without serious error for girders carrying moving loads:

$$c = L\sqrt{\frac{a}{A}}$$

where c = length of cover plate, ft.
L = span, ft.
A = gross area of compression flange or net area of tension flange, sq. in., at center of girder
a = gross or net area of cover plate to be cut plus that of all plates further from the flange angles (net area is used for tension flange)

The flange width is an important feature and should be carefully considered in selecting angles and plates. The compression flange should be supported laterally, this being accomplished in half-through bridges by brackets attached to the floor beams and in deck spans by cross frames and horizontal bracing. The allowable unit stresses in compression flanges are based upon the ratio of unsupported flange lengths to widths. For the sake of appearance, it is usual to select cover plates of sufficient width to project slightly beyond the flange angles on either side. They should, however, project not more than 2 in. For example, flanges with 6- × 6-in. angles should have plates not less than 13 in. and not more than 16 in. in width. Plates with a width in even inches should preferably be chosen.

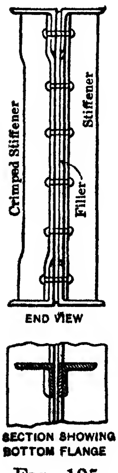


FIG. 105.

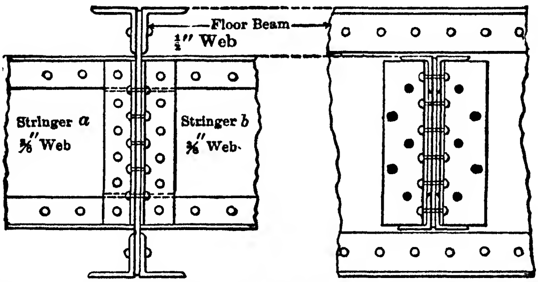
74. Connection Angles and Fillers.—It is necessary either to use fillers under plate-girder stiffeners or else to crimp the stiffener angles over the flange angles. There is but little difference in the cost of the two methods, but the former is generally used and should always be used when concentrated loads are carried by the stiffeners. Both of these types are illustrated in Fig. 105.

One objection to the use of fillers is that unless the filler is riveted to the web plate by an independent row of rivets, thus becoming practically a portion of the web (this type of filler is frequently called a *tight filler*), the rivets connecting the stiffener

to the web are reduced in strength since they have to carry stress through the loose filler plate and thus are subjected to some bending. This is of no importance in intermediate stiffeners, which serve to stiffen the web, but should be considered in the case of stiffeners carrying a concentrated load into the web. In such cases if loose fillers are employed, an excess of rivets, say 50 per cent, should be used.

The use of tight fillers is also advisable in some cases in order to increase the bearing value of rivets that otherwise would be limited by bearing on the web instead of by shear. The following example illustrates this:

Problem: Determine whether sufficient rivets are used in the connection of stringers to floor beams shown in Fig. 106.



Maximum end shear: stringer a=40,000 lbs.; stringer b=30,000 lbs.
Maximum reaction on floor beam from both stringers=65,000 lbs.

FIG. 106.

Allowable unit stress per square inch upon rivets,

	Bearing	Shear
Machine..................................	27,000	13,500

For ⅞-in. rivets, the units above give the following working values:

Bearing on ⅜-in. plate = 8,850 lb.
Shear = 8,100 lb.

Assume that the rivets shown in Fig. 106 are all that can be used in the angles. Field rivets are shown thus: (•).

Solution: To carry to the hitch angles the reaction of 40,000 lb. in stringer a, there are required 40,000/8,850 = 5 rivets to connect the stiffener angles to the web, provided that the rivets are limited by bearing on the web. As indicated in the figure, the largest number of rivets that can be used is six, but it is inadvisable to count upon those in the flanges, which are frequently fully stressed by the flange stress, and additional rivets should be added if the filler is to be a loose one; hence, it is necessary to use either

wider hitch angles with two rows of rivets or else a wide filler to increase the bearing value of the connection rivets. The latter would be cheaper and consequently advisable. If the filler, therefore, is made wider and connected to the web by two extra rivets, an additional stress equal to that which two rivets can carry can be taken from the web into the filler and by that carried into the rivets connecting the stiffeners. As these rivets would, however, have to carry a considerable bending moment in addition to the direct shear, it is advisable to make a liberal allowance; hence, it would be well in this case to use in the fillers four rivets placed directly opposite those in the stiffeners. The stringer would then be as shown in Fig. 107.

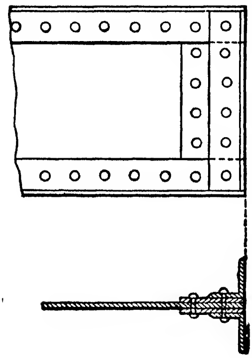


FIG. 107.

One other point yet remains to be considered, *viz.*, the shearing strength of the rivets themselves. The connection has so far been designed to carry the stress from the web plate into the rivets. Can the rivets carry this stress into the angles? As the thickness of the angles is not restricted, sufficient bearing area can be obtained, but can the rivets carry the stress without shearing off?

As the rivets are in double shear, there are needed 40,000/16,200 or three rivets. Hence, the number needed for bearing is also sufficient for shear; otherwise it might be necessary, despite the wide filler, to use wider stiffener angles.

The connection of stringers to floor beam may be treated in a similar manner. Ten rivets are shown. These have to carry in single shear the maximum shear in a single stringer, *i.e.*, 40,000 lb. They also have to carry 65,000 lb. in bearing upon the web.

$$\frac{40,000}{8,100} < 10$$

Hence, there are enough in shear.

$$\frac{65,000}{8,850} = 8$$

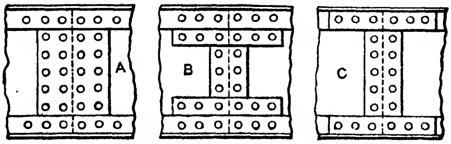
Hence, there are also enough in bearing.

75. Web Splices.—Owing to the limited length of plates obtainable, it is frequently necessary to splice the webs of long and deep girders. Figure 108 shows several methods of making such splices.

Of these, type A is best in appearance and is recommended, with the number of rows of rivets on each side of the splice to be determined by computation as illustrated later.

The design of such splices requires two distinct operations, *viz.*, the determination of the size of the splice plates, and the

determination of the number and location of the splice rivets. The former question involves the selection of plates that are of sufficient strength to carry not only the shear at the section where the splice is to be located, but also the bending resistance of the web as given by formula 17, *viz.*, one-twelfth its gross area multiplied by the product of the allowable unit stress and the dis-



Web Splices.

In all cases splice plates are used on both sides of the web

FIG. 108.

tance between centers of gravity of flanges. For a splice of the type shown by *A*, Fig. 108, both these considerations are usually satisfied by plates of the minimum allowable thickness, although for thick web plates or shallow girders the thickness of the plates should be carefully computed by the method used in the following example. The width of the plates is usually determined by the number of rivets needed and requires no computation. The rivets must be sufficient to carry the shear and bending moment that the splice plates are required to

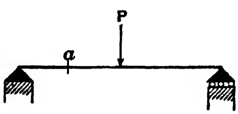


FIG. 109.

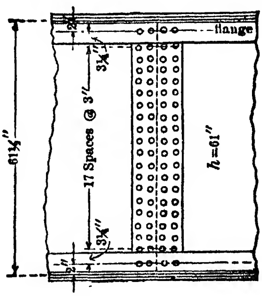


FIG. 110.

resist. Their computation involves the application of the method given in Art. 67 for the strength of rivets in torsion.

One of the well-recognized and important rules of good design is so to proportion the member that it will be as strong at joints and other critical sections as in its main portion. The application of this rule to the design of web splices involves making the splice of sufficient strength to carry all the shear and bending moment that the web plate is capable of carrying. For many girders, this would give excessive strength since the web is not

called upon to resist maximum moment and shear simultaneously. It is, however, a safe rule to follow in all cases and should not be deviated from unless the location of the splice can be so fixed that it will surely come at a point where it will not be subjected to maximum conditions of both kinds simultaneously.[1]

The example that follows illustrates the design of a web splice for a girder, the web plate of which is supposed to be fully stressed in shear and bending.

Problem: Design a web splice for a girder with a 61- by $\frac{7}{16}$-in. web, 6- by $3\frac{1}{2}$- by $\frac{9}{16}$-in. flange angles with $3\frac{1}{2}$-in. legs vertical, and two 16- by $\frac{3}{8}$-in. cover plates on each flange, using the unit values specified in Art. 18 and $\frac{7}{8}$-in. rivets.

Solution: First assume the number and location of rivets that can be used in one vertical row. As shown by Fig. 110, 18 rivets may be used, spaced 3 in. apart. Had the minimum allowable spacing of three times the rivet diameter been adopted, a few more rivets could have been inserted in a row, but it is inadvisable to use the minimum pitch if it can be avoided.

Next, assume that the minimum allowable thickness of material is $\frac{5}{16}$ in., and determine whether or not this thickness will prove sufficient for the splice plates. If these plates are assumed to be fitted to the edges of the vertical legs of the flange angles, their length will be $54\frac{1}{2}$ in. and the net area of the two splice plates through a row of rivets will equal $2(54\frac{1}{2} - 18)\frac{5}{16} = 22.8$ sq. in. The net area of the girder web equals $(61 - 20)\frac{7}{16} = 17.9$ sq. in.; hence, $\frac{5}{16}$-in. plates are ample to carry the shear.

To determine whether the strength of the splice plates in bending is sufficient, the allowable resistance of the girder web as used in formula (17) should first be determined. This equals $(\frac{1}{12} \times 61 \times \frac{7}{16})sh = 2.22 \times 61.1 \times s = 135.64s$. If there were no rivet holes in the splice plates, their resistance to bending would be given by the formula

$$M = \frac{1}{6}s \times bh^2 = \frac{1}{6}s\,(\frac{5}{8})(54\frac{1}{2})^2 = 309s$$

As already stated, the allowance for rivet holes is approximately equal to that obtained by using a coefficient of $\frac{1}{8}$ in the formula for M instead of $\frac{1}{6}$. Making this allowance gives the following value for the resistance[2] to bending:

$$M = \frac{1}{8}sbh^2 = \frac{3}{4}\,309s = 232s$$

The value of I for the gross area $= \frac{1}{12} \times \frac{5}{8} \times (54.5)^3 = 8{,}431$; hence, the reduction made by using the coefficient $\frac{1}{8}$ instead of $\frac{1}{6}$ is ample.

[1] Such a condition could exist at section a, in the girder shown by Fig. 109, provided that a cover plate stops well beyond this point.

[2] The actual effect of the rivet holes in the tension half of the splice plates in this case is to reduce the value of I for the gross area by the following amount: $\frac{5}{8}(25.5^2 + 22.5^2 + 19.5^2 + 16.5^2 + 13.5^2 + 10.5^2 + 7.5^2 + 4.5^2 + 1.5^2) = \frac{5}{8}(2{,}180) = 1{,}362$.

This value is much larger than necessary; hence, the $\frac{5}{16}$-in. plates are of sufficient strength to carry bending and shear.

To determine the number of rows of rivets, the allowable shearing and bending resistance of the girder web must be computed. These values are as follows:

Shear, $17.9 \times 11{,}000 = 196{,}600$ lb.

Bending, $135.64 \times 18{,}000 = 2{,}441{,}000$ in.-lb.

If two rows of rivets are assumed on each side of the splice, the vertical load per rivet will equal approximately $\frac{196{,}000}{2 \times 18} = 5{,}444$ lb.

The value of a rivet in bearing on the $\frac{7}{16}$-in. web $= 27{,}000 \times \frac{7}{16} \times \frac{7}{8} = 10{,}340$ lb.

The method of Art. 67 may now be applied but is somewhat laborious, and no essential error will be made if the resistance of each rivet to torsion is assumed to vary with its distance from the central axis of the girder, instead of with its distance from the centroid of the group of rivets, and to act in a direction parallel to this axis. Upon making this assumption, the resistance of the uppermost rivet to bending will equal

$$\sqrt{10{,}340^2 - 5{,}444^2} = 8{,}800 \text{ lb.}$$

The value of I in formula (18) has already been computed for a half row of rivets (see footnote). The resistance to bending of the two rows of rivets may now be written

$$R = 2\left(\frac{8{,}800}{25.5}\right)(2 \times 2{,}180) = 3{,}000{,}000 \text{ in.-lb.}$$

This is somewhat more than the value previously found for the allowable web resistance, *viz.*, 2,441,000 in.-lb. Were the resistance of the rivets to bending materially less than the allowable bending moment of the web, another row of rivets might be used or the splice located at a point where the web is not fully stressed in bending and shear simultaneously. A good location in such a case would be at a point toward the center of the girder from the end of a cover plate. At such a point, a portion of the cover-plate area would be in excess and could be counted upon to make up the deficiency in the strength of the splice.

The method of calculation illustrated by the previous example is not exact, and probably less so than for the cases given in Art. 67, where the number of rivets in a vertical row was much less.

In practice, it may be found desirable to use splice plates thicker than those required by computation. If the splice plate is used as a filler, it should be as thick as the flange angles. It is, however, possible in such a case to make up the total thickness required by the use of a $\frac{5}{16}$-in. splice plate and a filler, an arrangement frequently used.

76. Flange Splices.—In very long girders, it may be necessary to splice the flange angles. When this is to be done, only one angle in each flange should be spliced at a section. A good method is to splice the top angle on one side of the girder and the bottom angle on the other side at a section a little to one side of the center of the girder and to reverse this process for a corresponding section on the other side of the center. The splice should always be made by another angle the cross section area of which should be equal to that of the angle to be spliced. In order to simplify the construction the splice angles for the tension flange should be exactly like those for the compression

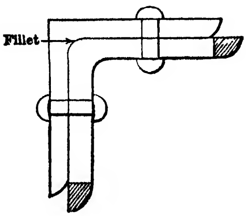


FIG. 111.

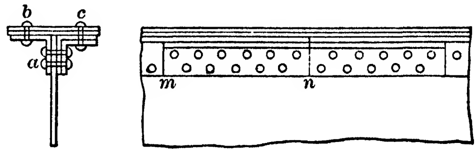


FIG. 112.

flange; hence, the net area of the splice angle should equal the net area of the main angle.

In order to obtain a splice of neat appearance and answering the requirements above, it is usually necessary to select an angle with the same width of legs as the main angle, but $\frac{1}{16}$ in. or $\frac{1}{8}$ in. thicker, and to plane off the projecting legs so that they may be flush with the main angle.

The following example illustrates this: Determine the splice angle required for a 6- by 6- by $\frac{1}{2}$-in flange angle. The net area of the main angle $= 5.75 - 1.00 = 4.75$ sq. in. The net area of a 6- by 6- by $\frac{9}{16}$-in. angle planed to fit the 6- by 6- by $\frac{1}{2}$-in angle = approximately $6.43 - 1.12 - 2(\frac{9}{16} \times \frac{1}{2}) + 0.20 = 4.95$ sq. in.; hence, this angle has sufficient area and should be used.

Figure 111 shows by crosshatching the portion of the angle to be cut off. The outer corner must also be rounded off as indicated to fit the fillet of the main angle.

The number of rivets required in the splice angle may be determined, if there are no cover plates, by computing the stress that the angle can bear and dividing it by the value of one

rivet, the strength of the rivet being generally limited by single shear. If the angles are equal-legged, one-half the number of rivets needed should be used in each leg. If the legs are unequal, each leg should have its proportional part of the total rivets required; *e.g.*, if a 6- by 4-in. angle is to be spliced and if 20 rivets are needed in all, put $6/10 \times 20 = 12$ rivets in the 6-in. leg, and $4/10 \times 20 = 8$ rivets in the 4-in. leg.

Cover plates are generally required in girders that need flange-angle splices, and in such girders the necessary number of splice rivets may be somewhat in excess of the number required to carry the total stress that the angles are good for, since the increment in stress in a distance equal to the length of the splice angle must be taken care of. With reference to Fig. 112, it is evident that the horizontal rivets at *a* must carry from the flange into the splice angle in the distance *mn* one-half the increment in flange stress in that distance (the other half going through the same rivet to the flange angle on the left-hand side) plus one-half the stress in the main angle at *m* (since the angle is equal-legged). The rivets at *c* should be computed to carry the same amount, since it is proper to assume that all the increment in flange stress is carried by the cover plates, the angle being fully stressed before cover plates are added. The rivets at both *a* and *c* are limited usually by single shear and should be designed accordingly.

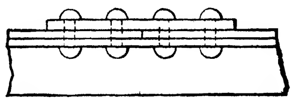


FIG. 113.

Since the splice would ordinarily be placed near the center of the girder where the increment in flange stress is small, it is usually sufficient to determine the number of rivets required to splice the angle, on the assumption that it is stressed to its full value, and to add one or two rivets to carry the flange-stress increment. If no cover plates are needed, it is unnecessary to consider the increment in stress since if the splice rivets are determined for the full value of the angle they will surely be sufficient to carry the stress in the angle at *m* plus the increment in *mn*.

A cover-plate splice may always be made by the addition of a splice plate of the same size as the plate to be cut and by the use of sufficient rivets to transmit the full stress from one plate to another, with the addition of a liberal percentage of extra rivets, say 33 per cent for each plate intervening between the

plate to be spliced and the splice plate. Such a splice is shown in Fig. 113.

The disadvantage of long rivets, subjected perhaps to bending moment because of the intermediate plates, together with the unslightly appearance of such a splice, makes it desirable if the girder has more than one cover plate to splice one cover plate by means of another. This may be done by properly choosing the section where the splice is to be made.

This is illustrated by Fig. 114, in which the lower cover plate is to be spliced. If this plate is cut at *a*, where the top cover plate should begin, and if the top plate is extended to *b*, making the distance *ab* such that enough rivets can be put between *a* and *b* to carry the stress that the plate is good for, the splice will be

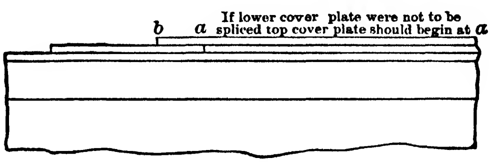


FIG. 114.

properly made. If the top plate were thinner than the bottom plate, the splice would have to be located nearer the end of the girder at a point where one cover plate of the thickness of the top plate would be just sufficient to carry the stress.

It will be observed that this method is based upon the transfer of all the stress from the end section of the plate to be spliced into the plate immediately above it. The intermediate section of the spliced plate instead of the upper plate will then take the additional increment of flange stress.

77. Welded Structures.—At the time of writing (March, 1939), field-welded buildings and other structures are being used to a limited extent in the United States and numerous welded highway bridges are being built in European countries. The author knows of no main-line welded railroad bridge yet constructed. The determination of stresses in welded girders and other structures is accomplished in the same manner as in the case of riveted structures, but the make-up of the members may be different and the amount of metal required reduced somewhat. The failure of several welded structures recently has made United States engineers hesitant to employ welding for main members of bridges, particularly in view of the difficulty involved in inspect-

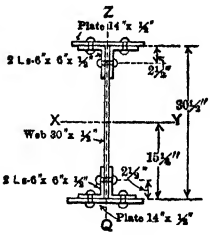
ing the welding and the scarcity of welders competent to do satisfactory field welding on large structures. In one large welded water tank for which the author had engineering responsibility, not more than one man in thirty of those who applied for work as welders was able to qualify. Specifications for arc welding have been prepared by the American Welding Society, and the American Association of State Highway Officials has issued specifications entitled "Arc Welding for Metal Bridge Structures." These various specifications are, however, still in the transition stage.

Problems

40. *a.* Compute I/c for this girder with respect to the neutral axis and to the axis ZQ, and compute maximum fiber stress for a total uniformly distributed vertical load of 3,200 lb. per foot over the entire girder. Allow for $^{15}\!/_{16}$-in. holes in bottom flange, assuming three in one section. Span = 40 ft.

b. Compute the maximum fiber stresses in both flanges for the loading given in *a*, using formula (17), Art. 62.

c. Compute by both the approximate and the exact methods the required pitch of the horizontal and vertical flange rivets at end of top flange, assuming all the load to be applied directly through the flange, and one cover plate on each flange to extend to end of girder. Use unit values given in Art. 18 and $\frac{7}{8}$ in. rivets.



PROB. 40.

d. Determine distance from end at which the cover plates may be cut if desired.

e. Determine whether intermediate stiffeners are needed.

f. Determine the size of stiffeners needed on the girder to support the top flange under a concentrated load of 200,000 lb. Allow 18,000 lb. per square inch bearing on stiffeners, and assume that their outstanding legs only are effective.

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CHAPTER VI

SIMPLE TRUSSES

78. Trusses Defined.—A truss is a structure consisting of separate bars designed to carry direct tension or direct compression. These bars are connected at their ends and occasionally at intermediate points, the points of connection being called *joints*. The connections are sometimes made by riveting the members directly together and sometimes by riveting them to a common steel plate, the truss in either case being called a *riveted truss*. The connections may also be made by fastening together with a large steel pin all the members meeting at a joint; such a truss is called a *pin truss*. The outer forces should be applied at the joints only, since the members are not intended to carry bending. This is accomplished by the use of floor beams in a bridge and purlins in a roof. As the depth of plate girders is limited by the available width of plates and by the inability to ship by rail single pieces wider than 10 ft. 6 in., or thereabouts, it is necessary to use trusses where economy or rigidity requires greater depths. The common practice in the United States at the present time is to use beams or girders up to lengths of 100 to 125 ft.; riveted trusses above these lengths up to 400 to 500 ft.; and pin trusses above this length. The use of shorter pin-truss spans for railroad bridges has been given up because of lack of rigidity and consequent early wearing out of the bridge.

A typical truss is illustrated by Fig. 4.

79. Classification.—All trusses may be divided into two general classes based upon the methods necessary for the determination of the stresses in the members; if these stresses can be determined by statics, the truss is statically determined; otherwise, it is statically undetermined. It should be noted that a truss may be statically undetermined with respect to the outer forces, *i.e.*, the reactions cannot be determined by statics, and yet be statically determined with respect to the inner forces, and vice versa. The former is usually the case with drawbridges,

the latter with the double intersection trusses sometimes used in simple span bridges.

80. Theory.—The theory upon which the computation of truss stresses is based assumes that the center-of-gravity lines of the members meeting at a joint intersect at a point and are held together at that point by a frictionless pin. It follows that the stresses in the various members will all be direct stresses. That this deviates considerably from the truth for riveted trusses is evident; the error in pin trusses is less but is not negligible; hence, the common theory of trusses is by no means a precise one. The secondary stresses produced by resistance to motion at the joints are, however, small in well-designed trusses, as compared with the primary stresses, as the stresses computed by the assumption above may be called, and experience shows that for simple spans of ordinary length these primary stresses are sufficiently exact to be used in designs where the ordinary factor of safety is applied.

81. Methods.—The methods necessary for the computation of the stresses in statically determined trusses are very simple and consist merely of the application of the three equations of equilibrium to portions of the truss, these portions being chosen in such a way as to enable the stress in a given bar to be easily found. There are in common use three methods of accomplishing this result: the method of *joints*, the method of *moments*, and the method of *shears*. All these are applications of the general method and differ only in detail. In the computation of a truss, it is often advantageous to employ all three methods, choosing for each bar that which is best adapted to it. The method of joints is the most general of these methods and will be considered first.

82. Analytical Method of Joints Described.—Figure 115 represents a simple truss carrying a load at the apex. Let a section be taken around the joint at a and the remainder of the truss removed. As the entire truss is in equilibrium, the portion enclosed by this section, shown by a circle in the figure, must also be in equilibrium; hence, the two equations of equilibrium $\Sigma H = 0$ and $\Sigma V = 0$, will suffice to determine the unknown stresses in ab and ac, the horizontal and vertical components of each of which are

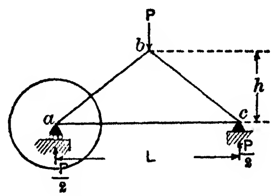


FIG. 115.

functions of the bar stress and are to each other as the horizontal and vertical projections of the bar. In a similar manner, other portions of the truss may be cut off by similar sections taken at a sufficient number of joints to permit the determination of all the unknown stresses. Figure 116 shows the condition that exists at joint a, on the assumption that the stresses in the bars are axial stresses, this being in accordance with the general theory. It should be carefully observed that this method deals with the stresses in the bars rather than with the bars themselves.

With reference to Fig. 116, it is evident that as there are but two unknown forces, S_1 and S_2, the two equations of equilibrium, $\Sigma H = 0$ and $\Sigma V = 0$, will be just sufficient to determine these, and since all the forces meet at a point the equation $\Sigma M = 0$ will be satisfied by any value of S_1 and S_2 and need not be considered. With the stresses in bars ab and ac thus determined, a section may next be taken at either joint b or joint c, and the stress in bar bc computed in a similar manner, the necessary computations for this truss being thus completed.

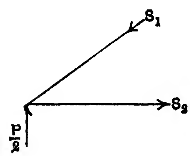


FIG. 116.

It should be observed that if the moment at a joint does not equal zero, the number of unknowns will be increased, and additional equations which cannot be derived from statics would be necessary to obtain a solution.

83. Character of Stress.—The determination of the character of the stress is often more important than that of its magnitude, as a bar designed for tension may fail if the stress is compression even if its magnitude is small. To determine the character of the stress in any bar, it is sufficient to assume arbitrarily the direction of the stresses before applying the equations of equilibrium. If the solution gives a positive result for a stress, it shows that this stress acts in the direction originally assumed.

In this connection, it should be carefully observed that the *internal* stresses in a bar subjected to tension continually tend to pull the ends together, *i.e.*, to shorten the bar; hence, tension in a bar always acts *away* from the joints at *both* ends, and *compression toward* the joints. Figure 116 illustrates this. S_2 is shown acting away from the joint, *i.e.*, in tension; S_1 on the other hand is assumed in compression and is shown acting *toward* the joint. If the joint at the top of the truss to which the diagonal

bar is attached should next be investigated, it would be necessary to represent S_1 as acting toward that joint also, since the computations give a positive value for S_1, thus indicating that it acts in the assumed direction.

It is safer for the beginner to assume the stress in each bar as tension, or away from the joint. Positive values will then indicate tension and negative values compression. This is in accordance with the common but not universal convention of representing tension (which increases the length of a bar) by a *plus* sign.

84. Determinate and Indeterminate Trusses.—For the truss shown in Fig. 115, there are three unknown bars. The stresses in two of them have been determined by considering one joint only; the stress in the other may be found by taking either of the other joints. Since by taking both of the other joints there would be four equations and only one other unknown bar, it would seem as if there were too many equations. This is a fallacy, however, as these equations must suffice to determine the unknown reactions as well as the unknown bars, since equilibrium of each joint involves equilibrium of the entire structure; *i.e.*, for this particular structure and in general for all planar structures that are statically determined with respect to the reactions, there must be three more equations than there are bars. In other words, the six equations of joints for such a truss are not independent but are related in such a manner as to satisfy the three general equations of equilibrium for the truss as a whole, *viz.*: $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma M = 0$, which may for most cases be replaced by the more common equations

$$\Sigma H = 0, \qquad \Sigma V = 0, \qquad \Sigma M = 0$$

There are therefore for the truss shown in Fig. 115 but three *independent* equations that can be used in determining the bar stresses; hence, these stresses are determinate.

In general it may be said for *planar* end-supported trusses which are statically determined with respect to the outer forces that, if n equals the number of joints, $2n - 3$ equals the number of bars which the structure must have to be determinate. If it has more bars, the stresses cannot be computed by statics; if less, it will not be rigid and will collapse except under special conditions. The same criterion must also be applicable to any

portion of the truss. For nonplanar trusses the conditions are somewhat different and are considered in Chap. XVI.

If it is desired to build a structure that because of the number of points of support or for other reasons would ordinarily be statically undetermined with respect to the outer forces, it may be possible to make the structure determinate in all respects by properly choosing the number of members. For example, in the case of a cantilever truss, a diagonal over a pier is sometimes omitted for this reason. In swing spans, diagonals are often omitted or made very small in order to reduce the numbers of

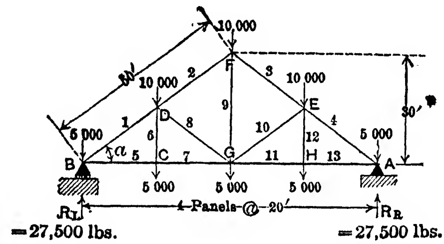


FIG. 117.

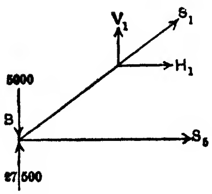


FIG. 118a.

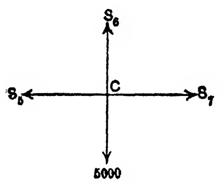


FIG. 118b.

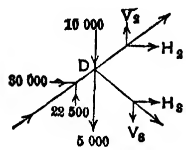


FIG. 118c.

unknowns. If the unknown components of the reactions are four, it is evident that there can be only $2n - 4$ bars if the structure is to be made determinate.

85. Mode of Procedure. Analytical Method of Joints.—In the solution of problems by the analytical method of joints the following mode of procedure should always be adopted:

1. Compute reactions.
2. Select a joint at which only two bars meet.
3. Assume the stresses in these bars to be tension, *i.e.*, to act away from the joint, and apply the equations of equilibrium, placing all the numerical terms on one side of the equation and zero on the other side. If the stress in either bar is found to

be negative, it indicates that the bar is in compression instead of tension.

4. Consider any other joint at which only *two* unknown bars meet, determine the stresses in these bars in the same manner, and proceed thus until all the stresses have been determined.

86. Application of Analytical Method of Joints.—The following numerical example has been worked out to show the application of this method:

Joint B. $\Sigma V = 0: V_1 + 27{,}500 - 5{,}000 = 0$

$$V_1 = -22{,}500 \text{ lb.} \qquad H_1 = {}^{40}\!/_{30} V_1 = -30{,}000 \text{ lb.}$$
$$S_1 = {}^{50}\!/_{30} V_1 = -37{,}500 \text{ lb.}$$
$$\Sigma H = 0: H_1 + S_5 = 0$$
$$S_5 = +30{,}000 \text{ lb.}$$

Joint C. $\Sigma V = 0: S_6 - 5{,}000 = 0$

$$S_6 = +5{,}000 \text{ lb.}$$
$$\Sigma H = 0: 30{,}000 - S_7 = 0$$
$$S_7 = +30{,}000 \text{ lb.}$$

Joint D. $\Sigma V = 0: 22{,}500 - 15{,}000 + V_2 - V_8 = 0$

$$\Sigma H = 0: 30{,}000 + H_2 + H_8 = 0$$

But

$$H_8 = {}^{20}\!/_{15} V_8 = {}^{4}\!/_{3} V_8$$

and

$$H_2 = {}^{40}\!/_{30} V_2 = {}^{4}\!/_{3} V_2$$

Therefore, from equations above may be obtained the following:

$$7{,}500 + V_2 - V_8 = 0$$
$$30{,}000 + {}^{4}\!/_{3} V_2 + {}^{4}\!/_{3} V_8 = 0$$

Solving gives

$$V_2 = -15{,}000 \text{ lb.} \qquad H_2 = -20{,}000 \text{ lb.}$$
$$V_8 = -\ 7{,}500 \text{ lb.} \qquad H_8 = -10{,}000 \text{ lb.}$$

Therefore,

$$S_2 = -15{,}000 \times {}^{5}\!/_{3} = -25{,}000 \text{ lb.}$$

and

$$S_8 = -7{,}500 \times {}^{25}\!/_{15} = -12{,}500 \text{ lb.}$$

The computation for a bar such as (8) may sometimes be advantageously referred to other than horizontal and vertical axes. If, in this particular case, the X-axis is taken along the upper chord BF and the Y-axis perpendicular to it, the condition at the joint will be as shown in Fig. 119. It is clear that in this

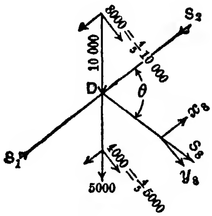


FIG. 119.

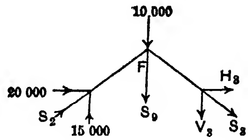


FIG. 119a.

case the value of Y_8 is given at once by the equation $\Sigma Y = 0$ and equals $-12{,}000$ lb.

$$\text{The actual stress in the bar} = -\frac{12{,}000}{\sin\theta} = -\frac{12{,}000}{2\sin a\cos a}$$
$$= -\frac{12{,}000 \times 25}{24} = -12{,}500 \text{ lb.}$$

It will be noticed that the stress in this case has been determined without reference to stresses S_1 and S_2 and is a direct function of the stress in bar 6 and the panel load at D.

Joint F. $\Sigma V = 0: 15{,}000 - 10{,}000 - V_3 - S_9 = 0$

$\Sigma H = 0: 20{,}000 + H_3 = 0$

Therefore,

$$H_3 = -20{,}000 \text{ lb.}$$

But

$$V_3 = \tfrac{3}{4}H_3$$

Therefore,

$$V_3 = -15{,}000 \text{ lb.}$$

Hence,

$$S_9 = +20{,}000 \text{ lb.}$$

and

$$S_3 = -15{,}000 \times \tfrac{5}{3} = -25{,}000 \text{ lb.}$$

Since the truss is symmetrical and the loads are also symmetrical, the stresses in the bars of one-half the truss are identical

with those in the bars of the other half; hence, further computations are unnecessary.

As a check, consider joint G, at which forces will act as shown in Fig. 120. It is evident that these forces are in equilibrium; hence, the stresses are checked to a certain extent. In practice, further checks should be applied.

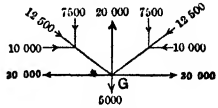


FIG. 120.

87. Graphical Method of Joints Described. The analytical method just given is perfectly general but too laborious to use in determining the stresses in all the bars of some trusses, though it may be used with great advantage for certain specific members. A graphical method based upon the same principles is well adapted for many types of trusses, particularly roof trusses with nonparallel chords, and should be thoroughly understood. This method consists of drawing polygons of forces for each joint in succession, the polygons being so combined as considerably to reduce the labor that would be required if each joint were to be considered separately. The stresses in the bars can be obtained by scaling the sides of the polygons. Like other graphical processes, this method is less precise than analytical methods, and errors in scaling the stresses are easily made and difficult of detection. A closure of the figure, however, would indicate that no error of importance had been made in the graphical work.

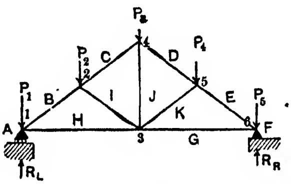


FIG. 121.

88. Mode of Procedure. Graphical Method of Joints.

1. Draw a sketch of the structure to any suitable scale, and show on it all the outer forces including reactions.

2. Designate each force and each bar by two letters so located that each force and each bar will lie between two letters and only two, as illustrated by Fig. 121.

3. Draw a polygon of outer forces. This should be drawn to a scale of sufficient size to give the desired accuracy, and the forces should be plotted in the order in which they are reached by going around the figure in a clockwise direction and should be lettered at the ends by the letters in the order obtained by this clockwise rotation. This polygon should close if the reactions have been correctly determined.

4. Draw a polygon of forces for each joint, beginning at any joint where an outer force and two bars only meet, and proceeding thence, joint by joint, selecting the joints in such an order that at no joint will there be more than two undetermined forces to consider. The sides of these polygons representing the outer forces are the sides of the force polygon. The sides representing bar stresses should be lettered at the ends by the letters obtained by going around each joint in a clockwise direction. Capital letters are used in the truss diagram and lower-case letters in the stress diagram in order that bars and stresses may be distinguished by the letters used. The diagram thus drawn should form a closed figure.

5. Determine the magnitude and character of the bar stresses from the diagram. The magnitude of the stress in any member equals the length of the line of the diagram parallel to the bar in question measured to the scale of the force polygon; its character is determined by the order in which the letters are reached in going about any joint in a clockwise direction. For example, to determine the character of the stress in bar CI of Fig. 121, note that ci in the stress diagram, Fig. 122, acts downward to the left, as determined by the order in which the letters are reached in going around joint 2; hence, the stress in CI also acts downward to the left, or toward the joint, since the bar is above the joint, and is therefore compression. A similar result is obtained by considering joint 4. For this joint, clockwise reading gives the designation of the bar IC, and ic in the stress diagram acts upward to the right, *i.e.*, toward joint 4, since the bar is below this joint.

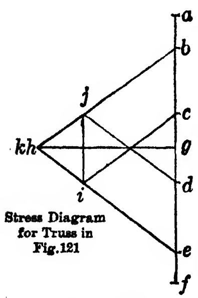


FIG. 122.

The example which follows represents clearly the application of this method and shows by the closure of the diagram that no error of importance has been made in the graphical work.

89. Application of Graphical Method of Joints.—Figure 121 shows the structure drawn to scale and with all the outer forces represented in direction and point of application. The force polygon is *abcdefga* in Fig. 122; this is a straight line, since all the forces are vertical. In it, $ab = P_1$, $bc = P_2$, etc. The reactions

R_L and R_R, represented by *ga* and *fg*, may be determined analytically, or graphically by methods given later.

The triangle of forces is first drawn for joint 1. The forces that act at this joint are R_L, P_1, the stress in bar *BH*, and the stress in bar *HG*, and these forces must be in equilibrium. Of these, P_1 and R_L are known in magnitude and direction. Their resultant equals *gb*. The stresses in *BH* and *HG* are known in direction but not in magnitude; hence, there are but two unknown quantities at this joint, and it is evident that the value of these may be found by drawing a polygon of forces. The figure *gabhg* is such a polygon and is obtained by drawing from *b* a line parallel to *BH* and from *g* a line parallel to *HG*. The line *bh* measured to the same scale as the force polygon gives the magnitude of the stress in the bar *BH*, and the line *hg* gives the magnitude of the stress in the bar *HG*. It remains to determine the character of these stresses. Considering joint 1 and reading around it in a clockwise direction, starting with *B*, gives *bh* acting downward to the left, *i.e.*, from *B* toward *H*, thus showing compression. In the same manner, the stress in *HG* is found to be tension, since it acts from *h* toward *g* or away from the joint. This method would not be correct had not the external forces been plotted by going around the figure in a clockwise direction, but it is evident that if this is done the method is correct, since in order to have *ga*, *ab*, *bh*, and *hg* in equilibrium the stresses in *BH* and *HG* must act as stated.

The next joint to be considered is joint 2, since there are now but two unknown forces acting there and they can therefore both be determined. To obtain them, draw *ci* and *ih* in the force polygon parallel, respectively, to the corresponding bars in the truss; they will intersect at *i*. *ci* acts toward joint 2, and *ih* also acts toward this joint; hence, compression occurs in both these bars. In a similar manner the stresses in the other bars may be determined.

90. Ambiguous Cases.—The method of joints, graphical or analytical, is perfectly general and applicable to all trusses; but in order to apply it successfully to some types of trusses, it is necessary to choose the method of procedure with care. For example, in solving by the analytical method the truss shown in Fig. 123, it is not possible to consider the joints in succession beginning at the abutment; but after solving for the bars *BL*,

CM, *LM*, *MN*, *LK*, and *NK*, it is advisable to determine the stress in *PQ*. To do this, apply the equations of equilibrium to joint at which P_4 is applied, using for axes the top chord and a line perpendicular to it. By using as axes the bar *OR* and a line at right angles to that, *po* may now be determined. It will then be possible to figure the stresses in the undetermined bars of that half of the truss. (The application of the method of moments as explained in Arts. 91 to 93 is simpler for such cases.)

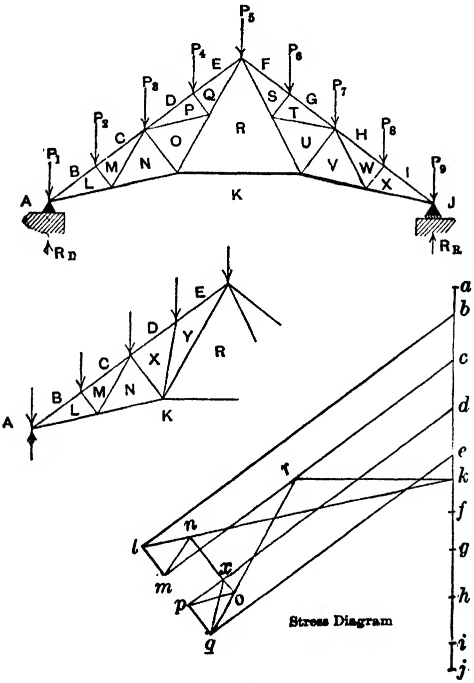


FIG. 123.

In the graphical solution of this structure a similar difficulty also arises. After the stresses in bars *BL*, *CM*, *LK*, *LM*, *MN*, and *NK* and in the corresponding bars in the other half of the truss have been determined, no joint exists at which only two unknown stresses act. To overcome this difficulty the following device may be employed: Consider the truss partially shown below as the original truss in Fig. 123, in which bars *PQ*, *QR*, *RO*, and *OP* of the original truss have been replaced by bars *XY* and *YR*. The truss is still determinate since one joint and two bars

have been eliminated. Moreover, the stress in EY equals that in EQ since if the stress in EQ is computed by the analytical method of joints in the manner just described, but working from the right end of the truss, its value is clearly seen to be independent of any possible arrangement of bars on the left. The stress diagram for the new truss may now be continued and the point y located. This corresponds to q for the original truss and is so lettered; hence, p is at the intersection of qp and dp, and the remainder of the construction may be made without difficulty. The stress diagram for the left half of the truss is shown. That for the right half would be similar and is omitted.

It is usually simpler to solve such a truss by a combination of graphical and analytical methods, the stresses being determined analytically in such bars as are necessary by the method of moments as hereinafter described and those values plotted in the diagram.

91. Analytical Method of Moments Described.—This method of finding truss stresses is based upon the application of the equation $\Sigma M = 0$. It is very useful for determining stresses in special bars of many trusses but is not so general as the method of joints and is frequently inapplicable to many bars even in the simplest trusses. Like the method of joints, it is also a method of sections, the truss being considered as divided into two or more portions by a section and the equilibrium of one of these portions being considered. It can be used to determine the stress in a given bar when all the undetermined bars cut by the section except the one in question, or their prolongations, meet at a point, which point should be taken as the origin of moments.

92. Mode of Procedure. Method of Moments.

1. Assume the truss to be divided into two or more parts by an assumed section, which may be straight or curved. This section should cut the bar in which the stress is to be determined, and all the other bars cut by it should meet at a point[1] which should be neither on the bar in which it is desired to determine the stress nor on its prolongation.

2. Apply the equation $\Sigma M = 0$ to one of the portions of the truss isolated by the section, using the point of intersection

[1] This statement is correct but not comprehensive, since the moment is in reality taken about an axis passing through the point and it is sufficient to have the bars in which the stress is not wanted pass through this axis.

described under (1) as the origin and considering that portion of the truss giving the simpler equation. The equation must include the moment of all the outer forces acting on the portion of the truss under consideration, together with the moment of the unknown bar stress which should be assumed as tension. Clockwise moments should be considered as positive. The section is commonly taken as cutting but three bars, of which two meet at a point and the third is the bar under consideration. It makes no difference, however, how many bars are cut if all but one pass through the origin of moments. It is sometimes simpler to deal with the moments of the components than with that of the forces themselves, particularly when the force may be resolved at a point such that the lever arms of one of the components is zero.

3. Solve the equation for the unknown stress. A positive result shows that the bar is in tension.

93. Application of Method of Moments.—The application of this method is clearly illustrated by the following numerical example for the truss shown in Fig. 124.

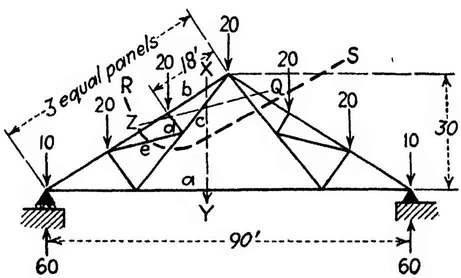


FIG. 124.

Bar a. For this bar the XY section fulfills the required conditions; *i.e.*, it cuts three bars, two of which meet at a point, and the other is the bar a. If that portion of the truss to the left of the section is now considered, it is evident that it will be held in equilibrium by the outer forces and the stresses in bars a, b, and c and that the moment of the outer forces and the bar stresses about any point in the plane of the truss must equal zero. If moments are taken about the intersection of b and c, the moment of the stresses in these bars will be zero; hence, the equation of moments will include only the known outer forces and the

unknown stress in bar a, which can, in consequence, be readily computed. The equation will be as below, stress in a being assumed as tension.

$$(60 - 10)45 - 20 \times (15 + 30) - 30S_a = 0$$

Therefore,

$$S_a = +45$$

Bar b. The stress in bar b may be found by taking the origin of moments at the intersection of bar a and bar c, produced, using the same section as for bar a.

Bar d. In both the cases previously considered the section XY was assumed to be vertical or nearly so; this, however, is not necessary, and a horizontal or inclined section may be used provided that the bars cut by it fulfill the stated conditions. For example, the stress in bar d may be computed by this method, by using the section ZQ and taking moments about the apex of the truss of the forces above the section. The following equation is obtained for this case:

$$20 \times 15 + Sd \times 18 = 0$$

whence

$$Sd = -20 \times {}^{15}\!/_{18} = -16\tfrac{2}{3}$$

Bar e. The stress in bar e may be readily solved by applying $\Sigma M = 0$ about an origin at the apex of the truss of the forces applied to the portion of the truss above section RS.

It should be observed that the method of moments is inapplicable to the determination of the stresses in the web members of a parallel-chord truss since in such trusses the origin of moments for the web member stress would be at infinity and the equations would be indeterminate.

94. Method of Shear Described.—This is another special method that can often be used to great advantage in the determination of the stresses in certain bars and particularly in diagonals of parallel-chord trusses. In the truss shown in Fig. 125, it is clear that, if the stresses in all the bars are axial, the resultant forces perpendicular to the chords on either side of section ZQ must be carried entirely by the diagonal U_1L_2, and the application of the equation $\Sigma Y = 0$ to all the forces on the portion of the truss to the left of the section gives at once an equation between

the component in U_1L_2 parallel to the axis OY and the corresponding component of the outer forces. This is illustrated by Fig. 126, in which the application of $\Sigma Y = 0$ gives the following equation:

$$\frac{P}{2} - y_3 = 0$$

whence

$$y_3 = \frac{P}{2}$$

95. Mode of Procedure. Method of Shear.—The treatment of the previous article shows the correctness of the following rules:

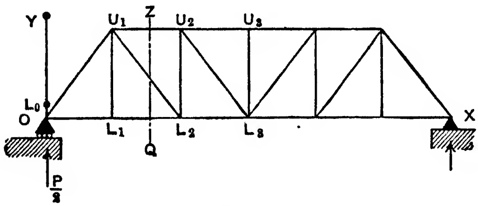


FIG. 125.

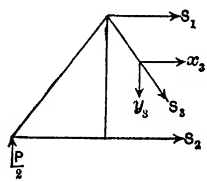


FIG. 126.

1. Divide the truss into two parts by a section passing through the bar in question. This section may cut any number of bars provided that all are parallel except the one under consideration, but in general it should be so chosen as to cut not more than three bars.

2. Refer the forces to two axes, parallel and perpendicular, respectively, to the parallel bars cut by the section. Let the axis perpendicular to these bars be known as the Y-axis. Determine the Y components of all the outer forces acting on that portion of the truss that has the fewer[1] outer forces acting on it, and apply $\Sigma Y = 0$. The equation should include the Y components of *all* the outer forces acting upon the portion of the truss selected, and the Y component of the unknown bar stresses which should be assumed as tension.

3. Solve the equation thus obtained for the unknown Y component. A positive result shows that the stress in the bar is tension.

[1] The side with the fewer forces acting on it is taken to save labor. The final result would be the same if the other side had been taken.

In most bridge trusses, these conditions involve merely the application of $\Sigma V = 0$ to the portion of the truss considered; *i.e.*, the shear on the section equals the vertical component of the stress in the given diagonal, and hence this method is ordinarily called the *method of shear*.

96. Application of Method of Shear.—The following example clearly illustrates the application of this method to the determination of the stresses in the web members of the simple bridge truss, with horizontal chords and carrying a uniform dead and live load, shown in Fig. 127.

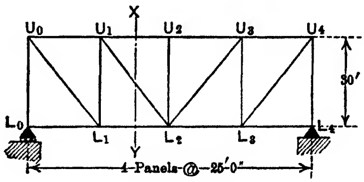


FIG. 127.

Let the dead load be taken as 1,000 lb. per foot, all on the bottom chord, and the live load as 2,000 lb. per foot, also on the bottom chord. The panel loads will then be 25,000 lb. dead and 50,000 lb. live, and the positive dead shear on the section XY will be 12,500 lb. To get the maximum live shear on XY, assume full panel loads at panel points L_2 and L_3 and no load at panel point L_1. This gives a live shear in the panel of +37,500 lb. The vertical component in the bar U_1L_2 is therefore +12,500 lb. dead and +37,500 lb. live. With the vertical component known the actual stress can be easily computed. The forces acting upon that portion of the truss to the left of the section will be as shown in Fig. 127*a*, from which it is readily seen that, by assuming the stress in the diagonal to be tension and applying the equation $\Sigma V = 0$, V_1 will be found to have a positive value.

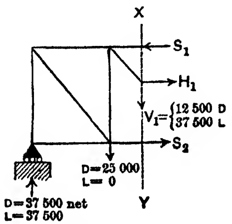


FIG. 127*a*.

Had the truss been inclined instead of horizontal, the proper course to pursue would have been to resolve the vertical forces into normal and tangential components and apply $\Sigma Y = 0$ to the normal forces.

97. General Rules for Determination of Truss Stresses.—The student should note carefully that the two latter methods, *viz.*, the method of moments and the method of shears, are both methods of sections and that in their application it is always necessary to assume a section through the truss and write an

equation of equilibrium between the outer forces acting upon the truss on one side of the section and the forces in the bars cut by the section; the method of joints is also a method of sections, the section in each case being taken around the joint. In computing a bar by analytical methods the first step is to determine the method to use. It should next be decided where to take the section and what portion of the truss to consider. Finally the proper equations should be applied between all the outer forces acting on the portion selected and the stresses in the bars cut.

A combination of the three methods that have been explained, joints, moments, and shears, enables us to compute readily the stress in any or all members of a statically determined truss. In order, however, to figure the stresses in the simplest manner, it may be necessary to study with considerable care some of the members, in order to determine which method should be adopted. In bridges, however, the forms of trusses that are in common use for simple spans are not numerous, and the best methods to adopt can be readily learned by the study of conventional types. For roof trusses the graphical method of joints will usually be found most convenient especially if supplemented by computing the stresses in certain bars by one of the analytical methods.

98. Counters.—In pin trusses the main diagonals usually are flat eyebars which can carry little or no compression and which are so placed in the truss as to be in tension under the dead load. Certain positions of the live load will, however, always tend to produce compression in some of the diagonals. This frequently overbalances the dead tension, especially when impact is added. Such is usually the case in panels near the center of a bridge truss where the dead stresses in the diagonals are small and the live stresses proportionally great. To prevent danger of collapse when this occurs, it is necessary either to make the main diagonals of such a shape that they will carry compression or else to relieve them by auxiliary diagonals, called *counters*, which are so placed that they will be brought into tension by that loading which would tend to put the main diagonal into compression. This latter method is the common practice, although in recent years recognition of the importance of rigidity as well as of strength in railroad bridges has induced many engineers to use the former method, even at the sacrifice of simpler and less expensive details.

For trusses such as the Howe truss, described in the following article, in which the main diagonals are compression members but are sometimes unsuited on account of end details to transmit tension, counters are needed to resist tension rather than compression. In recent years the development of metal connecting devices for wooden trusses has made it possible to transmit tension into the diagonals, and counters may be omitted for such trusses.

With riveted trusses, it is usually desirable to make all the web members of such shapes that they can carry both tension and compression, and the question of counters does not arise.

To determine whether counters are needed, the live loads should be placed in the position consistent with maximum compression in each diagonal, or in the case of the Howe truss with maximum tension, beginning with that in the panel nearest the center and proceeding toward the end, and the live stress in the diagonal computed. If this stress when combined with a reasonable allowance for impact equals or exceeds the dead stress of the opposite character in the bar, a counter is needed. It is wise to use a high allowance for impact in such a case, as the consequence of an increase in the live loads sufficient to overbalance the dead stress would be more serious here than for a bar where such an increase would tend merely to increase the unit stress in the member.

In trusses without counters it is necessary to make similar computations; for if reversal of stress occurs in a bar, it should be designed with a lower unit stress than would otherwise be adopted, at least, if the reversal of stress occurs suddenly and frequently, as in a railroad bridge.

The reason for beginning at the center and working toward the end in making these computations is to save labor. The ratio between the maximum live stress and the dead stress in the web members is always greater at the center (*i.e.*, for the ordinary end-supported truss) and grows less near the end. In consequence, after the panel in which a counter has first been found unnecessary is reached, no further investigation is required.

Illustrations of the computations required to determine whether or not counters are needed are given in examples that follow. It may be helpful, however, to state here that, in the ordinary parallel-chord end-supported truss, counters are needed wherever

the negative live shear plus impact equals or exceeds the positive dead shear.

99. Types of Truss.—The forms of simple bridge truss most frequently adopted are shown by Figs. 128 to 134.

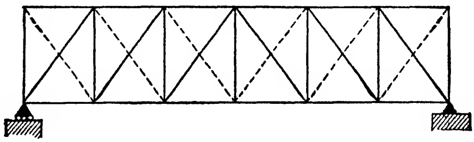


FIG. 128.—Howe truss.

The Howe truss is usually built with chords, diagonals and end verticals of wood, and intermediate verticals of steel. Stresses in diagonals will be compression and in intermediate verticals tension. The diagonal members shown by the dotted lines are

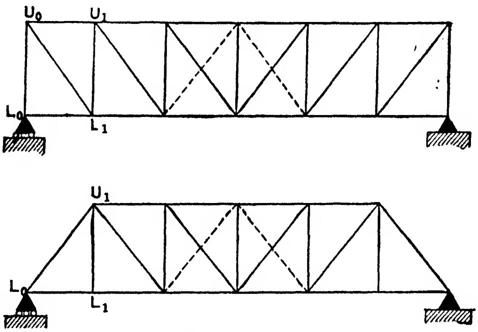


FIG. 129.—Pratt truss

counters. Often in such trusses a counter is used in every panel though not needed to carry shear, its size being made one-half that of the main diagonal; *i.e.*, if two equal sticks are used for the main diagonal, the counter would be made one stick of the same size.

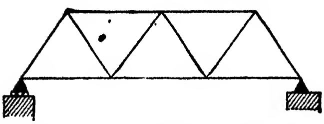


FIG. 130.—Warren truss.

The Pratt truss with its various modifications is the most common type of bridge truss. It is usually built of steel and has tension diagonals and compression verticals. The truss with end verticals shown in the upper portion of Fig. 129 is not commonly employed for through bridges since it is less economical of material than the other form in which the compression members L_0U_0 and U_0U_1 and the tension member U_0L_1 are

replaced by the one compression member L_0U_1. Counters are shown dotted and may be required in more panels than those shown.

The Warren truss is very commonly adopted for riveted trusses. No counters are used, and the diagonals in panels where negative shears occur are compression members. It is evident, also, from the arrangement of the diagonals that every

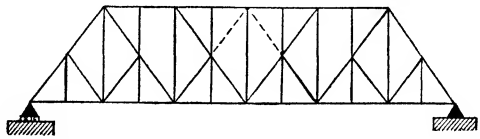


FIG. 131.—Subdivided Pratt truss commonly known as the Baltimore truss.

other one would, in any case, have to be a compression member to withstand the positive shear.

In all steel bridges, in order to obtain economy of material, it is essential that the ratio of depth of truss to length of span should be within certain limits, approximately $\frac{1}{5}$ to $\frac{1}{8}$, and that the diagonals should make angles of approximately 45° with the horizontal. To obtain both of these results, it is clear that the panel length would have to vary with the span. As it is undesir-

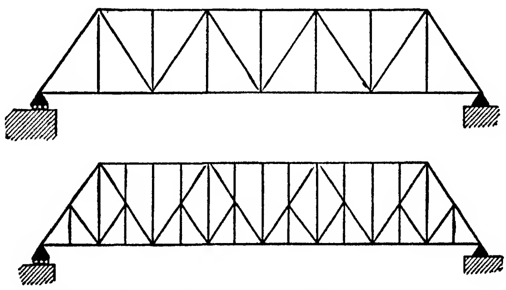


FIG. 132.—Subdivided Warren trusses.

able to use very long panels on account of the bending stresses produced in the chord bars by their own weight, and because also of the increase in weight per foot of the stringers as their span increases, the panel length is seldom made in excess of 35 ft., though in some spans of recent construction panel lengths greater than this have been used. In order to obtain panels of reasonable length in long spans, trusses are sometimes subdivided by a secondary system, as indicated in Figs. 131 and 132, though

such subdivision may cause, in some of the members, secondary stresses of considerable magnitude and should generally be avoided for this reason and for the unpleasant effect of so many members.

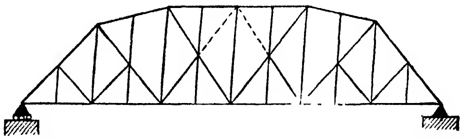


FIG. 133.—Subdivided Pratt truss with inclined top chord commonly known as the Petit truss.

For long spans, it is usually economical to make the truss deeper at the center than at the ends. If the depth is increased in proportion to the increase in moment, it is evident that the

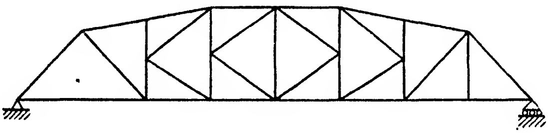


FIG. 134.—K truss.

chord stresses would remain essentially constant throughout the entire length of the span and that the chords would, in consequence, be much lighter at the center than if the end depth

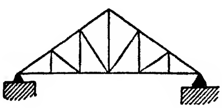


FIG. 135.

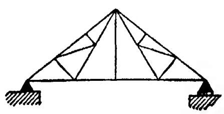


FIG. 136.

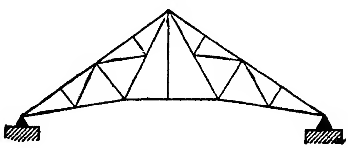


FIG. 137.

Common Types of Roof Trusses.

were to be continued throughout the span. The stresses in the diagonals would be increased in such a case, but the net result would be a saving of material; hence, if minimum weight alone

were to be the governing element, it would be desirable to make all trusses of varying height. It is necessary, however, to consider, also, economy of labor. Since trusses of varying depth are more expensive of labor, it is evident that they should be used only for structures in which the saving of cost due to reduction of weight balances or exceeds the increased cost of construction. This point is usually reached only in spans of considerable length, say 200 ft. and over. The types of truss often used in such spans are shown by Figs. 133 and 134.

Roof trusses are necessarily made of many forms to suit the varying shapes of buildings. Figures 135, 136, and 137 illustrate only a few of the more usual forms.

Figure 135 shows a common type of roof truss which is built of steel or of wood with steel verticals. It has no special name but is of the Pratt truss type. Figures 136 and 137 are Fink roof trusses.

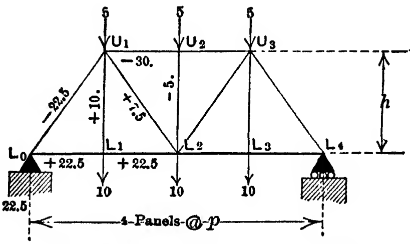
100. Systems of Loading.—In the computation of bridge-truss stresses, it is desirable to combine the various methods given in the preceding articles. Methods of doing this for the more common types of truss and for simple loadings are clearly shown in the numerical examples that follow. As it is the writer's purpose in this chapter to lay particular emphasis upon truss action rather than upon the consideration of moving concentrated load systems, which have been thoroughly treated in Chap. III, the live loading used in most of the examples is taken as a uniform load with a locomotive excess, *i.e.*, with a single concentrated live load which may be applied at any panel point. The magnitude of the locomotive excess load equals the difference between the maximum floor-beam load due to the actual locomotive and the floor-beam load due to the uniform load. The process of finding the maximum live stress in a member with such a loading consists of computing the maximum stress due to the uniform load and adding to it the maximum stress due to the locomotive excess. The dead load is also treated as a uniform load, this being nearly correct for trusses of ordinary span. For trusses of great length or of unusual weight, it is advisable to estimate the actual dead weight acting at each panel point.

It should be remarked that for parallel-chord trusses the determination of the live stresses due to concentrated load systems involves merely the computation of the maximum

moment at each panel point and the maximum shear in each panel. From the moments the chord stresses may be figured by the method of moments and from the shears the web stresses by the method of shears. If the student thoroughly understands truss action, as illustrated in the examples that follow, and the method of using concentrated load systems, he should have no difficulty whatsoever in the computation of trusses under concentrated load systems.

The fact that the locomotive-excess method is used for the determination of truss stresses should not be considered as indicative of the writer's belief that such a method is sufficiently precise for actual use in design. It is used here merely because of its value in illustrating truss action without complicating the theory with unnecessary computations.

101. Index Stresses.—For many bridge trusses the dead stresses and the stresses under full uniform live load can be most readily obtained by a special application of the method of



Loads shown are dead panel loads.

FIG. 138.

joints, involving the use of so-called *index stresses.* A clear understanding of the method of obtaining these index stresses and of what they signify may be gained from a study of the following example:

Let it be desired to determine the dead stresses in the simple truss shown in Fig. 138.

It is evident that the stress in U_2L_2 may be determined by the method of joints, the joint at U_2 being used, and that its value is -5.

Since the truss and loads are *symmetrical,* the stresses in U_1L_2 and L_2U_3 are equal; hence, the vertical component in each may be found by considering joint L_2. Its value $= +\frac{1}{2}(5 + 10) = +7.5$. The stress in U_1L_1 is found to equal $+10$, the joint at the

bottom of the member being used, and the vertical component in $U_1L_0 = -(5 + 10 + 7.5) = -22.5$, considering the joint at U_1.

These vertical components of the web stresses are the web index stresses and may be written directly on the truss diagram and the dead stresses computed from them by the slide rule with great rapidity.

It should be noticed that the vertical component in L_0U_1 equals the left reaction (*i.e.*, the net reaction, the panel load at L_0 being neglected) and that a very good check upon the web stresses is thereby obtained.

Had the truss or the loads been *unsymmetrical*, it would have been desirable to start by writing first the vertical component

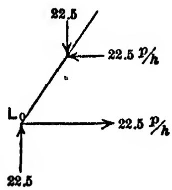


FIG. 139.

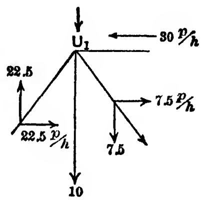


FIG. 140.

in L_0U_1 and proceeding thence to the right end of the truss, checking there with the right reaction. For symmetrical structures symmetrically loaded, it is better to begin at the center and work toward the end, as illustrated by the previous example.

To obtain the chord stresses, begin at L_0. The conditions necessary for equilibrium at that point are shown by Fig. 139. The actual stress in L_0L_1 is found to be $+22.5\frac{p}{h}$.

The condition at joint U_1 is shown by Fig. 140, and the stress in U_1U_2 is found to equal $30\frac{p}{h}$.

The numerical factors of the chord stresses are called the *chord index stresses.* For the truss in question, it is evident that their determination requires merely the progressive addition, joint by joint, of the index stresses in the diagonals of the web system and that this would be the case in all simple equal-paneled trusses of the Howe, Warren, or Pratt types. For the subdivided trusses of the Baltimore or Petit type the effect of the secondary diagonals must be carefully considered, since the index stresses in these must sometimes be added and sometimes be subtracted to obtain the chord index stresses.

It should be observed that in the case of a parallel-chord equal-paneled truss of height equal to panel length the chord index stresses equal the actual chord stresses and that for other heights the latter vary inversely as the truss height.

For trusses in which the diagonals do not all have the same slope the web index stresses must all be reduced to a standard slope before writing the chord index stresses. The method of doing this is fully explained later in the article on trusses with nonparallel chords and will not be given here.

The chord index stress in a bar adjoining the center panel point, or in the center bar, if the truss has an uneven number of panels should be verified by comparing the actual stress as obtained from it with that obtained in the same bar by the method of moments, the formula $\frac{1}{8}wL^2$ being used for the center moment if the load per foot is uniform. If the two results agree, it is evident that not only the index stress in this bar will be checked, but also the index stresses in all the other members of the same chord. Moreover, the index stresses in the members of the other chord may be verified so easily by comparison with those in the chord already checked that no excuse need exist for errors.

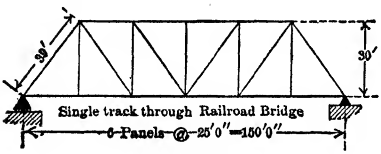


FIG. 141.

This method is so advantageous, from the standpoint both of accuracy and of rapidity, that it should invariably be used for simple bridge trusses. The numerous examples that follow illustrate it fully and should be carefully studied.

102. Computation of Stresses. Pratt Truss. *Uniform Load with Locomotive Excess.*

Problem: Determine the maximum stresses in all the members of the truss shown in Fig. 141 with the following loads:

Dead weight of bridge,

400 lb. per ft. per truss, top chord = 10,000 lb. per panel

1,000 lb. per ft. per truss, bottom chord = 25,000 lb. per panel

Uniform live load.

3,000 lb. per ft. per truss, bottom chord = 75,000 lb. per panel

Locomotive excess,[1] bottom chord = 40,000 lb. per panel

[1] The locomotive excess is computed for Cooper's E_{60} loading (see Fig. 11). For this loading the maximum floor-beam load for 25-ft. panels occurs

Index Stresses.—These are shown for the dead loads in units of 1,000 lb. in Fig. 142 and can be written directly on the diagram, the necessary computations being done mentally. The vertical component in L_0U_1 equals the net left reaction, as should be the case.

The stress in U_2U_3 as found from the index stress

$$= 157.5 \times 25/30 = 131.25 \text{ thousands of lb.}$$

The stress in the same bar by the method of moments,

$$= \tfrac{1}{8}(1{,}000 + 400)\frac{150^2}{30} = 131{,}250 \text{ lb.}$$

This agrees with the value found from the index stress and hence checks that stress and all the other index stresses involved in its determination.

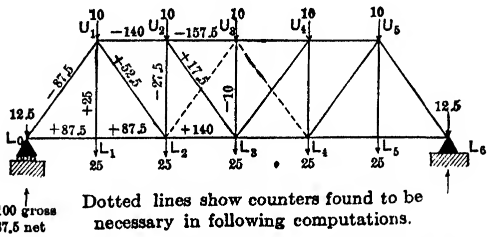


FIG. 142.—Dead panel loads and index stresses. Single-track through Pratt truss railroad bridge.

The index stress in chord L_2L_3 plus the index stress in U_2L_3 equals the index stress in U_2U_3, as it should; hence, the index stress in L_2L_3 is also correct.

Position of Loads for Maximum Live Stresses.—Since the truss is symmetrical, the maximum stresses need be determined in the bars of one-half of the truss only. The maximum live stress in any one of the chord bars occurs for the loading giving the maximum moment about some panel point, since the method of moments may be used for each of these bars, the origin in every case being at a panel point. It follows from this that for maximum chord stresses the uniform live load should extend

with load 4 at the floor beam and equals 227,000 lb. The corresponding load for the uniform live load = 6,000 × 25 = 150,000 lb. The total excess is, therefore, 227,000 − 150,000 = 77,000 lb. The excess per truss = 77,000/2 = 38,500, or, say 40,000 lb.

over the entire structure since only under this condition will the moment at a panel point be a maximum. In fact, it may be stated as a general rule that *the maximum chord stresses due to a uniform load in any end-supported truss will occur only when the uniform load extends over the entire span.*

It is evident, therefore, that the maximum chord stresses due to the uniform live load may be obtained directly from the dead stresses by multiplying the latter by the ratio between the live panel load and the combined dead panel loads on top and bottom chords. If desired, however, the index stresses may be written for the uniform live load just as for the dead load and the actual stresses computed independently.

The maximum chord stresses due to the locomotive excess should be determined by the method of moments and will evidently, in this form of truss, occur with the excess load located in a vertical line through the panel point which is the origin of moments for the bar in question. Its position for the various bars will be as follows:

Bars L_0L_1 and L_1L_2, E at L_1.

Bars U_1U_2 and L_2L_3, E at L_2.

Bar U_2U_3, E at L_3.

For the web stresses the uniform live load and locomotive excess should generally be so placed as to give maximum shear in the various panels. The only exception to this for the truss in question is that bar U_1L_1 has its maximum stress with a full panel load at L_1. It should be noticed that the stress in a vertical like U_2L_2 will be a maximum when the stress in the diagonal U_2L_3 is a maximum, since by the application of the method of joints it is evident that the stress in U_2L_2 equals the vertical component in U_2L_3. Were this condition not to exist, as would be the case if the live load should be distributed between the top and bottom chords, then the position of loads for maximum stress in U_2L_2 would have to be that which would give maximum shear in a diagonal section through U_1U_2 and L_2L_3. It should also be noticed that live stress in bar U_3L_3, if the load is applied to the bottom chord, occurs only when a counter is in action.

The position of loads for maximum stresses in the various bars will now be given, it being understood that the shear due to uniform live load will be treated by the approximate method hitherto used.

Bar L_0U_1, uniform live load over entire structure, E at L_1.

Bar U_1L_1, full uniform live panel load at L_1, E at L_1.

Bar U_1L_2, uniform load from right up to and including L_2, E at L_2.

Bars U_2L_3 and U_2L_2, uniform load from right up to and including L_3, E at L_3.

Bars U_3L_3 and U_3L_4, (counter) uniform load from right up to and including L_4, E at L_4.

Maximum Stresses.—The actual stresses may now be computed. These are given with all the necessary computations in the table on the following page.

It should be noted that it is simpler to determine the vertical components in all the diagonal bars before determining the actual stresses, particularly if the slide rule is used.

In addition to the bar stresses the maximum truss reactions must be determined. These occur for full loading, but their values depend upon whether an end floor beam is used. If an end floor beam is not used, the reaction equals the maximum shear in the end panel, the expression for the value of which has already been found in determining the stress in L_0U_1. If an end floor beam is used, the locomotive excess should be placed at L_0 and the end reaction determined accordingly.

103. Computation of Stresses. Warren Truss. *Uniform Load with Locomotive Excess.*

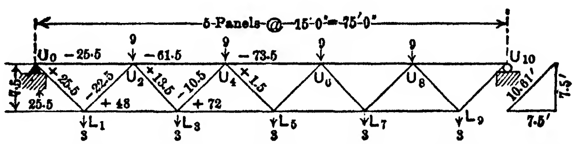


FIG. 143.—Dead panel loads and index stresses. Single-track deck Warren truss railroad bridge.

Problem: Determine the maximum stresses of both kinds for all the bars of the truss shown in Fig. 143 with the following loads:

Dead weight of bridge,		
600 lb. per ft. per truss, top chord	=	9,000 lb. per panel
200 lb. per ft. per truss, bottom chord	=	3,000 lb. per panel
Uniform live load,		
2,000 lb. per ft. per truss, top chord	=	30,000 lb. per panel
Locomotive excess, top chord	=	25,000 lb. per panel

MAXIMUM STRESSES FOR PRATT TRUSS IN ART. 102, UNITS OF 1,000 POUNDS

Bar	Index stress	Multiplier	Dead stress	Live stress due to uniform load	Live stress due to locomotive excess	Total live stress
L_0L_1	+ 87.5	$^{25}\!/_{30}$	+ 72.9	$+72.9 \times {}^{75}\!/_{35} = +156.2$	$+\frac{5}{6} \times 40 \times {}^{25}\!/_{30} = +27.8$	+184.0
L_1L_2	+ 87.5	$^{25}\!/_{30}$	+ 72.9	$+72.9 \times {}^{75}\!/_{35} = +156.2$	$+\frac{5}{6} \times 40 \times {}^{25}\!/_{30} = +27.8$	+184.0
U_1U_2	−140.0	$^{25}\!/_{30}$	−116.7	$-116.7 \times {}^{75}\!/_{35} = -250.0$	$-\frac{4}{6} \times 40 \times {}^{50}\!/_{30} = -44.4$	−294.4
L_2L_3	+140.0	$^{25}\!/_{30}$	+116.7	$+116.7 \times {}^{75}\!/_{35} = +250.0$	$+\frac{4}{6} \times 40 \times {}^{50}\!/_{30} = +44.4$	+294.4
U_2U_3	−157.5	$^{25}\!/_{30}$	−131.2	$-131.2 \times {}^{75}\!/_{35} = -281.1$	$-\frac{3}{6} \times 40 \times {}^{75}\!/_{30} = -50.0$	−331.1
L_0U_1	− 87.5	$^{39}\!/_{30}$	−113.8	$-113.8 \times {}^{75}\!/_{35} = -243.9$	$-\frac{5}{6} \times 40 \times {}^{39}\!/_{30} = -43.3$	−287.2
U_1L_2	+ 52.5	$^{39}\!/_{30}$	+ 68.3	$+\frac{10}{6}75 \times {}^{39}\!/_{30} = +162.5$	$+\frac{4}{6} \times 40 \times {}^{39}\!/_{30} = +34.7$	+197.2
U_2L_3	+ 17.5	$^{39}\!/_{30}$	+ 22.8	$+\frac{6}{6}75 \times {}^{39}\!/_{30} = +97.5$	$+\frac{3}{6} \times 40 \times {}^{39}\!/_{30} = +26.0$	+123.5
U_1L_1	+ 25.0	1.0	+ 25.0	$+75.0 \times 1.0 = +75.0$	$= +40.0$	+115.0
U_2L_2	− 27.5	1.0	− 27.5	$-\frac{6}{6}75 \times 1.0 = -75.0$	$-\frac{3}{6} \times 40 = -20.0$	− 95.0
U_3L_3	− 10.0	1.0	− 10.0	$-\frac{3}{6}75 \times 1.0 = -37.5$	$-\frac{2}{6} \times 40 = -13.3$	− 50.8
U_3L_4	− 17.5	$^{39}\!/_{30}$	− 22.8	$+\frac{3}{6}75 \times {}^{39}\!/_{30} = +48.8$	$+\frac{2}{6} \times 40 \times {}^{39}\!/_{30} = +17.3$	+ 66.1
	Counter not in action under dead load only					

The bottom panel loads at L_1 and L_9 would really be somewhat less than shown, but they are taken the same as the other bottom panel loads for convenience.

Index Stresses.—These and the panel loads are shown in Fig. 143 in units of 1,000 lb. The net reaction at the left end evidently equals 25.5 which checks the index stress in U_0L_1.

To check the index stress in the center member of the top chord, moments should be taken about L_5. Since this is not a panel point for both chords, this moment does not equal $\frac{1}{8}pL^2$ but may be found from the moment of the reaction minus the

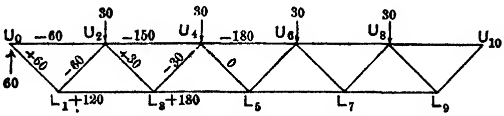


FIG. 144.—Panel loads and index stresses. Full live load.

moment of the panel loads. By this method the stress in the center top-chord bar equals

$$\left(25.5 \times 2\frac{1}{2} - 3 \times 3 - 9 \times 2\right)\frac{15}{7.5} = 73.5$$

Since the diagonals make an angle of 45° with the horizontal, the chord index stress equals the actual stress and is therefore correct.

The index stress in the center member of the bottom chord plus the index stress in U_4L_5 = 73.5 = the index stress in the center member of the top chord which is therefore seen to be correct.

For this truss the live stresses cannot be computed from the dead index stresses since the bottom-chord joints are not directly under the top-chord joints; as the chord stresses for the uniform live load have maximum values for full loading, they can, however, be determined by the method of index stresses, and Fig. 144 shows these stresses for a full uniform live load.

The moment at U_4 for this case equals $(\frac{24}{25}) \times (\frac{1}{8} \times wL^2)$, the method of Art. 43 being used; hence, the tension in bar L_3L_5 in thousands of pounds

$$= \frac{24}{25} \times \frac{1}{8} \times 2 \times \frac{75 \times 75}{7.5} = 180$$

This equals the index stress in this member and also in U_4U_6, as should be the case, since the index stress in diagonal U_4L_5 = 0.

Position of Loads for Maximum Live Stresses.

U_0L_1 and L_1U_2, full uniform load with E at U_2.

U_2L_3 and L_3U_4, uniform load from right up to and including U_4, E at U_4.

U_4L_5, uniform load from right up to and including U_6, E at U_6.

L_5U_6 (= maximum compression in U_4L_5), uniform load from right up to and including U_6, E at U_6.

U_6L_7 (= maximum tension in U_4L_3), full panel load and locomotive excess at U_8.

U_0U_2 and L_1L_3, full uniform load, E at U_2.

U_2U_4 and L_3L_5, full uniform load, E at U_4.

U_4U_6, full uniform load, E at U_4 or U_6.

Maximum Stresses.—These may now be computed and are given in 1,000-lb. units in the following table in which all the necessary computations are shown:

MAXIMUM STRESSES—WARREN TRUSS

Bar	Dead stress, 1,000-lb. units	Maximum live stress, 1,000-lb. units
U_0U_2	-25.5	$-(60 + \frac{4}{5}25) = -80.0$
U_2U_4	-61.5	$-(150 + \frac{3}{5}25 \times 3) = -195.0$
U_4U_6	-73.5	$-(180 + \frac{2}{5}25 \times 5) = -230.0$
L_1L_3	$+48.0$	$+(120 + \frac{4}{5}25 \times 2) = +160.0$
L_3L_5	$+72.0$	$+(180 + \frac{3}{5}25 \times 4) = +240.0$
U_0L_1	$+25.5 \times \frac{10.61}{7.5} = +36.1$	$+\left(60 + \frac{4}{5}25\right)\frac{10.61}{7.5} = +113.1$
L_1U_2	$-22.5 \times \frac{10.61}{7.5} = -31.8$	$-\left(60 + \frac{4}{5}25\right)\frac{10.61}{7.5} = -113.1$
U_2L_3	$+13.5 \times \frac{10.61}{7.5} = +19.1$	$+\left(\frac{6}{5}30 + \frac{3}{5}25\right)\frac{10.61}{7.5} = +72.1$
L_3U_4	$-10.5 \times \frac{10.61}{7.5} = -14.8$	$-\left(\frac{6}{5}30 + \frac{3}{5}25\right)\frac{10.61}{7.5} = -72.1$
U_4L_5	$+1.5 \times \frac{10.61}{7.5} = +2.1$	$+\left(\frac{3}{5}30 + \frac{2}{5}25\right)\frac{10.61}{7.5} = +39.6$
L_5U_6*	Same as U_4L_5 $= +2.1$	$-\left(\frac{3}{5}30 + \frac{2}{5}25\right)\frac{10.61}{7.5} = -39.6$
U_6L_7*	Same as U_4L_3 $= -14.8$	$+\frac{1}{5}(30 + 25)\frac{10.61}{7.5} = +15.6$

* In this truss no counters are used; hence, it is necessary to compute the maximum stresses of both kinds in all diagonals in which the live stress may tend to reverse the dead stress. This is easily done in the manner shown above.

104. Computation of Stresses. Subdivided Warren Truss. *Uniform Load with Locomotive Excess.*

Problem: Determine the maximum stresses of both kinds in all the bars of the truss shown in Fig. 145 with the following loads:

Dead weight of bridge,
1,000 lb. per ft. per truss, top chord = 25,000 lb. per panel
500 lb. per ft. per truss, bottom chord = 12,500 lb. per panel
Uniform live load,
2,000 lb. per ft. per truss, top chord = 50,000 lb. per panel
Locomotive excess = 30,000 lb.

Index Stresses.—These are shown in Fig. 145. Their computation involves no difficulty.

To check the index stress in U_4U_5, use the method of moments as follows:

$$\text{Stress in } U_4U_5 = \frac{1}{8} \times 1{,}500\frac{250 \cdot 250}{30} = 390{,}625 \text{ lb.}$$

From the index stress the actual stress in this bar = 468,750 × $^{25}\!/_{30}$ = 390,625; hence, index stresses are correct.

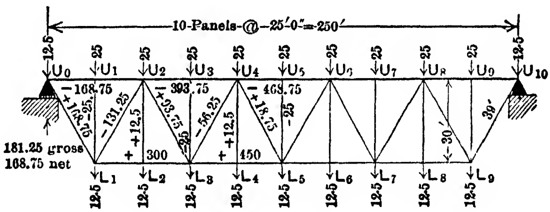


FIG. 145.—Dead panel loads and index stresses. Single-track deck subdivided Warren truss railroad bridge.

Position of Loads for Maximum Live Stresses.—For the chords the uniform live load should extend over the entire span and the stresses due to it may be computed directly from the dead stresses. The locomotive excess should be placed as follows:

Bars U_0U_1 and U_1U_2,	E at U_1.	Bars U_2U_3 and U_3U_4,	E at U_3
Bars L_1L_2 and L_2L_3,	E at U_2.	Bars L_3L_4 and L_4L_5,	E at U_4.
		Bar U_4U_5,	E at U_5.

For the diagonals the uniform load and locomotive excess should be placed to give maximum shear in the different panels, *i.e.*, full uniform live panel loads to the right of the panel containing the bar in question, and the locomotive excess at the nearest panel point to the right. For the verticals, it is evident that the

MAXIMUM STRESSES—SUBDIVIDED WARREN TRUSS

Bars	Index stress	Multiplier	Dead stress, 1,000-lb. units	Live stress due to uniform load, 1,000-lb. units	Live stress due to locomotive excess, 1,000-lb. units	Total live stress, 1,000-lb. units
U_0U_1, U_1U_2	−168.75	$\frac{25}{30}$	−140.6	$-140.6 \times \frac{50}{37.5} = -187.5$	$-\frac{9}{10} \times 30 \times \frac{25}{30} = -22.5$	−210.0
U_2U_3, U_3U_4	−393.75	$\frac{25}{30}$	−328.1	$-328.1 \times \frac{50}{37.5} = -437.5$	$-\frac{7}{10} \times 30 \times \frac{75}{30} = -52.5$	−490.0
U_4U_5	−468.75	$\frac{25}{30}$	−390.6	$-390.6 \times \frac{50}{37.5} = -520.8$	$-\frac{5}{10} \times 30 \times \frac{125}{30} = -62.5$	−583.3
L_1L_2, L_2L_3	+300.00	$\frac{25}{30}$	+250.0	$+250.0 \times \frac{50}{37.5} = +333.3$	$+\frac{8}{10} \times 30 \times \frac{50}{30} = +40.0$	+373.3
L_3L_4, L_4L_5	+450	$\frac{25}{30}$	+375.0	$+375.0 \times \frac{50}{37.5} = +500.0$	$+\frac{6}{10} \times 30 \times \frac{100}{30} = +60.0$	+560.0
U_0L_1	+168.75	$\frac{39}{30}$	+219.4	$+219.4 \times \frac{50}{37.5} = +292.5$	$+\frac{9}{10} \times 30 \times \frac{39}{30} = +35.1$	+327.6
L_1U_2	−131.25	$\frac{39}{30}$	−170.6	$-50 \times \frac{36}{10} \times \frac{39}{30} = -234.0$	$-\frac{8}{10} \times 30 \times \frac{39}{30} = -31.2$	−265.2
U_2L_3	+ 93.75	$\frac{39}{30}$	+121.9	$+50 \times \frac{28}{10} \times \frac{39}{30} = +182.0$	$+\frac{7}{10} \times 30 \times \frac{39}{30} = +27.3$	+209.3
L_3U_4	− 56.25	$\frac{39}{30}$	− 73.1	$-50 \times \frac{21}{10} \times \frac{39}{30} = -136.5$	$-\frac{6}{10} \times 30 \times \frac{39}{30} = -23.4$	−159.9
U_4L_5	+ 18.75	$\frac{39}{30}$	+ 24.4	$+50 \times \frac{15}{10} \times \frac{39}{30} = +\ 97.5$	$+\frac{5}{10} \times 30 \times \frac{39}{30} = +19.5$	+117.0
L_5U_6*	+ 18.75	$\frac{39}{30}$	+ 24.4	$-50 \times \frac{10}{10} \times \frac{39}{30} = -\ 65.0$	$-\frac{4}{10} \times 30 \times \frac{39}{30} = -15.6$	− 80.6
U_6L_7*	− 56.25	$\frac{39}{30}$	− 73.1	$+50 \times \frac{6}{10} \times \frac{39}{30} = +\ 39.0$	$+\frac{3}{10} \times 30 \times \frac{39}{30} = +11.7$	+ 50.7
L_7U_8*	+ 93.75	$\frac{39}{30}$	+121.9	$-50 \times \frac{3}{10} \times \frac{39}{30} = -\ 19.5$	$-\frac{2}{10} \times 30 \times \frac{39}{30} = -\ 7.8$	− 27.3
U_1L_1, U_3L_3, U_5L_5	− 25.0	1.0	− 25.0	$= -\ 50.0$	−30.0	− 80.0
U_2L_2, U_4L_4	+ 12.5	1.0	+ 12.5	0.0	0.0	0.0

* The live stresses in these bars equal maximum stresses of the opposite character to those occurring in the corresponding bars in the other half of the truss.

maximum stress in all odd-numbered bars like U_3L_3 will occur with full live panel load and locomotive excess at top panel point, whereas the even-numbered verticals will have no live stress.

Maximum Stresses.—All necessary computations for these, together with the final values, are given in the preceding table in units of 1,000 lb.

105. Computation of Stresses. Bridge Trusses with Nonparallel Chords. *Uniform Load with Locomotive Excess.*—To compute the stresses in such trusses, it is necessary to modify somewhat the procedure adopted in the simple parallel-chord trusses hitherto treated. This is due to the fact that the web stresses can no longer be directly determined by the method of shear, owing to the influence of the inclined top chord. Although the modification is in mode of procedure rather than in principle, it seems desirable to illustrate the necessary computations for such a truss; hence, the following example is given.

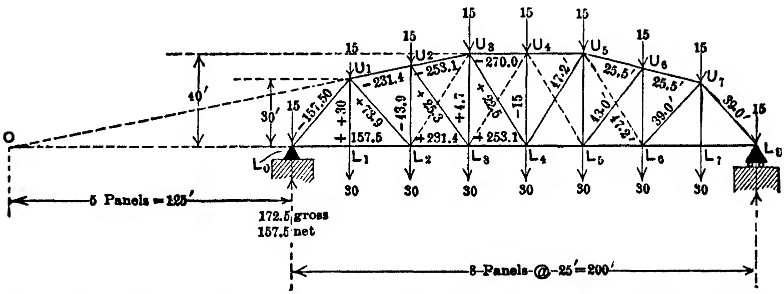


FIG. 146.—Dead panel loads and index stresses. Single-track through nonparallel-chord Pratt truss.

Problem: Compute the maximum stresses for the truss shown in Fig. 146 with the following loads:

Dead weight of bridge,

600 lb. per ft. per truss, top chord	=	15,000 lb. per panel
1,200 lb. per ft. per truss, bottom chord	=	30,000 lb. per panel

Uniform live load,

3,000 lb. per ft. per truss, bottom chord	=	75,000 lb. per panel
Locomotive excess	=	40,000 lb.

The determination of index stresses for this truss requires some explanation. The inclination of top-chord members adds vertical forces at joints U_1, U_2, and U_3; hence, the vertical com-

ponents in the inclined chord members must be determined before the index stresses for bars meeting at these joints can be written. In the trusses previously considered, all the diagonals had the same slope, and multiplication of the chord index stresses by the ratio of horizontal to vertical projection of the diagonal gave actual chord stresses. It is obvious that in order to follow this same method in the truss under consideration, some modification must be adopted. The simplest method in this case is to modify the index stresses in bars U_2L_3 and U_3L_4, before writing chord index stresses. The best method of accomplishing this is to multiply the index stress in each of these bars by the inverse ratio between its vertical projection and that of diagonal U_1L_2, *i.e.*, multiply the index stress in U_2L_3 by $6/7$ and multiply that in U_3L_4 by $3/4$.*

The effect of this is to establish the chord index stresses at the values they would have if all the diagonals had the same slope as U_1L_2.

The computation of the vertical components due to dead load in the inclined top-chord members follows:

$$\text{Bar } U_1U_2 - \text{V.C.} = (157.5 \times {}^{50}\!/_{35} - 45 \times {}^{25}\!/_{35})\,{}^{10}\!/_{50} = 38.6$$
$$\text{Bar } U_2U_3 - \text{V.C.} = (157.5 \times {}^{75}\!/_{40} - 45 \times {}^{75}\!/_{40})\,{}^{10}\!/_{50} = 42.2$$

With these known, the vertical components in the web members may be written at once beginning at the center in the usual manner,

* The correctness of this method is illustrated by the following example: Let V_1, V_2, and V_3 be the vertical components in the diagonals of the truss shown in Fig. 147, and H_1, H_2, and H_3 the stresses in the bottom chord. Evidently

$$H_1 = V_1 \times \frac{p}{h}, \qquad H_2 = (V_1 + V_2)\frac{p}{h}$$

and

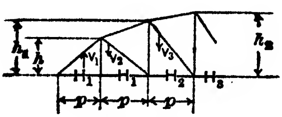


FIG. 147.

$$\begin{aligned} H_3 &= (V_1 + V_2)\frac{p}{h} + V_3 \times \frac{p}{h_1} \\ &= (V_1 + V_2)\frac{p}{h} + V_3\left(\frac{p}{h_1}\right)\frac{h}{h} \\ &= (V_1 + V_2)\frac{p}{h} \times \left(V_3 \times \frac{h}{h_1}\right)\frac{p}{h} \\ &= \left(V_1 + V_2 + V_3 \times \frac{h}{h_1}\right)\frac{p}{h} \end{aligned}$$

For trusses in which the height is constant but the panel length variable the same method may be applied, with the panel lengths substituted for the heights.

a check being obtained at the end where the vertical component in the inclined end post L_0U_1 is found to equal the net dead reaction. It will be noticed that the effect of the vertical component in U_2U_3 is to cause tension in bar U_3L_3. In a parallel-chord Pratt truss, this member, with the other verticals except those at panel point 1, would always be in compression under dead load. From the index stress the dead web stresses can be readily computed. See table on next page.

The corrected values of the diagonal index stresses that are to be used to determine the chord index stresses are as follows:

$$\text{Bar } U_2L_3 = 25.3 \times 6/7 = 21.7$$
$$\text{Bar } U_3L_4 = 22.5 \times 3/4 = 16.9$$

These values should be substituted for the actual diagonal index stresses in determining the chord index stresses. For example, the index stress in Bar $L_3L_4 = 231.4 + 21.7 = 253.1$, and that in $U_3U_4 = 253.1 + 16.9 = 270$.

To obtain a final check of these index stresses, the top-chord dead stress, as computed by the method of moments, should be compared with the value as obtained from the index stresses.

Dead stress in U_3U_4 by method of moments

$$= (1/8 \times 1{,}800 \times 200 \times 200)1/40 = 225{,}000 \text{ lb.}$$

Dead stress in U_3U_4 from index stress

$$= 270 \times 25/30 = 225 \text{ thousands of lb.}$$

Position of Loads for Maximum Live Web Stresses.—The dead stresses and chord and inclined end-post stresses due to uniform live load may be determined directly from the index stresses and will be given later. To determine the *position* of the live loads for maximum web stresses, the method of shear previously used should be replaced by the method of moments. In determining the *actual* stress, once the position of loads is known, the method of shear may be used, provided that the shear is corrected by the amount of the vertical component in the top chord, or the method of moments may be used directly. The individual bars will now be considered.

Bar U_1L_1: Maximum live stress with full load at L_1, E at L_1.

Bar U_1L_2: Place load to give maximum counterclockwise moment about the intersection O of the inclined top chord, prolonged, and the bottom chord. Evidently a load at L_1 will cause a clockwise moment about this origin, since the moment

of the reaction due to a load at L_1 will be less than the moment of the load itself, as its lever arm and magnitude are both less; hence, this point should not be loaded, but all other panel points from the right, up to and including L_2, should be loaded (note that this conclusion would not necessarily be correct for a concentrated load system). The locomotive excess should be placed at L_2.

Bars U_2L_2 and U_2L_3: In this case, load from right, up to and including L_3, with E at L_3, since this condition produces the maximum counterclockwise moment about O.

Bars U_3L_3 and U_3L_4: Load from right, up to and including L_4, with E at L_4.

DEAD WEB STRESSES, UNITS OF 1,000 POUNDS[a]

Bar	Index, stresses	Dead stresses	Bar	Dead stress
U_1L_2	+73.9	$73.9 \times \frac{39}{30} = +96.1$	U_1L_1	+30.0
U_2L_3	+25.3	$25.3 \times \frac{43}{35} = +31.1$	U_2L_2	−43.9
U_3L_4	+22.5	$22.5 \times \frac{47.2}{40} = +26.6$	U_3L_3	+ 4.7
U_4L_5 / U_4L_3	−22.5[b]	$22.5 \times \frac{47.2}{40} = -26.6$	U_4L_4[c]	−15.0
U_5L_6 / U_3L_2	−28.9[d]	$28.9 \times \frac{47.2}{40} = -34.1$		

[a] This table shows all necessary computations once the index stresses are known.

[b] With live load placed to produce maximum stress in U_4L_5 the main diagonal, U_5L_4 will be thrown out of action and dead shear in panel 4–5 will be carried by the counter U_4L_5. This would tend to produce compression in this bar the vertical component of which is 22.5, but this is balanced by some of the live stress; hence, the bar does not actually carry compression, as its sign would seem to indicate. A similar condition exists with bar U_5L_6.

[c] Owing to the counter action, the dead load when truss is loaded to produce maximum live compression in bar U_4L_4 tends to cause in the bar a tension of 22.5 − 15.0 = 7.5. This value should be combined with the live stress to obtain maximum stress.

[d] $28.9 = 157.5 - 90 - \left(\frac{157.5 \times 50 - 45 \times 25}{35}\right)\frac{10}{50}$

Bar U_4L_4 and bar U_4L_5 (counter): Load from right, up to and including L_5, with E at L_5.

Bar U_5L_6 (counter): Load from right, up to and including L_6, with E at L_6.

The tables on pages 186, 188, and 189 show the maximum stresses in all bars with all necessary computations.

For convenience, it is common to write the maximum stresses in a diagram called the *stress diagram,* and Fig. 148 is given to illustrate such a diagram.

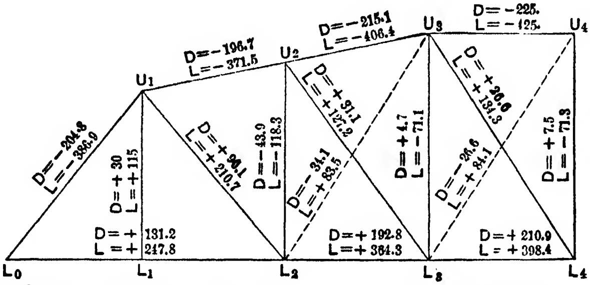


FIG. 148.—Stress diagram, maximum live and dead stresses. Uniform live load with locomotive excess.

106. Computation of Stresses. Bridge Trusses with Nonparallel Chords. *Concentrated Load Systems.*—In the parallel-chord trusses hitherto studied, the stresses due to concentrated load systems were not determined, since this would have involved merely the substitution of maximum shears and moments caused by such loads for the maximum shears and moments due to uniform loads, and the method of determining such shears and moments was discussed with sufficient thoroughness in Chap. III. For nonparallel-chord trusses, however, the conditions differ sufficiently to warrant special consideration; in consequence, the computations for the maximum live stresses due to Cooper's E_{40} loading will be given for the truss shown by Fig. 146. The computations for all the bars will be given, to make the solution complete, although in a number of cases it involves nothing more than the determination of maximum moment or shear. The moment diagram of Fig. 75 is used in all computations, and units are in thousands of pounds.

Position of Loads for Maximum Stresses.—(Panel length $= p.$)

Bars L_0U_1, L_0L_1, and L_1L_2: Maximum stress occurs when shear in panel $0 - 1$ is a maximum.

MAXIMUM CHORD AND INCLINED END-POST STRESSES, UNITS OF 1,000 POUNDS*

Bar	Index stresses	Dead stresses	Live stresses: Uniform load	Live stresses: Locomotive excess	Total maximum live stress
L_0U_1	157.5	$157.5 \times \frac{39}{30} = -204.8$	$204.8 \times \frac{75}{45} = -341.4$	$\frac{7}{8}40 \times \frac{39}{30} = -45.5$	-386.9
U_1U_2	231.4	$231.4 \times \frac{25}{30} \times \frac{25.5}{25} = -196.7$	$196.7 \times \frac{75}{45} = -327.8$	$\frac{3}{4}40 \times \frac{50}{35} \times \frac{25.5}{25} = -43.7$	-371.5
U_2U_3	253.1	$253.1 \times \frac{25}{30} \times \frac{25.5}{25} = -215.1$	$215.1 \times \frac{75}{45} = -358.6$	$\frac{5}{8}40 \times \frac{75}{40} \times \frac{25.5}{25} = -47.8$	-406.4
U_3U_4	270.0	$270 \times \frac{25}{30} = -225.0$	$225 \times \frac{75}{45} = -375.0$	$\frac{1}{2}40 \times \frac{100}{40} = -50.0$	-425.0
L_0L_1 L_1L_2	157.5	$157.5 \times \frac{25}{30} = +131.2$	$131.2 \times \frac{75}{45} = +218.6$	$\frac{7}{8}40 \times \frac{25}{30} = +29.2$	$+247.8$
L_2L_3	231.4	$231.4 \times \frac{25}{30} = +192.8$	$192.8 \times \frac{75}{45} = +321.4$	$\frac{3}{4}40 \times \frac{50}{35} = +42.9$	$+364.3$
L_3L_4	253.1	$253.1 \times \frac{25}{30} = +210.9$	$210.9 \times \frac{75}{45} = +351.5$	$\frac{5}{8}40 \times \frac{75}{40} = +46.9$	$+398.4$

* This table shows all necessary computations.

MAXIMUM LIVE WEB STRESSES, UNITS OF 1,000 POUNDS*

Bar	Reaction = shear	Components, top-chord stress loading giving maximum stress in bar				Vertical component, maximum stress	Maximum live stress
		Horizontal		Vertical			
U_1L_1	....	....	....	....	....	....	$+115.0$
U_1L_2	$\frac{21}{8}75 + \frac{3}{4}40 = 226.9$	Bar U_1U_2	$226.9 \times \frac{50}{35} = 324.1$	U_1U_2	64.8	$226.9 - 64.8 = 162.1$	$+162.1 \times \frac{39}{30} = +210.7$
U_2L_2	$\frac{15}{8}75 + \frac{5}{8}40 = 165.6$	Bar U_1U_2	$165.6 \times \frac{50}{35} = 236.6$	U_1U_2	47.3	$165.6 - 47.3 = 118.3$	-118.3
U_2L_3	Same as $U_2L_2 = 165.6$	Bar U_2U_3	$165.6 \times \frac{75}{40} = 310.5$	U_2U_3	62.1	$165.6 - 62.1 = 103.5$	$+103.5 \times \frac{43}{35} = +127.2$
U_3L_3	$\frac{10}{8}75 + \frac{40}{2} = 113.8$	Bar U_2U_3	$113.8 \times \frac{75}{40} = 213.3$	U_2U_3	42.7	$113.8 - 42.7 = 71.1$	-71.1
U_3L_4	Same as $U_3L_3 = 113.8$	....	....	....	....	113.8	$+113.8 \times \frac{47.2}{40} = +134.3$
U_4L_4	$\frac{6}{8}75 + \frac{3}{8}40 = 71.3$	....	....	....	....	71.3	-71.3
U_4L_5 / U_4L_3	Same as $U_4L_4 = 71.3$	....	....	....	....	71.3	$+71.3 \times \frac{47.2}{40} = +84.1$
U_5L_6 / U_3L_2	$\frac{3}{8}75 + \frac{1}{4}40 = 38.1$	Bar U_5U_6	$38.1 \times \frac{150}{35} = 163.3$	U_5U_6	32.7	$38.1 + 32.7 = 70.8$	$+70.8 \times \frac{47.2}{40} = +83.5$

* This table shows all necessary computations.

Try load 3 at L_1, and move up load 4.

$$(284 + 2 \times 79)\frac{5}{8p} + \delta > 50 \times \frac{5}{p}$$

Try load 4 at L_1, and move up load 5.

$$(284 + 2 \times 84)\frac{5}{8p} + \delta < 70 \times \frac{5}{p}$$

Therefore, load 4 at L_1 gives maximum.

Bar U_1L_1: Maximum stress occurs with the loads placed in the position giving maximum moment at the center of a beam 50 ft. long (see Art. 51).

Try load 3. $50 < 66$ Therefore, not a maximum.
Try load 4. $59 < 70$ Therefore, a maximum.
Try load 5. $59 < 70$ Therefore, not a maximum.
Therefore, load 4 at L_1 gives maximum.

Bar U_1L_2: Maximum stress occurs with the loading giving maximum counterclockwise moment about 0 (intersection of U_1U_2 prolonged and bottom chord). This position may be determined by method of moving up the loads. The distance from L_0 to 0 may be found as follows:

Chord bar U_1U_2 drops 5 ft. in one panel or 30 ft. in six panels; hence, its intersection with the bottom chord is six panel lengths from L_1 or five panel lengths from L_0.

Start with load 2 at L_2, and move up load 3.

Increase in moment about 0 of left reaction

$$= (284 + 2 \times 49)\frac{5}{8p} \times 5p + \delta = 382\left(\frac{25}{8}\right) + \delta$$

Increase in moment about 0 of reaction at floor beam L_1

$$= 30 \times \frac{5}{p} \times 6p = 900\frac{p}{p} = 900$$

Since $382(25/8) + \delta > 900$, load 3 gives greater moment than load 2.

Start with load 3 at L_2, and move up load 4.

$$(284 + 2 \times 54)\frac{5}{8p} \times 5p + \delta < 50 \times \frac{5}{p} \times 6p$$

Therefore, load 4 should not be moved up and load 3 at L_2 gives maximum.

Bars L_2L_3 and U_1U_2: Maximum stress occurs with loading, giving maximum moment at L_2. This position is found in the usual manner, as follows:

Trial load	Average load on left	Greater or less than	Average load on right	Maximum?
Load 6 to left of L_2. .	$\frac{103}{2}$	<	$\frac{284 + 73 \times 2 - 103}{6} = \frac{327}{6}$	No
Load 7 to left of L_2. .	$\frac{116}{2}$	>	$\frac{430 + 10 - 116}{6} = \frac{324}{6}$	Yes
Load 8 to right of L_2	$\frac{116}{2}$	>	$\frac{440 + 12 - 116}{6} = \frac{336}{6}$	No
Load 11 to right of L_2	$\frac{102}{2}$	<	$\frac{284 + 2 \times 105 - 152}{6} = \frac{342}{6}$	Yes
Load 11 to left of L_2	$\frac{122}{2}$	>	$\frac{342 - 20}{6} = \frac{322}{6}$	
Load 12 to right of L_2	$\frac{102}{2}$	<	$\frac{322 + 2 \times 5}{6} = \frac{332}{6}$	Yes
Load 12 to left of L_2	$\frac{122}{2}$	>	$\frac{332 - 20}{6} = \frac{312}{6}$	
Load 13 to right of L_2	$\frac{102}{2}$	<	$\frac{312 + 10}{6} = \frac{322}{6}$	Yes
Load 13 to left of L_2	$\frac{122}{2}$	>	$\frac{322 - 20}{6} = \frac{302}{6}$	
Load 14 to right of L_2	$\frac{122}{2}$	>	$\frac{302 + 10}{6} = \frac{312}{6}$	No

A maximum moment may occur with load 7, 11, 12, or 13 at L_2. Computations show that load 7 at L_2 gives maximum, as would be expected from inspection of the loading.

Bars U_2L_2 and U_2L_3: Maximum stress occurs with maximum counterclockwise moment about origin 0 of all forces to left of a vertical section in panel 2–3 (or a diagonal section cutting bars U_1U_2, U_2L_2, and L_2L_3). Determine the position for maximum moment by method of moving up the loads.

Start with load 2 at L_3, and move up load 3.

$$(284 + 24 \times 2) \times \frac{5}{8p} \times 5p + \delta > 30 \times \frac{5}{p} \times 7p$$

$$\delta = \frac{2 \times 5 \times 2.5}{8p} \times 5p$$

In this case the slight increase in moment due to the term δ is sufficient to cause a larger moment with load 3 than with load 2. Therefore, maximum stress occurs with load 3 at L_3.

Bars L_3L_4 and U_2U_3: Maximum stress occurs with loading causing maximum moment at L_3. This position is found in the usual manner, as follows:

Trial load	Average load on left	Greater or less than	Average load on right	Maximum?
Load 11 to right of L_3	$\frac{152}{3}$	<	$\frac{284 + 80 \times 2 - 152}{5} = \frac{292}{5}$	Yes
Load 11 to left of L_3	$\frac{172}{3}$	>	$\frac{272}{5}$	
Load 12 to right of L_3	$\frac{172}{3}$	>	$\frac{272 + 10}{5} = \frac{282}{5}$	No
Load 13 to right of L_3	$\frac{192}{3}$	>	$\frac{282 - 20 + 10}{5} = \frac{272}{5}$	No

Therefore, load 11 at L_3 gives maximum.

Bar U_3L_3: Maximum compression occurs with loading giving maximum moment about O, of forces to left of a diagonal section cutting bars U_2U_3, U_3L_3 and L_3L_4. This position may be determined by method of moving up the loads.

Start with load 1 at L_4, and move up load 2.

$$271 \times \frac{8}{8p} \times 5p + \delta > 10 \times 8 \times \frac{8p}{p}$$

Therefore, move up load 3.

$$284 \times \frac{5}{8p} \times 5p + \delta < 30 \times 5 \times \frac{8p}{p}$$

Therefore, load 2 at L_4 gives maximum stress. This loading is evidently consistent with main diagonals U_2L_3 and U_3L_4 being in action.

MAXIMUM LIVE WEB STRESSES, UNITS OF 1,000 POUNDS
(Cooper's E_{40} loading)

Bar	Position of loads	All necessary stress computations (L_R = left reaction. V.C. = vertical component)
L_0U_1	4 at L_1	Shear = $(16{,}364 + 284 \times 84 + 84 \times 84) \div 200 - 480/25$ $= 217.2$ V.C. stress = 217.2 stress = $217.2 \times 1.3 =$ -282.4
U_1L_1	4 at L_1	Stress = $10\left(\frac{7}{25}\right) + 20\frac{(15 + 20 + 25 + 20)}{25} + 13\frac{(11 + 6)}{25} = +75.6$
U_1L_2	3 at L_2	$L_R = [16{,}364 + (284 + 54)54] \div 200 = 173.1$ Floor-beam load at $L_1 = 230 \div 25 = 9.2$ Stress = $[(173.1 \times 5p - 9.2 \times 6p) \div 7p] \times 1.3 = +150.4$
U_2L_2	3 at L_3	$L_R = [16{,}364 + (284 + 29)29] \div 200 = 127.2$ Floor-beam load at $L_2 = 9.2$ Stress = $(127.2 \times 5p - 9.2 \times 7p) \div 7p = -81.7$
U_2L_3	3 at L_3	V.C. = $(127.2 \times 5p - 9.2 \times 7p) \div 8p = 71.5$ Stress = $(71.5 \times 43) \div 35 = +87.8$
U_3L_3	2 at L_4	$L_R = (16{,}364 - 284) \div 200 = 80.4$ Floor-beam load at $L_3 = (10 \times 8) \div 25 = 3.2$ Stress = $(80.4 \times 5p - 3.2 \times 8p) \div 8p = -47.1$
U_3L_4	3 at L_4	Shear = $[16{,}364 + (284 + 4)4] \div 200 - (230 \div 25)$ $= 78.4$ Stress = $(78.4 \times 47.2) \div 40 = +92.5$
U_4L_3 U_4L_5	2 at L_5	Shear = $(8{,}728 + 232 \times 4) \div 200 - (10 \times 8) \div 25$ $= 45.1$ Stress = $(45.1 \times 47.2) \div 40 = +53.2$
U_4L_4	2 at L_5	Stress = V.C. in bar $U_4L_5 = -45.1$
U_5L_6 U_3L_2	3 at L_6*	$L_R = [3{,}496 + 142 \times 15 + 20 \times 5] \div 200 = 28.6$ Floor-beam load at $L_5 = 230 \div 25 = 9.2$ Stress = $[(28.6 \times 13p - 9.2 \times 8p) \div 7p] \times 1.18$ $= +50.3$

* One locomotive followed by uniform load.

MAXIMUM LIVE CHORD STRESSES, UNITS OF 1,000 POUNDS
(Cooper's E_{40} loading)

Bar	Position of loads	All necessary stress computations (see previous table for some of the values used in table)
L_0L_1 L_1L_2	4 at L_1	$= (217.2 \times 25) \div 30 \quad = \quad +181.0$
L_2L_3	7 at L_2	Moment at $L_2 = [16{,}364 + (284 + 78)78] \times \frac{2}{8} - 2{,}155 = 8{,}995$ Stress $= 8{,}995 \div 35 \quad = \quad +257.0$
U_1U_2	7 at L_2	$257 \times 25.5 \div 25 \quad = \quad -262.1$
L_3L_4	11 at L_3	Moment at $L_3 = [16{,}364 + (284 + 80)80] \times \frac{3}{8} - 5{,}848 = 11{,}208$ Stress $= 11{,}208 \div 40 \quad = \quad +280.2$
U_2U_3*	11 at L_3	Stress $= 280.2 \times \dfrac{25.5}{25} \quad = \quad -285.8$
U_3U_4	13 at L_4	Moment at $L_4 = [16{,}364 + (284 + 65)65] \times \frac{1}{2} - 7{,}668 = 11{,}856$ Stress $= 11{,}856 \div 40 \quad = \quad -296.4$

* This stress would be incorrect if the loading used were to throw counter U_3L_2 or L_3U_4 into action. To decide whether this is the case, the shear in panel 2–3 due to this loading may be computed, and the vertical component in top chord U_2U_3 subtracted from it. If the result is positive, or negative, but less (with due allowance for impact) than the dead shear in panel, the counter will not be in action. The computations follow, making use of previous computations and the moment diagram.

$$\text{Shear} = \frac{45{,}484}{200} - 116 - \frac{8}{25}10 - \frac{16 + 21}{25}13 = +89.0$$

$$\text{V.C.} \quad U_2U_3 = 280.2 \times \frac{10}{50} = -56.0.$$

Therefore, counter is not in action, and stresses are correctly determined.

Bar U_3L_4: Maximum stress occurs for loading giving maximum positive shear in panel 3–4 and is determined as follows:

Start with load 2 at L_4, and move up load 3.

$$284 \times \frac{5}{8p} + \delta > 30 \times \frac{5}{p}$$

Move up load 4.

$$292 \times \frac{5}{8p} + \delta < 50 \times \frac{5}{p}$$

Therefore, load 3 at L_4 gives maximum.

Bar $U_4L_5 = U_4L_3$ *and bar* U_4L_4: Maximum stress occurs for loading giving maximum positive shear in panel 4–5 and is determined as follows:

Start with load 2 at L_5, and move up load 3.

$$232 \times \frac{5}{8p} + \delta < 30 \times \frac{5}{p}$$

Therefore, load (2) at L_5 gives maximum.

Bar U_5L_6 *(counter)* $= U_3L_2$: Maximum stress occurs with loading giving maximum clockwise moment about O', the point of intersection of U_5U_6 prolonged and the bottom chord prolonged, of forces to left of vertical section through U_5L_6.

Determine position by method of moving up the loads. Start with load 1 at L_6, and move up load 2.

$$\left(142 \times \frac{8}{8p}\right)13p + \delta > 10 \times \frac{8}{p} \times 8p$$

Move up load 3.

$$\left(152 \times \frac{5}{8p}\right)13p + \delta > 30 \times \frac{5}{p} \times 8p$$

Move up load 4.

$$\left(152 \times \frac{5}{8p}\right)13p + \delta < 50 \times \frac{5}{p} \times 8p$$

Therefore, load 3 at L_6 gives maximum moment.

Note that one locomotive followed by uniform load will cause a larger stress than two locomotives.

Bar U_3U_4: Maximum stress occurs for loading giving maximum moment at L_4. This position is found in the usual manner, as follows:

Trial load	Average load on left	Greater or less than	Average load on right	Maximum?
Load 11 to left of L_4	$17\frac{2}{4}$	<	$22\frac{2}{4}$	No
Load 12 to left of L_4	$19\frac{2}{4}$	<	$21\frac{2}{4}$	No
Load 13 to left of L_4	$21\frac{2}{4}$	>	$20\frac{2}{4}$	Yes
Load 14 to right of L_4	$21\frac{2}{4}$	=	$21\frac{2}{4}$	Yes

Maximum moment occurs with either load 13 or load 14 at L_4. Computations show that load 13 causes the maximum.

The necessary computations for maximum stresses in all bars are shown in the two tables on pages 193 and 194.

107. Computation of Stresses. Bridge Trusses with Parabolic Chord. *Uniform Load with Locomotive Excess.*—The methods used for the truss considered in the two previous articles were perfectly general and may be used for any nonparallel-chord truss. If the panel points on either chord or both chords lie upon a parabola passing through the end-panel points, the truss has, however, certain characteristics that may be taken advantage of in making the computations. Such trusses are not commonly used for bridges, but the same special features occur in certain trussed arches; hence, it seems desirable to give an example of the computations for such a truss.

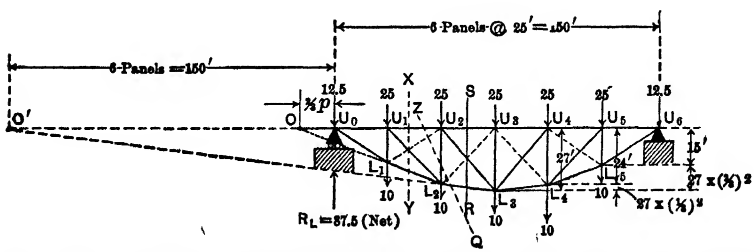


FIG. 149.—Truss with parabolic bottom chord showing dead panel loads.

Problem: Determine the maximum stresses in all the members of the truss shown in Fig. 149 with the following loads:

Dead weight of bridge,

1,000 lb. per ft. per truss, top chord	=	25,000 lb. per panel
400 lb. per ft. per truss, bottom chord	=	10,000 lb. per panel

Uniform live load,

2,000 lb. per ft. per truss, top chord	=	50,000 lb. per panel
Locomotive excess, top chord	=	25,000 lb.

In the computations for this truss the following points should be noted:

1. When the truss is loaded uniformly throughout its length, the ordinates representing the bending moments at the panel points are ordinates of a parabola; hence, the bending moment at each panel point divided by the depth of truss at the same panel point is constant.

2. Since the horizontal component of the stress in any bottom-chord member equals the moment at a panel point divided by the depth of the truss at the same panel point, the horizontal component of the bottom-chord stress under uniform load is constant throughout.

3. For the same reason the top-chord stress under uniform load is constant throughout.

4. It follows from (2) and (3) that under uniform load the horizontal components in the diagonals will be zero and the verticals will all carry compression equal to the top panel load.

5. Since under the dead load (if a uniform load) the stresses in the main diagonals are zero, it follows that the live load can always be so placed as to produce compression in any diagonal; hence, if the diagonals are to be tension members, counters will be required in every panel.

Dead Stresses.—For the given truss the dead stresses in units of 1,000 lb. will be as follows:

$$
\begin{aligned}
&\text{Top chord, stress} = \frac{1}{8}(1.4)\frac{(150)(150)}{27} &&= -145.8\\
&\text{Bottom chord, horizontal component} &&= +145.8\\
&\text{Diagonals, stress} &&= \quad 0\\
&\text{Verticals, stress} &&= -\ 25.0
\end{aligned}
$$

To confirm the correctness of the conclusions reached for web stresses the diagonal stresses will be computed in the usual manner.

Shear in panel 1–2	$= 87.5 - 35$	$= 52.5$
V.C. in bottom chord L_1L_2	$= \left(\dfrac{87.5 \times 25}{15}\right)\dfrac{9}{25}$	$= 52.5$
V.C. in diagonal U_1L_2	$= 52.5 - 52.5$	$= 0$
Shear in panel 2–3	$= 87.5 - 70$	$= 17.5$
V.C. in bottom chord L_2L_3	$= \left(\dfrac{87.5 \times 50 - 35 \times 25}{24}\right)\left(\dfrac{3}{25}\right)$	$= 17.5$
V.C. in diagonal U_2L_3	$= 17.5 - 17.5$	$= 0$

Counters. Parabolic Trusses.—It has been stated that counters are needed in every panel. The truth of this may easily be tested by actual computation. For example, to determine whether counters are required in panel 1–2, assume the section XY, and see if the live load can be so placed as to produce compression in bar U_1L_2. The stress in this bar may be computed by taking moments about the origin O. If a load is placed to the left of XY, it will produce a reaction less than itself, and the

MAXIMUM STRESSES DUE TO LOCOMOTIVE EXCESS, UNITS OF 1,000 POUNDS

Bar	Position of load	Computations
U_0L_1	E at U_1	H.C. stress $= \frac{5}{6}25 \times \frac{25}{15} = +34.7$
U_0U_1	E at U_1	stress $= \frac{5}{6}25 \times \frac{25}{15} = -34.7$
L_1L_2*	See footnote	H.C. stress $= \frac{4\frac{2}{7}}{6}25 \times \frac{25}{15} = +29.8$
U_1U_2	E at U_2	stress $= \frac{4}{6}25 \times \frac{50}{24} = -34.7$
L_2L_3*	See footnote	H.C. stress $= \frac{3\frac{3}{7}}{6}25 \times \frac{50}{24} = +29.8$
U_2U_3	E at U_3	stress $= \frac{1}{2}25 \times \frac{75}{27} = -34.7$

* The maximum stress in this bar occurs with E so located that there is no stress in either diagonal of the panel containing the bar. Under this condition the influence line then consists of two straight lines intersecting at the neutral point of the panel which is, therefore, the point where the ordinate to the influence line is a maximum. This conclusion does not apply to a truss of which the top chord is a parabola, as may readily be seen by comparing the expressions for the stress with a load at the neutral point with that for a load at a panel point.

moment of this reaction about O will be less than the moment of the load itself not only because of its smaller value but because its lever arm is less; hence, any load to the left of XY will produce clockwise moment about O of the forces to the left of XY and thereby cause compression in U_1L_2, and therefore a counter will be needed in that panel. As this method is perfectly general, it follows that counters are needed in every panel since the live load can always be placed so as to produce compression in the main diagonals, and the dead stress in each of these members is zero.

Live Chord Stresses.—The maximum live chord stresses occur with the uniform live load extending over the whole truss and can be computed from the dead stresses by multiplying the latter by the ratio of live load to dead load. The chord stresses due to the locomotive excess are as shown on page 198.

Live Web Stresses.—The maximum live web stresses occur with partial loading. The position of the loads may be determined by the methods previously used. The necessary computations for maximum stresses are given in the following table:

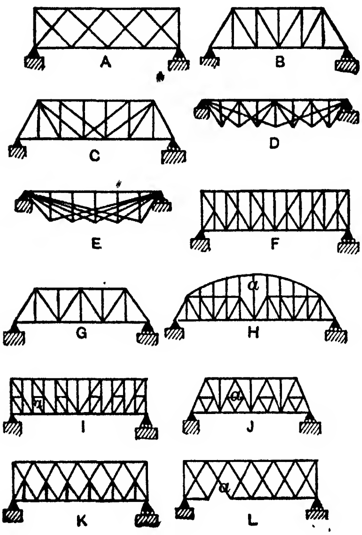
MAXIMUM LIVE WEB STRESSES, UNITS OF 1,000 POUNDS

Bar	Panel points loaded with uniform load	Position of E	Computations, vertical components of maximum live web stresses
U_1L_1	U_1 or U_1 to U_6	U_1	$50 + 25 \qquad = -75.0$
U_1L_2	U_2 to U_6	U_2	Shear in panel 1–2 $= \frac{10}{6}50 + \frac{4}{6}25 = 100.0$ V.C. in L_1L_2 $= 100 \times \frac{25}{15} \times \frac{9}{25} = 60.0$ V.C. in U_1L_2 $= 100 - 60 = +40.0$
U_2L_2	U_2 or U_1 to U_5	U_2	-75.0
U_2L_3	U_3 to U_6	U_3	Shear in panel 2–3 $= \frac{6}{6}50 + \frac{3}{6}25 = 62.5$ V.C. in L_2L_3 $= 62.5 \times \frac{50}{24} \times \frac{3}{25} = 15.6$ V.C. in U_2L_3 $= 62.5 - 15.6 = +46.9$
U_3L_3	U_3 or U_1 to U_4	U_3	-75.0
U_3L_4	U_4 and U_5	U_4	Shear in panel 3–4 $= \frac{3}{6}50 + \frac{2}{6}25 = 33.3$ V.C. in L_3L_4 $= 33.3 \times \frac{75}{27} \times \frac{3}{25} = 11.1$ V.C. in U_3L_4 $= 33.3 + 11.1 = +44.4$
U_4L_5	U_5	U_5	Shear in panel 4–5 $= \frac{1}{6}75 = 12.5$ V.C. in L_4L_5 $= 12.5 \times \frac{100}{24} \times \frac{9}{25} = 18.75$ V.C. in U_4L_5 $= 12.5 + 18.75 = +31.25$

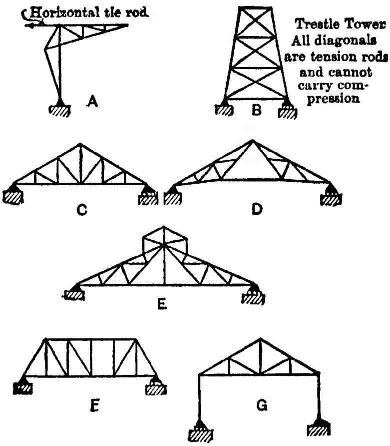
For actual locomotive loads the computations for this truss should present no more difficulty than for the truss of the previous example.

Problems

41. State which of the trusses shown in the figure are statically determined with respect to the inner forces, and give reasons.



PROB. 41.



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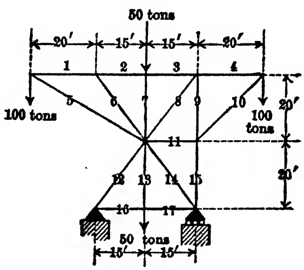

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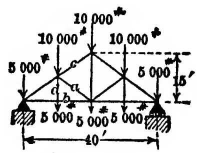

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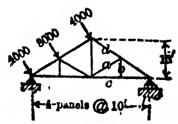


PROB. 45.

42. State which of the structures shown in the figure are statically undetermined with respect to the inner forces, and give reasons.

43. *a.* Compute by the analytical method of joints the vertical components in all diagonal members and the actual stress in all other members of this structure. Tabulate results in order according to

bar numbers. Designate tension by (+) and compression by (−).

b. Determine stress in all members by graphical method of joints (Bow's notation). Tabulate results in same order as in (*a*).

44. Compute by method of moments the stress in bars *a*, *b*, *c*, and *d*, and state whether stress is tension or compression.

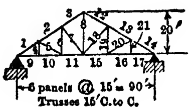
45. *a.* Compute by method of moments the stress in bars *a*, *b*, *c*, and *d*, and state whether tension or compression.

b. Same as (*a*), but direction of reaction is not fixed by rollers. (Assume both reactions to act parallel to direction of applied loads.)

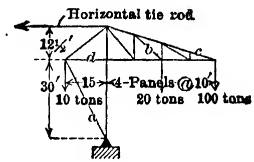
46. Compute maximum stress in each member of this intermediate roof truss due to following loads applied at top chord:

1. Dead, 30 lb. per horizontal square foot.
2. Snow, 20 lb. per horizontal square foot.
3. Wind, 20 lb. per square foot normal to surface.

Tabulate stresses for each kind of loading, and determine maximum stresses, arranging results according to bar numbers as given on diagram. Indicate tension by (+) and compression by (−).



PROB. 46.

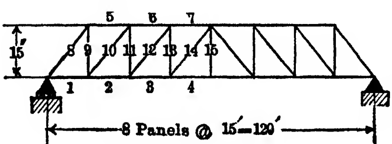


PROB. 47.

47. Compute stresses in tons, and state whether tension or compression for the following bars:

Bar *a*, by method of joints.
Bar *b*, by method of moments.
Bar *c*, by method of joints.
Bar *d*, by method of moments.

48. Uniform live load, 2,000 lb. per foot, on bottom chord.
Locomotive excess, 20,000 lb. on bottom chord.
Impact by formula (7).
Dead load, bottom chord, 600 lb. per foot.
Dead load, top chord, 200 lb. per foot.



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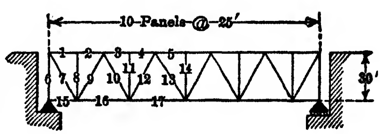

PROB. 49.

Determine panels in which counters are needed, and compute maximum stress in each member of this truss due to foregoing loads. Number bars as shown in figure, and arrange results in order according to bar numbers.

Stresses to be given in pounds. Tension to be denoted by (+) and compression by (−).

49. Uniform live load, 2,000 lb. per foot on top chord.
Locomotive excess, 20,000 lb. top chord.
Impact by method given in Art. 16.
Dead load, 1,000 lb. per foot top chord.
Dead load, 500 lb. per foot bottom chord.

No counters are to be used. Compute maximum stresses of both kinds in all members of this truss due to foregoing loads. (Rules as to arrangement of results, etc., as in previous problems.)

CHAPTER VII

BRIDGE TRUSSES WITH SECONDARY WEB SYSTEMS, INCLUDING THE BALTIMORE AND PETIT TRUSSES

108. Secondary Systems Described.—The bridge trusses heretofore treated have all been of such simple types that the application of the ordinary methods of joints, moments, and shear required no special explanation. For spans of considerable length, however, the frequent subdivision of the main panels and the addition of a secondary[1] set of diagonals and verticals produce complications the effect of which will be explained in this chapter. The Baltimore and Petit trusses, illustrated by Figs. 132 and 134, are the common forms of such trusses and will alone be considered. An examination of one of these trusses shows that the stresses in the secondary verticals may easily be determined by the method of joints, the real complication occurring in the secondary diagonal stresses.

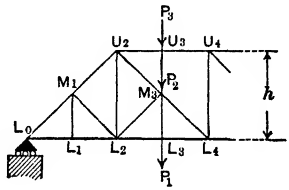

Fig. 150.

In order to study the stress in one of these diagonals, consider the portion of such a truss shown in Fig. 150, and let the problem be the determination of the stress in diagonal L_2M_3. In order that the case may be perfectly general, let it be assumed that the dead loads are applied at the middle as well as at the top and bottom panel points, although such an accurate division of the dead panel loads is not generally required.

An examination of the forces acting at joint M_3 shows that the function of the secondary diagonal M_3L_2 is to support the main diagonal U_2L_4 under the loads P_1, P_2, and P_3, which, if the secondary diagonal were not inserted, would cause a large bending moment in the members U_2L_4. If no loads are applied at the secondary panel points U_3, M_3, or L_3, there will be no stress in

[1] Secondary members are bars which are stressed by loads acting at specific panel points. Principal members are stressed by loads acting at any panel point.

either the secondary diagonal M_3L_2 or the secondary verticals U_3M_3 and M_3L_3, since these members take no part in the transmission to the abutment of the loads at other panel points. It is therefore necessary to consider only the loads P_1, P_2, and P_3, in determining the stresses in the secondary members meeting at the joint M_3, and the effect of these members upon the main diagonal stresses. The stresses in the secondary verticals are evidently equal to the panel loads applied at their ends, *i.e.*, the compression in $U_3M_3 = P_3$, which ordinarily is merely the dead weight of the top chord acting at this point, and the tension in $M_3L_3 = P_1$. This is equivalent, so far as the secondary diagonal is concerned, to the application at M_3 of a resultant downward vertical force equal to $P_1 + P_2 + P_3$. For simplicity this resultant will hereafter be called R and considered as acting directly at M_3. The stress in M_3L_2 may then be computed by the method of joints by resolving the force R along two axes coincident with U_2L_4 and L_2M_3. With this stress known, the effect of R upon U_2M_3 may also be readily determined by the same method. The stress in M_3L_4 may be determined by the method of shear in the usual manner, since its vertical component equals the shear in panel 3–4.

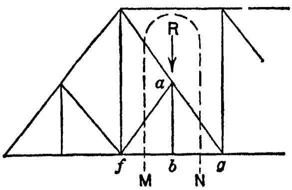

FIG. 151.

Although the method of joints for this case presents no special difficulty, the method of moments is much simpler, as is seen from the following discussion.

Consider the truss shown in Fig. 151, and apply the method of moments to the forces acting on the portion of the truss lying within the curved section MN. All the bars cut by this section except fa meet at joint g which should be taken as the origin of moments. If we now let V = vertical component of the stress in bar fa, assuming it to be tension, and write the equation for moments about point g, the following expression is obtained:

$$R(bg) + V(fg) = 0$$

Hence,

$$V = -R\frac{bg}{fg} = -(P_1 + P_2 + P_3)\frac{bg}{fg}$$

It is evident that the same method could be applied if the secondary diagonal were to extend from M_3 to U_4, Fig. 150, instead of from M_3 to L_2, but the section MN should in this case cut the top chord instead of the bottom chord and be inverted. The numerical value of the stress in the bar would be the same as for that just found, but it would be in tension instead of compression.

The following proposition may now be stated:

The *vertical component* of the *compression* in bar 1 in the case shown by Fig. 152 or the *vertical component* of the tension in bar 2 in the case shown by Fig. 153 $= \dfrac{P_1 + P_2 + P_3}{2}$.

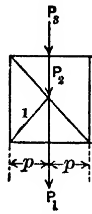

FIG. 152.

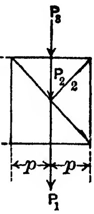

FIG. 153.

It follows from the foregoing rule that the vertical component of the maximum stress in a secondary diagonal of a Baltimore truss with equal panels and horizontal chords equals one-half the maximum panel load.

With the vertical component in the secondary diagonal known, the vertical component in the main diagonal in the same panel may be found by subtracting this value from the shear in the panel, if the bars are shown in Fig. 152, or by adding it to the shear for the case shown in Fig. 153, provided that in both cases the shear is positive. In case the shear is negative, the question of counters must be investigated in accordance with the methods of the following article.

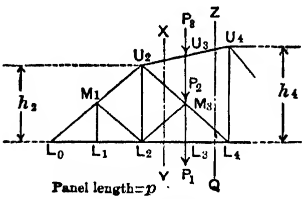

FIG. 154.

The demonstration just given is simple and equally applicable to trusses with nonparallel chords. The demonstration that follows is given, however, in order to illustrate a method which is often very useful in determining bar stresses in certain forms of trusses.

This method consists in first deriving an expression for the sum of the horizontal components in two bars such as L_2M_3 and L_2L_3, Fig. 154, called hereafter for convenience H.C. $(L_2M_3 +$

L_2L_3), and then subtracting from it the horizontal component of one of the two bars such as L_2L_3, called H.C. (L_2L_3).

Let M_2 = the moment of the forces to the left of XY about U_2.

M_2' = the moment of the forces to the left of ZQ about U_2.

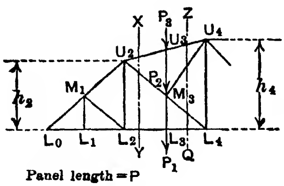

FIG. 155.

Then

$$\text{H.C. } (L_2M_3 + L_2L_3) = \frac{M_2}{h_2}$$

and

$$\text{H.C. } (L_3L_4) = \text{H.C. } (L_2L_3) = \frac{M_2'}{h_2}$$

Therefore,

$$\text{H.C. } (L_2M_3 + L_2L_3) - \text{H.C. } (L_2L_3) = \text{H.C. } (L_2M_3) = \frac{M_2 - M_2'}{h_2}$$

The only difference between M_2 and M_2' is the moment of the forces P_1, P_2, and P_3, acting between XY and ZQ, since otherwise the forces to the left of the two sections are identical; hence,

$$\frac{M_2 - M_2'}{h_2} = (P_1 + P_2 + P_3)\frac{p}{h_2} = \text{H.C. } (L_2M_3)$$

For the case shown by Fig. 154, M_2' will be larger than M_2; hence, the result above will be negative, showing compression in the secondary diagonal L_2M_3.

If the secondary diagonal is a tension member, as shown in Fig. 155, instead of the compression bar of Fig. 154, the same general method applies; but M_2 and M_2' should for this case be replaced by M_4' and M_4, the moments about L_4 of the forces to the left of ZQ and XY, respectively, and the expression should have h_4 in the denominator instead of h_2. The following expression results:

$$\text{H.C. } (U_3U_4 + M_3U_4) - \text{H.C. } (U_3U_4) = \frac{M_4' - M_4}{h_4}$$

Therefore,

$$\text{H.C. } (M_3U_4) = (P_1 + P_2 + P_3)\frac{p}{h_4}$$

This result will be positive, thereby indicating tension in the bar.

It should be noticed that in all these cases the intermediate panel point has been so located as to divide the main diagonal at the *center*, and the two halves of the latter member have been in the same straight line. Moreover, the chords have been straight between the panel points.

Were these conditions not to exist, the demonstration would not be true. For example, if the members were to be as shown in Fig. 156, it would be necessary to determine the value of the stress in the secondary diagonal by a special method. A general equation for this case will not be given, but for any given truss the stress in U_2M_3 may be readily obtained by the method of moments, L_2 and m being used for origins, with sections XY and ZQ as before. It should be noticed that in such a case the stress

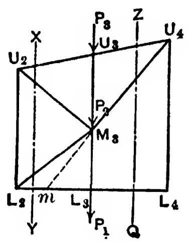

FIG. 156.

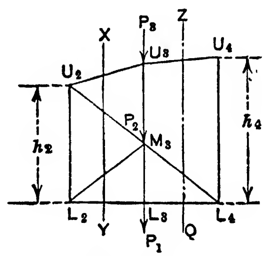

FIG. 157.

in U_2M_3 is a function not only of the panel loads P_1, P_2, and P_3, but also of the loads at all other panel points, since the moment about m of the outer forces to the left of XY differs from the moment of these same forces about L_2.

A somewhat similar case is shown by Fig. 157, where the top chord is not straight between main panel points. In this case the stress in the secondary diagonal is a function of the stress in the vertical M_3U_3 as well as of the panel loads P_1, P_2, and P_3. Since the stress in M_3U_3 is a function of the upper chord stresses, it is in consequence affected by all the loads on the structure. The simplest method of solution for this case for any given loading is to combine the stress in U_3M_3 with the panel load P_2 and then to determine the diagonal stresses. The stress in U_3M_3 may readily be obtained by applying the method of joints to the forces acting at U_3, the horizontal components of the top-chord stresses in U_3U_2 and U_3U_4 having first

been found in the usual manner by the method of moments, and from these their vertical components.

The stress in a main diagonal, such as U_2M_3 of a truss, like that shown in Fig. 154, can be easily computed, provided that the stress in the secondary diagonal is known. It should be observed, however, that the stress in the main diagonal depends not only upon the shear and the stress in the secondary member, but also upon the vertical component in the top chord. This case is more complicated than for the parallel-chord truss but is fully illustrated by the example given in the following article.

109. Computation of Maximum Stresses in Petit Truss. *Dead Loads and Concentrated Load System.*

Problem: Let the problem be the computation of the maximum stresses in all bars of the truss shown in Fig. 158 for the following loads.

Dead load on top chord per horizontal foot, 2,250 lb. per truss = 68,000, lb. per panel (approximately)

Dead load on bottom chord, 3,500 lb. per foot per truss = 105,000 lb. per panel

Live load, Cooper's E_{50} standard loading.

In this truss the dotted horizontal members are used to support the main verticals against buckling and are subjected

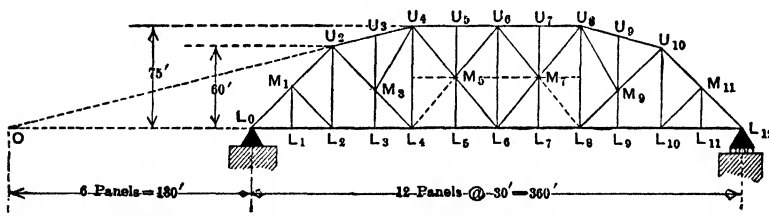

FIG. 158.—Double-track railroad bridge. Diagonals tension members, except M_1L_2 and $M_{11}L_{10}$. Trusses 33 ft., 0 in. c/c. Tracks 13 ft., 0 in. c/c.

to secondary stresses only, a common device in long-span trusses. The dotted diagonals represent counters and are not in action under dead loads. In the computations that follow, the moment diagram for Cooper's E_{40} loading given in Art. 52 has been used and the stresses for E_{50} obtained by multiplying by the ratio $^{50}/_{40}$. All units are in thousands of pounds.

Index Stresses.—In determining the index stresses, it is necessary, as in the previous example, to determine first the vertical component in the inclined top-chord bar and to correct

the diagonal stresses to conform to the slope of the end diagonals. As the stresses in the secondary members are independent of the stresses in the main members, it is advisable to write these first. For the other members the usual process will be pursued of beginning at the center and working toward the end, checking with the reaction at the end and with the chord stress as computed by moments at the center.

The index stresses are given in Fig. 159, and the necessary computations for bars in which the index stresses are at all com-

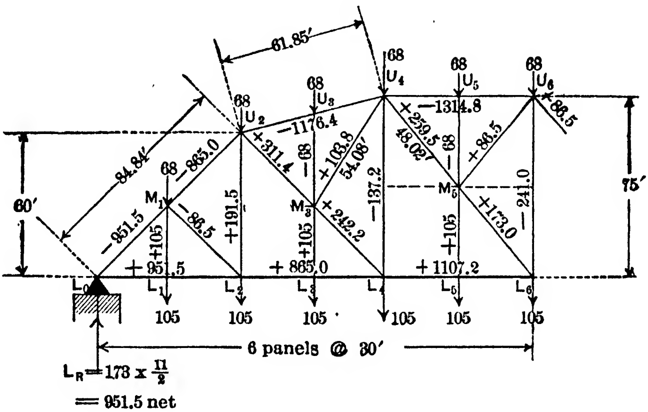

FIG. 159.—Index stresses and dead panel loads for truss shown in Fig. 158.

plicated follow. In determining the index stresses the diagonals sloping at 45° are taken as standard.

V.C. in $M_3U_4 = 173 \times \dfrac{30}{75} \times \dfrac{45}{30}$ $= 103.8$

V.C. in $U_2U_4 = \left(\dfrac{951.5 \times 120 - 173 \times 150}{75}\right)\dfrac{15}{60}$ $= 294.1$

V.C. in $U_4L_4 = 259.5 + 103.8 + 68.0 - 294.1$
(method of joints) $= 137.2$

V.C. in $U_2M_1 = 191.5 + 311.4 + 294.1 + 68.0$
(method of joints) $= 865.0$

Corrected index stress in M_3U_4, for use in determining

index stress in bar U_4U_5, $= 103.8 \times \dfrac{30}{45} = 69.2$

Corrected index stress in U_4M_5, for use in determining

index stress in bar U_4U_5, $= 259.5 \times \dfrac{60}{75} = 207.6$

To check top-chord stresses, determine combined horizontal component in U_5U_6 and M_5U_6 by dividing center moment by center height, and add to the value thus obtained the horizontal component in M_5U_6.

$$\text{Stress in } U_4U_5 = \frac{951.5 \times 180 - 173 \times 5 \times 90}{75} + 86.5 \times \frac{30}{37.5} = 1{,}314.8$$

This equals the index stress in U_4U_5, as should be the case, since the latter was determined for diagonals sloping at 45°.

Dead Stresses.—The actual dead stresses are given in the following table in which the columns headed "Ratio" give the length of each web member divided by its vertical projection and of each chord member divided by its horizontal projection.

DEAD STRESSES, UNITS OF 1,000 LB.

Bar	Index stress	Ratio	Stress	Bar	Index stress	Ratio	Stress
L_0M_1	− 951.5	1.414	−1,345.4	L_1M_1	+105.0	1.000	+105.0
M_1U_2	− 865.0	1.414	−1,223.1	L_3M_3	+105.0	1.000	+105.0
U_2U_3	−1,176.4	1.031	−1,212.9	L_5M_5	+105.0	1.000	+105.0
U_3U_4	−1,176.4	1.031	−1,212.9	U_3M_3	− 68.0	1.000	− 68.0
U_4U_5	−1,314.8	1.000	−1,314.8	U_5M_5	− 68.0	1.000	− 68.0
U_5U_6	−1,314.8	1.000	−1,314.8	M_1L_2	− 86.5	1.414	−122.3
L_0L_1	+ 951.5	1.000	+ 951.5	L_2U_2	+191.5	1.000	+191.5
L_1L_2	+ 951.5	1.000	+ 951.5	U_2M_3	+311.4	1.414	+440.3
L_2L_3	+ 865.0	1.000	+ 865.0	M_3L_4	+242.2	1.414	+342.5
L_3L_4	+ 865.0	1.000	+ 865.0	M_3U_4	+103.8	1.202	+124.8
L_4L_5	+1,107.2	1.000	+1,107.2	U_4L_4	−137.2	1.000	−137.2
L_5L_6	+1,107.2	1.000	+1,107.2	U_4M_5	+259.5	1.280	+332.2
				M_5U_6	+ 86.5	1.280	+110.7
				M_5L_6	+173.0	1.280	+221.4
				L_6U_6	−241.0	1.000	−241.0

Counters.—Before computing the live stresses, and even before determining the position of live loads for maximum stresses, it is necessary to decide to what panels counters are required.

Panels 4–5 and 7–8. Evidently, if the diagonals are all tension members, counters will be needed in these panels if the resultant

shear due to live, dead, and impact in panel 7–8 is ever positive, or negative but less than one-half the total panel load (live + dead + impact) simultaneously occurring at L_7. The application of the method of moving up the loads shows that load 2 at L_8 gives maximum positive shear in this panel. Its magnitude per truss for E_{50} equals

$$\frac{5}{4} \times 2\left[\frac{16{,}364 + (284 + 19)19}{360}\right] - \frac{5}{4} \times 2 \times \frac{80}{30} = 146.9$$

and panel load at L_7 equals

$$5/4 \times 2 \times 10 \times 8/30 = 6.7$$

If impact is computed by the method given in Art. 16 its value will be $\frac{23}{33} \times 18.95$ per cent of stress due to load on one track only; hence live shear plus impact in panel $= 146.9 + 19.0 = 165.9$, which is considerably less than the dead shear of -259.5. Hence the resultant shear in the panel is neither positive, nor negative and less than one-half the total panel load, and a counter is not necessary.

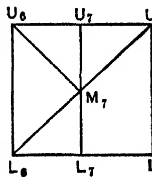

FIG. 160.

Panels 2–3 and 9–10. Counters will be needed in these panels if the live compression plus impact in bar M_9U_{10} exceeds its dead tension. The position of loads which will give the maximum compression in M_9U_{10} will be that which will give maximum clockwise moment of forces to left of vertical section through panel 9–10 about O', the intersection of top-chord bar $U_8U_9U_{10}$ prolonged and the bottom chord prolonged.

To determine this position, start with load 1 at L_{10}, and move up load 2, using p to represent the panel length.

$$152 \times \frac{8}{12p} \times 18p + \delta > 10 \times \frac{8}{p} \times 9p$$

Therefore, move up load 2.

Now try moving up load 3.

$$172 \times \frac{5}{12p} \times 18p + \delta > 30 \times \frac{5}{p} \times 9p$$

Therefore, load 3 at L_{10} gives maximum.

V.C. live stress in bar M_9U_{10} with load 3 at L_{10} for E_{60}

$$= 2 \times \frac{5}{4} \times \frac{1}{300}\left[(7,668 - 192) \times \frac{18p}{12p} - \frac{230}{p} \times 9p\right] = 76.2$$

This value is so much less than the vertical component of the dead stress in the bar that no counter is needed.

Position of Loads for Maximum Live Stress in all Members.

Bar U_2M_3: Load for maximum moment about O of loads to left of a vertical section through panel 2–3.

Start with load 2 at L_3, and move up load 3.

$$(284 + 169 \times 2)\frac{5}{12p} \times 6p + \delta > 30 \times \frac{5}{p} \times 8p$$

Move up load 4.

$$(284 + 174 \times 2)\frac{5}{12p} \times 6p + \delta < 50 \times \frac{5}{p} \times 8p$$

Therefore, load 3 at L_3 gives maximum.

Bar M_3L_4: Let M_4/h_4 = moment about U_4 of forces to left of a vertical section through panel 3–4 divided by height of truss

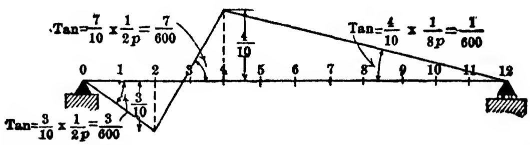

FIG. 161.—Influence line for horizontal component in M_3L_4.

at L_4. Let M_2/h_2 = moment about U_2 of forces to left of a vertical section through panel 2–3 divided by height of truss at L_2. Since the horizontal component of the stress in $M_3L_4 = M_4/h_4 - M_2/h_2$, the position of loads for maximum stress in the bar is that giving the maximum values of this quantity. Figure 161 shows the influence line for the horizontal component in this bar and that one of the loads should lie at L_4. To determine the position for maximum stress, use the method of moving up the loads, multiplying the loads to right of L_4 by the product of the distance moved and the tangent $1/600$ and those in panels 2–3 and 3–4 by the product of the distance moved and the tangent $7/600$.

Start with load 3 at L_4, and move up load 4.

$$(234 + 144 \times 2)5/600 + \delta > 50 \times 5 \times 7/600$$

Move up load 5.

$$(214 + 149 \times 2)5/600 + \delta > 70 \times 5 \times 7/600$$

Move up load 6.

$$(194 + 154 \times 2)9/600 + \delta < 90 \times 9 \times 7/600$$

Load 5 at L_4 gives a maximum.

It is possible that this bar may be brought into compression by loads coming on from left; hence, the position giving maximum compression should be determined.

Start with load 2 at L_2, and move up load 3, bringing loads on from left.

$$142 \times 5 \times 3/600 + \delta > 30 \times 5 \times 7/600$$

Move up load 4.

$$142 \times 5 \times 3/600 + \delta > 50 \times 5 \times 7/600$$

Move up load 5.

$$142 \times 5 \times 3/600 + \delta < 70 \times 5 \times 7/600$$

Therefore, load 4 at L_2 with loads coming on from left gives maximum compression.

Bar U_4M_5: Load for maximum shear in panel 4–5.

Start with load 2 at L_5, and move up load 3.

$$(284 + 109 \times 2)\frac{5}{12p} + \delta > 30 \times \frac{5}{p}$$

Move up load 4.

$$(284 + 114 \times 2)\frac{5}{12p} + \delta < 50 \times \frac{5}{p}$$

Therefore, load 3 at L_5 gives maximum.

Bar M_5L_6: Load to give the maximum value of the resultant of the positive shear in panel 5–6 and the vertical component in bar M_5U_6.

Start with load 3 at L_6, and move up load 4.

$$(284 + 84 \times 2)\frac{5}{12p} + \delta > \frac{1}{2}\left(50 \times \frac{5}{p}\right)$$

Move up load 5.

$$(284 + 89 \times 2)\frac{5}{12p} + \delta > \frac{1}{2}\left(70 \times \frac{5}{p}\right)$$

Move up load 6.

$$(284 + 94 \times 2)\frac{9}{12p} + \delta < \frac{1}{2}\left(90 \times \frac{9}{p}\right)^*$$

Therefore, load 5 at L_6 gives maximum.

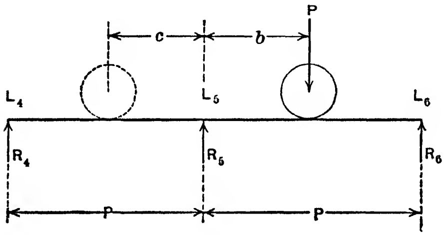

FIG. 162.

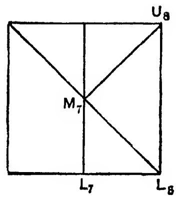

FIG. 163.

Bar M_7L_8: If this bar is in action, the condition shown in Fig. 163 will exist. Place loads so that the sum of the positive shear in panel 7–8 and the vertical component in bar M_7U_8 will be a maximum.

Start with load 2 at L_8, and move up load 3.

$$(284 + 19 \times 2)\frac{5}{12p} + \delta > \frac{1}{2}30 \times \frac{5}{p}$$

* The right-hand side of this inequality equals the increment in the sum of the panel load at L_4 and the vertical component of the stress in the secondary diagonal M_5U_6 due to the movement of the loads. If no load passes L_5, it is obvious that this change equals one-half the sum of the product of the loads moving in panel 5–6 and the distance that they move. That this is also true, provided that no load passes L_4, may be readily proved as follows:

Let the original position of a load P be as shown by the full circle in Fig. 162, and assume that, in moving the loads, P passes L_5 to the position shown by the dotted circle.

The following equations may then be written:

Original position of loads, $R_4 + \frac{R_5}{2} = \frac{P(p - b)}{2p}$

Second position of loads, $R_4 + \frac{R_5}{2} = P\frac{c}{p} + \frac{P}{2p}(p - c)$

The increase in $R_4 + \frac{R_5}{2} = P\frac{c}{p} + P\frac{(p - c)}{2p} - \frac{P(p - b)}{2p} = \frac{P(b + c)}{2p}$

This demonstration applies equally well to two corresponding panels in any other position of the truss.

Move up load 4.

$$(284 + 24 \times 2)\frac{5}{12p} + \delta > \frac{1}{2}50 \times \frac{5}{p}$$

Move up load 5.

$$(284 + 29 \times 2)\frac{5}{12p} + \delta < \frac{1}{2}70 \times \frac{5}{p}$$

Therefore, load 4 at L_8 gives a maximum.

Bars L_0M_1, L_0L_1, and L_1L_2: Load for maximum shear in panel 0–1.

Start with load 3 at L_1, and move up load 4.

$$(284 + 234 \times 2)\frac{5}{12p} + \delta > 50 \times \frac{5}{p}$$

Move up load 5.

$$(284 + 239 \times 2)\frac{5}{12p} + \delta < 70 \times \frac{5}{p}$$

Therefore, load 4 at L_1 gives maximum.

Bars M_1U_2, L_2L_3, and L_3L_4: Load for maximum moment at U_2.

Try load 7 at L_2. $^{624}/_{10} > {}^{116}/_{2}$. Not a maximum.

Try load 8 at L_2. $^{636}/_{10} > {}^{116}/_{2}$ and $^{623}/_{10} < {}^{129}/_{2}$. A maximum.

Try load 9 at L_2. $^{633}/_{10} < {}^{129}/_{2}$. Not a maximum.

Therefore, load 8 at L_2 gives a maximum.

Bar U_2L_2: This bar is really a part of the secondary system and is affected by loads at L_1 and L_2 only. The influence line for this bar is shown by Fig. 164 and has the same form as the influence line for moment at a point 30 ft. from the right end of an end-supported 90-ft. span; hence, the criterion for maximum moment may be applied to determine the position of loads which should be brought on from the left.

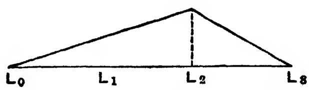

FIG. 164.—Influence line for stress in U_2L_2.

Try load 3 at L_2. $^{142}/_{60} > {}^{50}/_{30}$. Not a maximum.

Try load 4 at L_2. $^{142}/_{60} > {}^{70}/_{30}$. Not a maximum.

Try load 5 at L_2. $^{142}/_{60} < {}^{90}/_{30}$. A maximum.

Therefore, load 5 at L_2 gives a maximum with loads coming on from left.[1]

[1] With loads coming on from right, load 13 gives practically the same value as the other case.

Bars U_2U_3 and U_3U_4: Load for maximum moment about L_4 of forces to left of vertical section through panel 2–3. The influence line for the horizontal component of the stress in these bars is shown in Fig. 165. Evidently, for a maximum, one of the loads should lie at L_3.

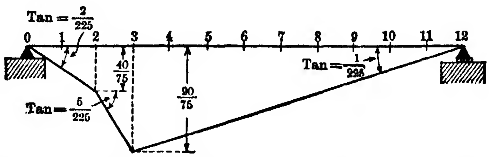

FIG. 165.—Influence line for horizontal component in U_2U_3 and U_3U_4.

While the influence line for the stress in this case is not composed of two straight lines and the criterion for maximum moment cannot be applied, it is evident that the loads will lie somewhat as for the ordinary case of maximum moment at a panel point, and one of the second-engine loads will probably give the maximum.

The following expression for the change in the stress may be written, the method of moving up the loads being used.

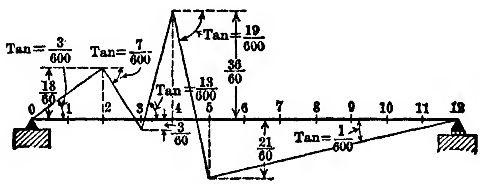

FIG. 166.—Influence line for stress in U_4L_4.

Start with load 11 at L_3, and move up load 12.

$$(112 + 225 \times 2)5/225 + \delta > 56 \times 5 \times 5/225 + 13(3 \times 5/225 + 2 \times 2/225) + 103 \times 5 \times 2/225$$

Move up load 13.

$$(92 + 230 \times 2)5/225 + \delta < 63 \times 5 \times 5/225 + 13(4 \times 5/225 + 1 \times 2/225) + 116 \times 5 \times 2/225$$

Therefore, load 12 at L_3 gives maximum.

Bar U_4L_4: On the assumption that counter M_5L_4 is not in action, the influence line for stress in this bar will be as given in Fig. 166 and shows that for maximum compression the load should come on from the right, and for maximum tension from the left.

Position for maximum compression, load coming on from right:

Start with load 1 at L_5, and move up load 2, making use of the tangents to the influence line, as was done with bar M_3L_4.

$$(274 + 101 \times 2)^{8}\!/_{600} + \delta > 10 \times 8 \times {}^{19}\!/_{600}$$

Move up load 3.

$$(254 + 109 \times 2)^{5}\!/_{600} + \delta < 30 \times 5 \times {}^{19}\!/_{600}$$

Therefore, load 2 at L_5 gives maximum compression.

Position for maximum tension, loads coming on from left:

For this case, heavy loads should be placed at both L_4 and L_2. These panel points are 60 ft. apart; hence, if the heavy loads of the first locomotive are placed near L_4, the heavy loads of the second locomotive will be located near L_2, this giving a favorable position for maximum stress.

Start with load 2 at L_4, and move up load 3 to right.

$$86 \times 5 \times {}^{13}\!/_{600} + (92 + 2 \times 19) \times 5 \times {}^{3}\!/_{600} + 20 \times 1 \times {}^{3}\!/_{600} + \delta > 30 \times 5 \times {}^{19}\!/_{600} + 56 \times 5 \times {}^{7}\!/_{600} + 20 \times 4 \times {}^{7}\!/_{600}$$

Move up load 4.

$$79 \times 5 \times {}^{13}\!/_{600} + (72 + 2 \times 24) \times 5 \times {}^{3}\!/_{600} + 20 \times 1 \times {}^{3}\!/_{600} + \delta < 50 \times 5 \times {}^{19}\!/_{600} + 63 \times 5 \times {}^{7}\!/_{600} + 20 \times 4 \times {}^{7}\!/_{600}$$

Therefore, load 3 at L_4 gives maximum.

The foregoing conditions for maximum stress will not be correct if in either case counter M_5L_4 is in action. That the counter is not in action for the position of loads giving maximum compression is, however, evident from inspection.

For the position for maximum tension the negative shear in panel 4–5 is given by the following expression:

Negative live shear in panel 4–5, load 3 at L_4, loads coming on from left

$$= 2 \times \frac{5}{4}\left[\frac{16{,}364 + (284 + 24)24}{360} - \frac{230}{30}\right] = 145.7$$

This value even without the addition of impact when combined with dead shear in the panel gives a total positive shear greater

than one-half the panel load at L_5 and hence is consistent with the assumption that M_5L_4 is not in action.[1]

A study of the influence line for L_4U_4 for the conditions existing when the counter L_4M_5 is in action shows that this condition gives less tension than the case already considered.

Bars U_4U_5 and U_5U_6: Load for maximum moment about L_6 of loads to left of vertical section through panel 4–5. The influence line for the stress in this case consists of three straight lines, as shown in Fig. 167, and shows that the maximum stress occurs with one of the loads at L_5.

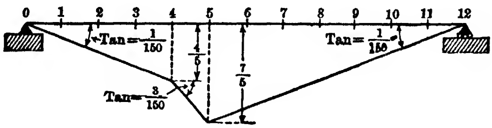

FIG. 167.—Influence line for stress in U_4U_5 and U_5U_6.

Start with load 14 at L_5, and move up load 15, making use of the tangents to the influence line.

$$(52 + 180 \times 2)\tfrac{9}{150} + \delta > 142 \times \tfrac{9}{150} + 10(\tfrac{2}{150} + 7 \times \tfrac{3}{150}) + 80 \times 9 \times \tfrac{3}{150}$$

Move up load 16.

$$(39 + 189 \times 2)\tfrac{5}{150} + \delta < 152 \times \tfrac{5}{150} + \delta + 93 \times 5 \times \tfrac{3}{150}$$

Therefore, load 15 at L_5 gives maximum.

Bars L_4L_5 and L_5L_6: Load for maximum moment at U_4 on the assumption that counter L_4M_5 is out of action.

Try load 14 to left of L_4. $^{23}\tfrac{2}{4} < {}^{47}\tfrac{2}{8}$. Not a maximum.
Try load 15 to left of L_4. $^{24}\tfrac{5}{4} > {}^{47}\tfrac{7}{8}$. A maximum.
Try load 16 to right of L_4. $^{24}\tfrac{5}{4} > {}^{48}\tfrac{7}{8}$. Not a maximum.
Therefore, load 15 at L_4 gives a maximum.
The shear in panel L_4L_5 for this condition

$$= 2 \times \frac{5}{4}\left[\frac{16{,}364 + (284 + 219)219}{360} - 245 - \frac{58}{30}13 - \frac{18}{30}4\frac{1}{2}\right] = +196.5$$

Hence, counter L_4M_5 is not in action for this loading.

[1] It should be noted that the tension in this bar should not be computed with full impact allowance since this would be inconsistent with assumed

Bar U_6L_6: Two cases must be considered for this bar. These are shown in Fig. 168.

Case 1. Maximum stress, if this case exists, will occur with the loading giving the maximum value of the algebraic sum of the positive shear on section XY and the vertical component in diagonal M_5U_6.

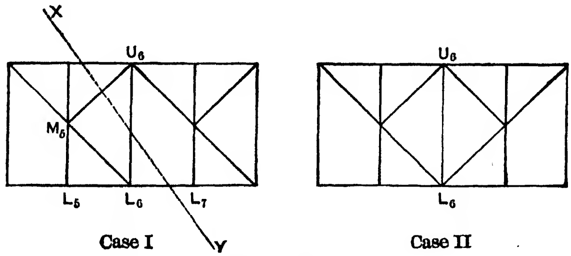

FIG. 168.

Try load 2 at L_7, and move up load 3.

$$(284 + 49 \times 2)\frac{5}{12p} + \delta > 30 \times \frac{5}{p}$$

Move up load 4.

$$(284 + 54 \times 2)\frac{5}{12p} + \delta < 50 \times \frac{5}{p}$$

Therefore, load 3 at L_7 gives a maximum.

Case 2. The maximum stress, if this case exists, cannot exceed twice the vertical component of the maximum stress in one of the secondary diagonals; *i.e.*, it will not exceed the maximum panel load. Since the stress in Case 1 is likely to be greater than this limiting value, the position of loads should not be determined until after the stress for Case 1 has been computed. If it then becomes necessary to determine the position, the method of influence lines will be used.

Bars M_1L_1, M_1L_2, M_3L_3, U_4M_3, M_5L_5, *and* U_6M_5: Maximum stress in these bars is a function of the maximum load at a secondary panel point. This has the same value in all cases and may be found for any one of these panel points, such as L_1, by placing the loads so as to give the maximum moment at the center of a 60-ft. span. (Continued on page 221.)

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conditions in the panel. The impact allowance should be made just sufficient to make the positive shear in L_4L_5 equal to one-half the panel load at L_5.

MAXIMUM LIVE STRESSES IN MAIN DIAGONALS, UNITS OF 1,000 LB. This table shows all necessary computations. (Note that 16,364 ÷ 360 = 45.45)

Bar	Position of loads	Computations
U_2M_3	Load 3 at L_3, maximum tension	Shear in panel 2–3 $= \frac{16{,}364 + (284 + 174)174}{360} - \frac{230}{30} =$ $= 266.8 - 7.7 \qquad = +259.1$ V.C. in U_2U_3 $- \frac{15}{60}(266.8 \times 4 - 7.7 \times 2)\frac{30}{75} \qquad = -105.2$ V.C. in U_2M_3 in tons for E_{49} $\qquad = +153.9$ Tension for $E_{60} = 153.9 \times 1.414 \times \frac{5}{4} \times 2 \qquad = +544.0$
M_3L_4	Load 5 at L_4, maximum tension	Shear in panel 3–4 $= \frac{16{,}364 + (284 + 154)154}{360} - \frac{830}{30}$ $= 232.8 - 27.7 \qquad = +205.1$ V.C. in U_3U_4 $= \frac{15}{60}(232.8 \times 4)\frac{30}{75} \qquad = -\ 93.1$ V.C. in M_3U_4 $= \frac{45}{30}\left(\frac{830}{30}\right)\left(\frac{30}{75}\right) \qquad = +\ 16.6$ V.C. in M_3L_4 in tons for E_{40} $= 205.1 - 93.1 + 16.6 \qquad = +128.6$ Tension for $E_{60} = 128.6 \times 1.414 \times \frac{5}{4} \times 2 \qquad = +454.6$
M_3L_4	Load 4 at L_2, maximum component, loads coming on from left	Shear in panel 3–4 $= \frac{8{,}728 - 212}{360} \qquad = -\ 23.6$ V.C. in U_3U_4 $= \frac{15}{60}\left(23.6 \times 8 + \frac{480}{30}\right)\frac{30}{75} \qquad = -\ 20.5$ V.C. in M_3U_4 $= \frac{480}{30} \times \frac{30}{75} \times \frac{45}{30} \qquad = +\ \ 9.6$ V.C. in M_3L_4 in tons for E_{40} $= 23.6 + 20.5 - 9.6 \qquad = -\ 34.5$ Compression for $E_{60} = 34.5 \times 1.414 \times \frac{5}{4} \times 2 = -121.9$ This is so much smaller than the dead tension that compression will never actually occur in this bar

MAXIMUM LIVE STRESSES IN MAIN DIAGONALS, UNITS OF 1,000 LB.—(*Continued*)

Bar	Position of loads	Computations
U_4M_5	Load 3 at L_5, maximum tension	Shear in panel 4–5 $= \frac{16{,}364 + (284 + 114)114}{360} - \frac{230}{30}$ $= 171.5 - 7.7 \quad = +163.8$ Tension for $E_{50} = 163.8 \times 1.280 \times \frac{5}{4} \times 2 \quad = +524.2$
M_5L_6	Load 5 at L_6, maximum tension	Shear in panel 5–6 $= \frac{16{,}364 + (284 + 94)94}{360} - \frac{830}{30}$ $= 144.1 - 27.7 \quad = +116.4$ V.C. in M_5U_6 $= \frac{27.7}{2} \quad = +13.8$ V.C. in M_5L_6 in tons for E_{40} $= 116.4 + 13.8 \quad = +130.2$ Tension for $E_{50} = 130.2 \times 1.280 \times \frac{5}{4} \times 2 \quad = +416.6$
M_7L_8	Load 4 at L_8, maximum tension	Shear in panel 7–8 $= \frac{16{,}364 + (284 + 29)29}{360} - \frac{480}{30}$ $= 70.7 - 16.0 \quad = +54.7$ V.C. in $M_7U_8 \quad = +8.0$ V.C. in M_7L_8 in tons for E_{40} $= 54.7 + 8.0 \quad = +62.7$ Tension for $E_{50} = 62.7 \times 1.280 \times \frac{5}{4} \times 2 \quad = +200.6$

Try load 12 at L_1. $86 > 56$ and $79 > 76$. Hence, not a maximum.

Try load 13 at L_1. $79 > 63$ and $72 < 83$. Hence, a maximum.

Try load 14 at L_1. $72 > 70$ and $52 < 90$. Hence, a maximum.

Therefore, maximum stress occurs with either load 13, or with load 14 at a secondary panel point. (Note that load 14 gives same moment as load 5.)

MAXIMUM LIVE STRESSES IN INCLINED END POSTS, CHORDS, AND MAIN VERTICALS, UNITS OF 1,000 LB.
(This table shows all necessary computations)

<table>
<tr><th>Bar</th><th>Position of loads</th><th>Computations</th><th></th></tr>
<tr><td>L_0M_1</td><td>Load 4 at L_1, maximum compression</td><td>Shear in panel 0–1
$= \frac{16,364 + (284 + 239)239}{360} - \frac{480}{30}$
$= 392.7 - 16.0$
Compression in L_0M_1 for E_{60}
$= 376.7 \times 1.414 \times \frac{5}{4} \times 2$</td><td>

$= 376.7$

$= 1,331.6$</td></tr>
<tr><td>L_0L_1
L_1L_2</td><td>Load 4 at L_1, maximum tension</td><td>Tension for $E_{60} = 376.7 \times \frac{5}{4} \times 2$</td><td>$= 941.8$</td></tr>
<tr><td>M_1U_2</td><td>Load 8 at L_2, maximum compression</td><td>Moment at L_2
$= \frac{16,364 + (284 + 234)234}{6} - 2,851$
H.C. in M_1U_2 in tons for E_{40}
$= \frac{20,078}{60} = 334.6$
Compression for E_{60}
$= 334.6 \times 1.414 \times \frac{5}{4} \times 2$</td><td>
$= 20,078$

$= 1,182.8$</td></tr>
<tr><td>L_2L_3
L_3L_4</td><td>Load 8 at L_2, maximum tension</td><td>Tension for $E_{60} = 334.6 \times \frac{5}{4} \times 2$</td><td>$= 836.5$</td></tr>
<tr><td>L_2U_2</td><td>Load 5 at L_2, loads coming on from left, maximum tension</td><td>Panel load at L_2,
$10 \times \frac{7}{30} + 80 \times \frac{22.5}{30} + 52 \times \frac{13}{30}$
Panel load at L_1,
$52 \times \frac{17}{30} + 80 \times \frac{11.5}{30} + 10 \times \frac{27}{30}$
Tension for $E_{60} = \left(84.9 + \frac{69.1}{2}\right)\frac{5}{4} \times 2$</td><td>
$= 84.9$

$= 69.1$
$= 298.5$</td></tr>
<tr><td>U_2U_3 and U_3U_4</td><td>Load 12 at L_3, maximum compression</td><td>Moment about L_4 of left reaction,
$\frac{16,364 + (284 + 230)230}{3}$
Moment about L_4 of loads to left of section,
$2,155 + 116 \times 62 + \frac{230}{30} \times 60$
$+26 \times \frac{23.5}{30} \times 60$
H.C. in bar in tons for E_{40}
$= \frac{44,861 - 11,029}{75} = 451.1$ tons.
Compression for $E_{60} = 451.1 \times 1.031 \times \frac{5}{4} \times 2$</td><td>
$= 44,861$

$= 11,029$

$= 1,162.7$</td></tr>
</table>

MAXIMUM LIVE STRESSES IN INCLINED END POSTS, CHORDS, AND MAIN VERTICALS, UNITS OF 1,000 LB.—(*Continued*)

Bar	Position of loads	Computations
U_4L_4	Load 2 at L_5, maximum compression	Moment of left reaction about 0 $= \dfrac{16{,}364 + (284 + 109)109}{12p} \times 6p = 29{,}600$ Panal load at $L_4 = \dfrac{80}{30} = 2.67$ Compression in bar for E_{60} $= \left(\dfrac{29{,}600}{300} - 2.67\right)\dfrac{5}{4} \times 2 = 240.0$
U_4L_4	Load 3 at L_4, maximum tension, loads coming on from left	Moment about L_4 of all forces to right of section through panel 3–4 $= \dfrac{8}{12}[16{,}364 + (284 + 24)24] - 230 = 15{,}607$ H.C. (bar $L_3L_4 + M_3L_4$) $= \dfrac{15{,}607}{75} = 208.1$ Moment about U_2 of loads to right of section through panel 2–3 $= \dfrac{10}{12}[16{,}364 + (284 + 24)24]$ $-(7{,}668 - 192) = 19{,}797 - 7{,}476 = 12{,}321$ Stress in L_2L_3 in tons for $E_{40} = \dfrac{12{,}321}{60} = 205.3$ H.C. in tons for E_{40} in M_3L_4 = V.C. = 2.8 tension Panel load at L_4 (load 3 at L_4) $= 50 - \dfrac{230}{30} + 20\left(\dfrac{25 + 20}{30}\right) + \dfrac{13(11 + 6)}{30}$ $= 79.7$ Tension in U_4L_4 for E_{50} $= 2 \times \dfrac{5}{4} \times (79.7 - 2.8) = 192.2$
U_4U_5 and U_5U_6*	Load 15 at L_5, maximum compression	Moment about L_6 of left reaction $= \dfrac{16{,}364 + (284 + 189)189}{2} = 52{,}880$ Moment about L_6 of loads to left of section $= 4{,}632 + 152 \times 62 + \dfrac{80 \times 16.5}{30} \times 60 = 16{,}696$ Compression in bar for E_{60} $= \dfrac{52{,}880 - 16{,}696}{75} \times \dfrac{5}{4} \times 2 = 1{,}206.1$

MAXIMUM LIVE STRESSES IN INCLINED END POSTS, CHORDS, AND MAIN VERTICALS, UNITS OF 1,000 LB.—(*Continued*)

Bar	Position of loads	Computations	
L_4L_5 and L_5L_6	Load 15 at L_4, maximum tension	Moment about U_4 $= \frac{16,364 + (284 + 219)219}{3} - 10,816$	$= 31,358$
		Tension in bar for E_{60} $= \frac{31,358}{75} \times \frac{5}{4} \times 2$	$= 1,045.3$
U_6L_6	Load 3 at L_7, maximum compression	Shear in panel 6–7 $= \frac{16,364 + (284 + 54)54}{360} - \frac{230}{30}$ $= 96.2 - 7.7$	$= 88.5$
		Stress in $M_5U_6 = 0$	
		Compression in bar $= 88.5 \times 2 \times \frac{5}{4}$	$= 221.2†$

* Note that shear for this loading in panel 4–5 is positive; hence, counter M_5L_4 is not in action.

† Note that this is larger than maximum panel load and hence is maximum stress.

MAXIMUM LIVE STRESSES IN SECONDARY MEMBERS, UNITS OF 1,000 LB.
(This table shows all necessary computations)

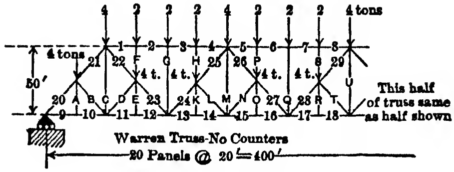
Bar	Position of loads	Computations
M_1L_1 M_3L_3 M_5L_5	Load 13 at L_1, L_3 or L_5, maximum tension	It has been previously determined that a maximum occurs with either load 13 or load 14 at a secondary panel point; hence, panel loading for each case is computed below Load 13, $\frac{13(4+16+11+5)}{30} + 10 \times \frac{12}{30} + 20\left(\frac{100}{30}\right) = 86.3$ Load 14, $\frac{13(21+16+10+5)}{30} + 10 \times \frac{7}{30} + 20\frac{(15+20+25+30)}{30} = 84.8$ Tension in bar for $E_{60} = 86.3 \times \frac{5}{4} \times 2 = 215.7$
M_1L_2	Load 13 at L_3, maximum compression	Compression in bar for $E_{60} = 86.3 \times \frac{5}{4} \times 1.414 = 152.5$
M_3U_4	Load 13 at L_1, maximum tension	Tension in bar for E_{60} $= 86.3 \times \frac{5}{4} \times \frac{45}{75} \times 2 \times 1.202 = 155.6$
M_5U_6	Load 13 at L_5, maximum tension	Tension in bar for $E_{60} = 86.3 \times \frac{5}{4} \times 1.280 = 138.1$

Problems

50. Uniform live load, 2,000 lb. per foot on bottom chord.
Locomotive excess, 20,000 lb.
Dead load, 800 lb. per foot on bottom chord.
Dead load on top chord and intermediate panel points as shown in figure.

Compute dead stress and maximum live stress of both kinds in each member, following rules given in previous problems as to arrangement of

computations, using special care to number and letter the bars exactly as in figure.

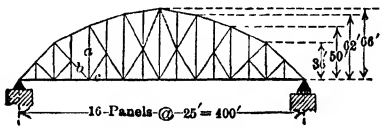


PROB. 50.

51. Uniform live load, 2,000 lb. per foot on bottom chord.
Locomotive excess, 20,000 lb.
Dead load, 1,200 lb. per foot on bottom chord.
Dead load, 600 lb. per foot on top chord.

a. Draw influence line for stress in bar *a* and compute the maximum tension in this bar for above loads.

b. Draw influence line for stress in bar *b* and compute the maximum tension in this bar for above loads.



PROB. 51.

c. Draw influence line for stress in bar *c* and compute the maximum tension in this bar for above loads.

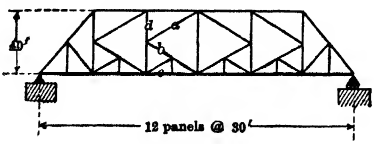
52. Draw influence line for stress in bar *a* of following trusses shown in Prob. 41.

a. Truss *I*. Truss has 12 panels at 25 ft. and height of 60 ft.

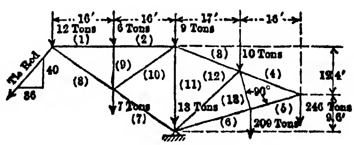
b. Truss *J*. Truss has 8 panels at 20 ft. and height of 30 ft.

53. Dead load, top chord, 2,250 lb. per foot per truss = 135,000 lb. per panel point (approximately).

Dead load, bottom chord, 3,500 lb. per foot per truss = 105,000 lb. per panel point.



PROB. 53.



PROB. 53½.

Uniform live load, bottom chord, 3,000 lb. per foot per truss.

Draw influence lines for stresses in *a*, *b*, *c*, and *d*, and compute values of maximum live stresses of both kinds.

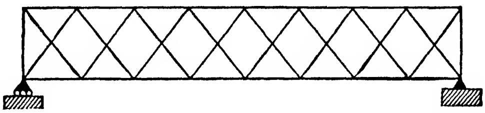
53½. *a.* Compute stresses in bars 10 and 12 by method of moments.

b. Compute stress in bar 11 by method of joints.

CHAPTER VIII

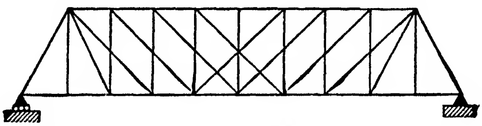
TRUSSES WITH MULTIPLE WEB SYSTEMS, LATERAL AND PORTAL BRACING, TRANSVERSE BENTS, VIADUCT TOWERS

110. Trusses with Multiple Web Systems.—Trusses of this type are statically undetermined but have been frequently built for spans of moderate length, as many engineers believe that more rigidity is thereby obtained. The trusses shown in Figs. 169 and 170 represent the more common types of such structures.



Double-system Warren Truss
This truss should always have an even number of panels
FIG. 169.

The fact that such trusses are indeterminate makes it impossible to determine correctly the stresses by methods previously given. Methods of accurately computing such stresses will be given later in full. But it may be said here that these methods can be applied only to trusses in which the areas of the various



Whipple Truss
This truss should always have an even number of panels
FIG. 170.

members are known or assumed in advance; hence, if used in design they must be applied through a series of approximations, the areas being first determined approximately, the stresses then computed, and the areas revised if necessary, this process being continued until a sufficiently accurate design is finally obtained.[1]

[1] Sometimes the principle of symmetry may be applied to the solution of such trusses; *e.g.*, the stresses in the Warren truss shown in Fig. 169 may be accurately determined for a *symmetrical loading* by using the principle of symmetry and the three equations of statics.

The accuracy of the approximate method ordinarily employed for such trusses is, however, sufficiently high to make unnecessary the employment of more exact methods for the simple types of trusses shown in this article.

The approximate method in common use consists of the separation of the web members into systems, each of which is considered to be entirely distinct from the others. This amounts in reality to dividing the truss into two or more separate trusses with common top and bottom chords. The maximum web stresses in each of these trusses may then be computed in the ordinary manner, each system being assumed to carry only such panel loads as are applied to it. The chord stress in any bar corresponding to each web system may then be computed, the total stress in any chord bar being the sum of the stresses determined for each system. This method of determining stresses is clearly illustrated by the example that follows:

111. Approximate Determination of Maximum Stresses in a Double-system Warren Truss.

Problem: Let the problem be the determination of the maximum stresses in all the bars of the truss shown in Fig. 171, with the following loads:

Dead weight of bridge.

800 lb. per foot per truss, top chord	= 12,800 lb. per panel
400 lb. per foot per truss, bottom chord	= 6,400 lb. per panel

Uniform live load.

3,000 lb. per foot per truss, top chord	= 48,000 lb. per panel
Locomotive excess	= 40,000 lb.

The two web systems into which this truss is assumed to be divided are shown by the full and dotted lines, respectively. The number of redundant bars in the structure may be determined in the usual manner by comparing the total number of bars with twice the number of joints less 3. This comparison shows that the number of bars is 1 in excess of the number needed for statical determination..

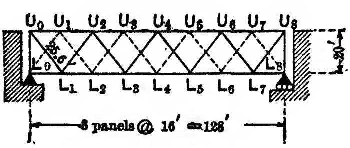


Fig. 171.

Index Stresses.—These may be written for each web system separately in the ordinary manner, the full system being considered to carry only such loads as act at even-numbered top-

chord panel points and odd-numbered bottom-chord panel points and the dotted system to carry all other panel loads. With the web index stresses known, the chord stresses may be written in the ordinary manner, by adding the diagonal stresses at each joint successively, both systems being considered. Figure 172 shows the index stresses for one-half the truss. Under this loading the structure is symmetrical with respect to loads and bars and hence need not be divided into systems to get the dead stresses.

Were this truss to have an odd number of panels, it would be necessary to write the index stresses for the web members in both halves of the truss, since neither system would be symmetrical.

The index stresses were written as usual by beginning at the center of the truss. The left reaction $= 4\frac{1}{2}(6.4) + 3\frac{1}{2}(12.8) = 73.6$, which checks the web index stresses. The chord index stresses may be checked by the method of moments as in the ordinary truss, provided that due allowance is made for the stress in the diagonal cut by the section selected. In this case the stress in the center panel of the bottom chord may be checked by computing the moments about U_4 of the external forces to left of section XY and subtracting from it the moment of the stresses in diagonal U_3L_4, the fact being used that the moment about U_4 of the stress in this diagonal equals the product of its vertical component, *i.e.*, its index stress, and the panel length.

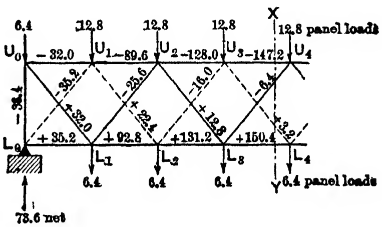


FIG. 172.

The stress in L_3L_4 as determined from the index stresses $= 150{,}400 \times {}^{16}\!/_{20} = 120{,}300$ lb. By the method of moments the stress in $L_3L_4 = \frac{1}{8} \times 1{,}200 \times \frac{128 \times 128}{20} - \frac{3{,}200 \times 16}{20} = 120{,}300$ lb. This value agrees with that obtained from the index stresses and consequently shows the correctness of these stresses. The actual dead stresses may be computed from the index stresses in the usual manner and will not be given.

Maximum Live Web Stresses.—To determine the maximum live web stresses, consider each system as an independent truss,

and determine the stresses in the usual manner by the method of shear. The panel loads will be those corresponding to the panel lengths of the actual truss.

MAXIMUM LIVE WEB STRESSES, UNITS OF 1,000 LB.

Bar	Truss system	Uniform load at panel points	Locomotive excess at panel points	Vertical component in bars	L/h	Stress
U_0L_0	Full	$U_0-U_2-U_4-U_6$	U_0	$\frac{12}{8}48 + 24 + 40 = 136.0$	1.00	-136.0
U_0L_1	Full	$U_2-U_4-U_6$	U_2	$\frac{12}{8}48 + \frac{6}{8}40 = 102.0$	1.28	$+130.6$
L_1U_2	Full	$U_2-U_4-U_6$	U_2	$\frac{12}{8}48 + \frac{6}{8}40 = 102.0$	1.28	-130.6
U_2L_3	Full	U_4-U_6	U_4	$\frac{6}{8}48 + \frac{4}{8}40 = 56.0$	1.28	$+71.7$
L_3U_4	Full	U_4-U_6	U_4	$\frac{6}{8}48 + \frac{4}{8}40 = 56.0$	1.28	-71.7
U_4L_5	Full	U_6	U_6	$\frac{2}{8}48 + \frac{2}{8}40 = 22.0$	1.28	$+28.2$
L_5U_6	Full	U_6	U_6	$\frac{2}{8}48 + \frac{2}{8}40 = 22.0$	1.28	-28.2
L_0U_1	Dotted	$U_1-U_3-U_5-U_7$	U_1	$\frac{16}{8}48 + \frac{7}{8}40 = 131.0$	1.28	-167.7
U_1L_2	Dotted	$U_3-U_5-U_7$	U_3	$\frac{9}{8}48 + \frac{5}{8}40 = 79.0$	1.28	$+101.1$
L_2U_3	Dotted	$U_3-U_5-U_7$	U_3	$\frac{9}{8}48 + \frac{5}{8}40 = 79.0$	1.28	-101.1
U_3L_4	Dotted	U_5-U_7	U_5	$\frac{4}{8}48 + \frac{3}{8}40 = 39.0$	1.28	$+49.9$
L_4U_5	Dotted	U_5-U_7	U_5	$\frac{4}{8}48 + \frac{3}{8}40 = 39.0$	1.28	-49.9
U_5L_6	Dotted	U_7	U_7	$\frac{1}{8}48 + \frac{1}{8}40 = 11.0$	1.28	$+14.1$

As the truss is a Warren truss, no counters are needed, but the maximum stress of each kind should be computed in all bars in which reversal of stress may occur, since the required area of such bars is dependent upon the magnitude of both kinds of stresses.

Maximum Live Chord Stresses.—For the maximum stresses due to the uniform live load, the index stresses should be written and the maximum stresses computed in the ordinary manner. It should be observed that for this truss the live stresses cannot be obtained from the dead stresses by multiplying by the ratio between the two loads since the live stress is not distributed in

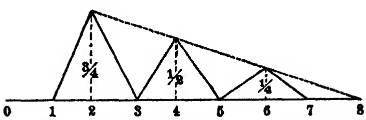
the same manner between the top and bottom chord. For the uniform live load covering the entire span the truss is determinate.

To determine the maximum stresses due to the locomotive excess, it is necessary to decide in which system the bar should be considered in order that the stress may have its maximum value. This can usually be settled by inspection, but if doubt exists the maximum stresses for both systems should be written and the larger value used.

The following table gives the maximum stresses due to locomotive excess in all chord bars:

Bar	System	Load at	Stress
U_0U_1	Full	U_2	$\frac{6}{8}40 \times \frac{16}{20} = -24$
U_1U_2	Dotted	U_3	$\frac{5}{8}40 \times \frac{32}{20} = -40$
U_2U_3	Full	U_4	$\frac{4}{8}40 \times \frac{48}{20} = -48$
U_3U_4	Dotted	U_5	$\frac{3}{8}40 \times \frac{64}{20} = -48$
L_0L_1	Dotted	U_1	$\frac{7}{8}40 \times \frac{16}{20} = +28$
L_1L_2	Full	U_2	$\frac{6}{8}40 \times \frac{32}{20} = +48$
L_2L_3	Dotted	U_3	$\frac{5}{8}40 \times \frac{48}{20} = +60$
L_3L_4	Full	U_4	$\frac{4}{8}40 \times \frac{64}{20} = +64$

Concentrated Load System.—The position of loads for the maximum stresses in this truss due to a concentrated load system may be determined by the use of influence lines. A complete solution for all bars will not be given, but the typical example which follows includes all the important points which are likely to arise.



Influence Line for Vertical Component in Bar U_0L_1

FIG. 173.

Bar U_0L_1. Position of Loads for Maximum Stress for Cooper's E_{40}.—The influence line for the vertical component in this case is shown in Fig. 173 and indicates that heavy loads should lie in panels 1–2 and 2–3, with one of the loads at point U_2. The method of moving up the loads, making use if necessary of the

tangents of the angles between the influence lines and the horizontal, will enable us to determine which load should lie at panel point U_2. As the loads in panels 1–2 and 2–3 will be of the most importance in deciding this question, it is advisable to determine first the position, the loads in these two panels only being considered, and then to investigate to see whether a change in position will diminish or increase the stress. Since the influence line for these two panels is composed of two straight lines, the loads in these panels should be placed so as to give maximum moment at the center of a 32-ft. span. It is evident from inspection that this occurs with load 3, at panel point U_2. With load 3 just to left of panel point U_2, the total load on the left panel of each of the other two-panel segments is greater than that on the right, and movement to the left until load 4 comes to panel point U_2 will not change this relation; hence, it is evident that load 4, at panel point U_2, gives a smaller stress than load 3. Movement to the right until load 2 is at the panel point will decrease materially the stress due to loads in panels 1–2 and 2–3 but will increase the effect of the loads in the other panels. This will probably decrease the stress in the bar; but as the effect of this change cannot be so readily determined by inspection as in the other case, both cases will be computed, as this is simpler than to attempt to determine the exact change by the process of moving up the loads.

Vertical component of stress in bar U_0L_1. Load 2 at U_2.

Load at panel point 2.

$$10 \times \frac{8}{16} + 20\left(\frac{16 + 11 + 6 + 1}{16}\right) = 47.5$$

Load at panel point 4. $\quad 13 \times \dfrac{8 + 13 + 13 + 8}{16} = 34.1$

Load at panel point 6. $\quad 20 \times \dfrac{8 + 13 + 14 + 9}{16} = 55.0$

V.C. in bar from influence line ordinate

$$47.5 \times \frac{3}{4} + 34.1 \times \frac{1}{2} + 55 \times \frac{1}{4} = 66.4$$

Load 3 at U_2.

Load at panel point 2.

$$10 \times \frac{3}{16} + 20 \times \frac{11 + 16 + 11 + 6}{16} = 56.9$$

Load at panel point 4.

$$13 \times \frac{3 + 8 + 14 + 13}{16} + 10 \times \frac{5}{16} = 34.0$$

Load at panel point 6.

$$20 \times \frac{3 + 8 + 13 + 14}{16} + 13 \times \frac{5}{16} = 51.6$$

V.C. in bar equals $56.9 \times \frac{3}{4} + 34.0 \times \frac{1}{2} + 51.6 \times \frac{1}{4} = 72.6$

This latter value is the maximum and should be used in the design. The position of load for the other web members may be determined in a similar manner.

Bar U_3 U_4. Position of Loads for Maximum Stress for Cooper's E_{40}.—The influence line for this bar is shown by full lines in Fig. 174. The values of the ordinates are given by the following computations, the bar in question being considered as a part

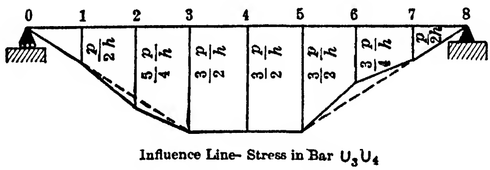


Fig. 174.

of the dotted system for loads at odd-numbered panel points, and as a part of the full-line system for loads at other panel points.

Load at 7: bar in dotted system—ordinate $= \frac{1}{8} \times 4\frac{p}{h} = \frac{p}{2h}$

Load at 6: bar in full system—ordinate $= \frac{2}{8} \times 3\frac{p}{h} = \frac{3p}{4h}$

Load at 5: bar in dotted system—ordinate $= \frac{3}{8} \times 4\frac{p}{h} = \frac{3p}{2h}$

Load at 4: bar in full system—ordinate $= \frac{4}{8} \times 3\frac{p}{h} = \frac{3p}{2h}$

Load at 3: bar in dotted system—ordinate $= \frac{3}{8} \times 4\frac{p}{h} = \frac{3p}{2h}$

Load at 2: bar in full system—ordinate $= \frac{2}{8} \times 5\frac{p}{h} = \frac{5p}{4h}$

Load at 1: bar in dotted system—ordinate $= \frac{1}{8} \times 4\frac{p}{h} = \frac{p}{2h}$

Inspection shows that for this case the moment will certainly increase as the loads come on from the right until load 6 reaches panel point 3. As the loads move still farther, it is more difficult to determine exactly the position for maximum moment. An approximate determination based upon the assumption that the sloping influence lines coincide with the dotted lines may be used, the error thus introduced being comparatively small. This condition being assumed, the position for maximum moment will occur when the load on panel points 5–8 equals that on panel points 1–3.

Try load 6 to left of 3.

Load on 1–3 = 103; load on 5–8 = 118. Therefore, move up load 7.

Try load 7 to left of 3.

Load on 1–3 = 116; load on 5–8 = 108. Therefore, load 7 at panel point 3 will probably give the maximum value.

It should be noticed that, for this position, load 12 is at panel point 5.

To determine the stress for this position, compute the panel loads at panel points 1, 2, 6, and 7. Compute also the panel loads at 3 and 5 due to loads in panels 2–3 and 5–6. Multiply each of these panel-point loads by the corresponding ordinate to the influence line, and multiply the loads in panels 3–4 and 4–5 by the influence line ordinate in these panels. The summation of these quantities gives the stress in the bar.

Stress in U_3U_4. Load 7 at panel point 3.

Load at panel point 1. $10 \times \dfrac{11}{p} + 20 \times \dfrac{13 + 8 + 3}{p} = \dfrac{590}{p}$

Load at panel point 2.

$$20 \times \frac{3 + 8 + 13 + 14}{p} + 13 \times \frac{5}{p} = \frac{825}{p}$$

Load at panel point 3 (loads in panel 2–3 only).

$$20 \times \frac{2}{p} + \frac{13 \times 11}{p} = \frac{183}{p}$$

Load at panel point 5 (loads in panel 5–6 only).

$$20 \times \frac{11 + 6}{p} = \frac{340}{p}$$

Load at panel point 6. $20 \times \dfrac{5 + 10}{p} + 13\dfrac{13 + 8 + 2}{p} = \dfrac{599}{p}$

Load at panel point 7.

$$13 \times \frac{3+8+14}{p} + \frac{13 \times 13}{p} + \frac{16 \times 4}{p} = \frac{558}{p}$$

$$\text{Stress in bar} = \left(\frac{590+558}{p}\right)\frac{p}{2h} + \frac{825}{p} \times \frac{5p}{4h} + \frac{599}{p} \times \frac{3p}{4h} + \left(\frac{183+340+89 \times 16}{p}\right)\frac{3p}{2h} = \frac{4{,}975}{20} = -248.7$$

As the method used for determining the position in this case was not a rigid one, the stress in the bar with load 13 at panel point 5 will be computed for comparison.

Load at panel point 1.

$$10 \times \frac{6}{p} + 20 \times \frac{14+13+8+3}{p} = \frac{820}{p}$$

Load at panel point 2.

$$20 \times \frac{3+8+13}{p} + 13\frac{10+5}{p} = \frac{675}{p}$$

Load at panel point 3 (loads in panel 2–3 only).

$$13 \times \frac{6+11}{p} = \frac{221}{p}$$

Load at panel point 5 (loads in panel 5–6 only).

$$20 \times \frac{11}{p} + 13 \times \frac{2}{p} = \frac{246}{p}$$

Load at panel point 6.

$$20 \times \frac{5}{p} + 13 \times \frac{14+13+7+2}{p} = \frac{568}{p}$$

Load at panel point 7.

$$13 \times \frac{3+9+14}{p} + 2 \times \frac{13 \times 6.5}{p} = \frac{507}{p}$$

$$\text{Stress in bar} = \left(\frac{820+507}{p}\right)\frac{p}{2h} + \frac{675}{p} \times \frac{5p}{4h} + \frac{568}{p} \times \frac{3p}{4h} + \left(\frac{221+246+96 \times 16}{p}\right)\frac{3p}{2h} = \frac{4938}{20} = -246.9$$

or less than the value previously obtained.

112. Approximate Determination of Maximum Stresses in a Whipple Truss.—The Whipple truss shown in Fig. 170 may be treated to a similar manner to the double-system Warren truss. The two systems into which the truss may be divided are shown in Fig. 175 by dotted and full lines, respectively.

This symmetrical truss is determinate under a symmetrical loading provided that it is assumed that the stress in each of the center diagonals in the full-system truss equals zero—the usual assumption in end-supported symmetrical trusses.

Under partial loading the truss has one redundant member, it being assumed that center diagonals are tension members and that only one can act at once. This truss has, however, one element of uncertainty which does not exist in the double-system Warren truss previously treated, *viz.*, that the end verticals U_1L_1 and U_9L_9 do not distinctly belong to either system. This ambiguity

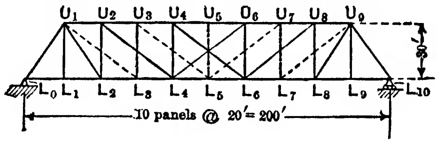


FIG. 175.

is troublesome in determining how to place the live load for maximum stresses. The usual solution in this case is to use these verticals in such a manner as to give the maximum stress in the bar under consideration. For example, if the problem is the determination of the maximum tension in bar U_2L_4, the bar L_9U_9 should be considered as a part of the full system and the bar U_1L_1 as a part of the dotted system and the truss loaded accordingly. The following example illustrates the method of solution for such a truss:

Problem: Let the problem be the determination of the maximum stresses in all the bars of the truss shown in Fig. 175.

Dead weight of bridge.
1,200 lb. per foot per truss, bottom chord = 24,000 lb. per panel
600 lb. per foot per truss, top chord = 12,000 lb. per panel
Uniform live load.
3,000 lb. per foot per truss, bottom chord = 60,000 lb. per panel
Locomotive excess, = 40,000 lb.

Index Stresses.—The dead index stresses shown in Fig. 176 were written for the dotted system by beginning at the center, the bar U_3L_5 carrying one-half of the center panel loads, the dotted system being symmetrical, and panel point 5 at its center. For the full system the shear in the center panel is zero, and the stresses in bars U_4L_6 and L_4U_6 will each be considered as zero. The index stresses present no special difficulty. The only

point to which attention should be called is the necessity for correcting the index stresses in the diagonals in the same manner as in the inclined-chord trusses previously considered.

In this problem the diagonal index stresses are corrected to conform to the slope of the diagonal U_1L_2; *i.e.*, the stresses in the other diagonals are each doubled before the chord index stresses are written: Check calculations.

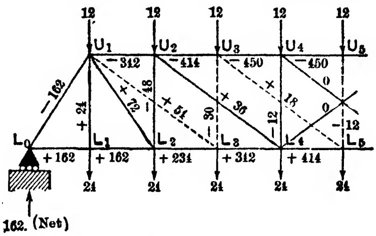


FIG. 176.

$$\text{Stress in } U_4U_5 \text{ by method of moments} = \frac{1}{8}\,\frac{1{,}800 \times 200 \times 200}{30}$$
$$= 300{,}000$$

Stress in U_4U_5 in 1,000-lb. units from index stresses $= 450 \times {}^{20}\!/\!_{30} = 300$

DEAD STRESSES, UNITS OF 1,000 LB.

Bar	Index stress	Ratio	Dead stress	Bar	Index stress	Ratio	Dead stress
L_0U_1	−162	1.201	−194.6	L_0L_1	+162	⅔	+108
U_1L_2	+ 72	1.201	+ 86.5	L_1L_2	+162	⅔	+108
U_1L_1	+ 24	1.000	+ 24.0	L_2L_3	+234	⅔	+156
U_2L_2	− 48	1.000	− 48.0	L_3L_4	+342	⅔	+228
U_3L_3	− 30	1.000	− 30.0	L_4L_5	+414	⅔	+276
U_4L_4	− 12	1.000	− 12.0	U_1U_2	−342	⅔	−228
U_5L_5	− 12	1.000	− 12.0	U_2U_3	−414	⅔	−276
U_1L_3	+ 54	⅗	+ 90.0	U_3U_4	−450	⅔	−300
U_2L_4	+ 36	⅗	+ 60.0	U_4U_5	−450	⅔	−300
U_3L_5	+ 18	⅗	+ 30.0				
U_4L_6	0	⅗	0				

The actual dead stresses are given in the table on page 237 in which the column headed "Ratio" gives for each web member its length divided by its vertical projection; and for each chord member the fraction $\frac{2}{3}$, which equals the horizontal projection of the diagonal U_1L_2 divided by its vertical projection.

Before computing the live stresses the necessity for counters will be investigated. To do this consider each system separately.

Maximum live compression in U_3L_5. Place full uniform live panel loads at L_1 and L_3 and E at L_3; then V.C. $= \frac{4}{10}60 + \frac{3}{10}40$. This is considerably larger than the corresponding figure for dead tension; hence, a counter L_3U_5 is required.

LIVE WEB STRESSES, UNITS OF 1,000 LB.
(This table shows all necessary computations)

Bar	Uniform load at panel points	E at panel point	Vertical component of maximum stress	Ratio	Stress
L_0U_1	L_1 to L_9 inclusive	L_1	$60 \times 4\frac{1}{2} + \frac{9}{10}40 = 306$	1.201	-367.5
U_1L_2	L_2, L_4, L_6, L_8, L_9	L_2	$\frac{21}{10}60 + \frac{8}{10}40 = 158$	1.201	$+189.8$
U_1L_1	L_1	L_1	$60 + 40 = 100$	1.000	$+100.0$
U_2L_4	L_4, L_6, L_8, L_9	L_4	$\frac{13}{10}60 + \frac{6}{10}40 = 102$	$\frac{5}{3}$	$+170.0$
U_2L_2	L_4, L_6, L_8, L_9	L_4	$\frac{13}{10}60 + \frac{6}{10}40 = 102$	1.000	-102.0
U_4L_6	L_6, L_8, L_9	L_6	$\frac{7}{10}60 + \frac{4}{10}40 = 58$	$\frac{5}{3}$	$+96.7$
U_4L_4	L_6, L_8, L_9	L_6	$\frac{7}{10}60 + \frac{4}{10}40 = 58$	1.000	-58.0
U_6L_8	L_8, L_9	L_8	$\frac{3}{10}60 + \frac{2}{10}40 = 26$	$\frac{5}{3}$	$+43.3$
U_1L_3	L_3, L_5, L_7, L_9	L_3	$\frac{16}{10}60 + \frac{7}{10}40 = 124$	$\frac{5}{3}$	$+206.7$
U_3L_5	L_5, L_7, L_9	L_5	$\frac{9}{10}60 + \frac{5}{10}40 = 74$	$\frac{5}{3}$	$+123.3$
U_3L_3	L_5, L_7, L_9	L_5	$\frac{9}{10}60 + \frac{5}{10}40 = 74$	1.000	-74.0
U_5L_7	L_7, L_9	L_7	$\frac{4}{10}60 + \frac{3}{10}40 = 36$	$\frac{5}{3}$	$+60.0$
U_5L_5	L_7, L_9	L_7	$\frac{4}{10}60 + \frac{3}{10}40 = 36$	1.000	-36.0

Maximum live compression in U_2L_4 — load L_1 and L_2 — E at L_2 V.C. = $\frac{3}{10}60 + \frac{2}{10}40 = 26$. This with impact added would be larger than the corresponding figure for dead tension; hence, a counter L_2U_4 should be used.

LIVE CHORD STRESSES, UNITS OF 1,000 LB.
(This table shows all necessary computations)

Bar	Live stress due to uniform load $= {}^{30}\!/_{18}$ of dead stress	Position of E	Stress due to E	Total maximum live stress
L_0L_1	$108 \times {}^{30}\!/_{18} = +180$	L_1	$\frac{9}{10}40 \times \frac{2}{3} = +24.0$	+204.0
L_1L_2	$108 \times {}^{30}\!/_{18} = +180$	L_1	$\frac{9}{10}40 \times \frac{2}{3} = +24.0$	+204.0
L_2L_3	$156 \times {}^{30}\!/_{18} = +260$	L_2	$\frac{8}{10}40 \times \frac{4}{3} = +42.7$	+302.7
L_3L_4	$228 \times {}^{30}\!/_{18} = +380$	L_3	$\frac{7}{10}40 \times \frac{6}{3} = +56.0$	+436.0
L_4L_5	$276 \times {}^{30}\!/_{18} = +460$	L_3	$\frac{7}{10}40 \times \frac{6}{3} = +56.0$	+516.0
U_1U_2	$228 \times {}^{30}\!/_{18} = -380$	L_3	$\frac{7}{10}40 \times \frac{6}{3} = -56.0$	−436.0
U_2U_3	$276 \times {}^{30}\!/_{18} = -460$	L_4	$\frac{6}{10}40 \times \frac{8}{3} = -64.0$	−524.0
U_3U_4	$300 \times {}^{30}\!/_{18} = -500$	L_5	$\frac{5}{10}40 \times \frac{10}{3} = -66.7$	−567.7
U_4U_5	$300 \times {}^{30}\!/_{18} = -500$	L_5	$\frac{5}{10}40 \times \frac{10}{3} = -66.7$	−567.7

The determination of the maximum stresses in a Whipple truss for a concentrated load system should be made in a manner similar to that employed for the Warren truss, influence lines being used to determine the position of loads. Computations for such loads will be omitted as involving no new methods.

113. Skew Bridges.—It is often necessary to construct bridges the abutments or piers of which are not at right angles to the bridge axis Plans of such bridges are shown in Figs. 177 and 178.

In structures of this sort the trusses are frequently unsymmetrical, as is evidently the case for the trusses shown in Fig. 177. The trusses shown in Fig. 178 are symmetrical, but the panel loads are affected somewhat by the skew of the ends If it is

desired to use inclined end diagonals for such trusses, they should both have the same inclination to the horizontal in order that the end portal may lie in a plane. For simplicity in construction the floor beams should be located at right angles to the trusses. In order to satisfy both these conditions, it is frequently desirable to place the end hangers at an inclination to the vertical, as shown in Fig. 179.

The computation of stresses in such trusses may be made in the same manner as in the trusses already considered and requires no special treatment. If difficulties occur in determining the position of the loads, they may usually be solved by using the influence line.

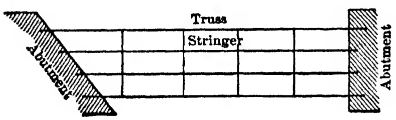


FIG. 177.

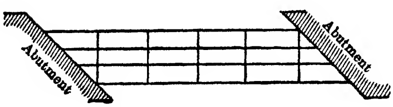


FIG. 178.

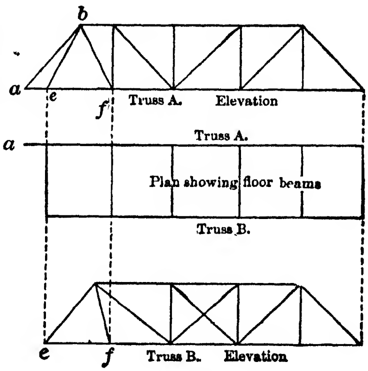


FIG. 179.

114. Lateral and Portal Bracing.—It is evident that a bridge in which the floor beams form the only connection between the trusses would be unstable laterally, especially if of long span. This instability would be due partly to its inability to withstand the force of the wind acting upon the truss itself and upon the train or other live load that may be upon the bridge and partly to the lateral vibration to which it may be subjected by the live load, this being especially severe for railroad bridges exposed to swift and heavy train service. In addition to the insecurity of such a structure as a whole, another disadvantage would be the fact that the top chords would have to be made much heavier than would be the case were they to be rigidly braced, since they would be in the condition of very long columns unsupported laterally, and the extra material used to give them

sufficient strength would, in most cases, be more than sufficient to provide for lateral bracing.

For these reasons, it is considered necessary to use lateral bracing in most bridges. In through bridges, this bracing should consist of a horizontal truss in the plane of the bottom chord, another in the plane of the top chord when the depth permits (trusses of insufficient depth to permit the use of overhead bracing are called *pony* trusses and are described on page 2), and vertical bracing between the verticals of as great a depth as the allowable clearance permits.[1] In deck bridges a horizontal truss

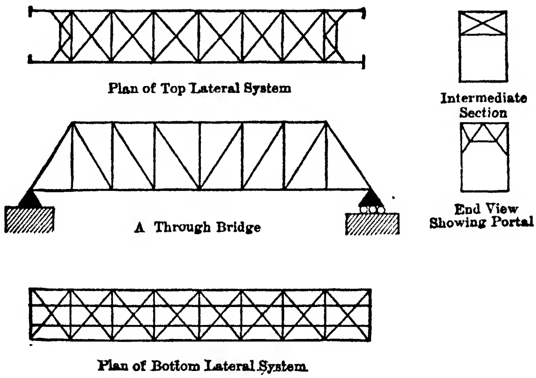


FIG. 180.

may be used in the plane of the upper chord and vertical sway bracing between the vertical members, no horizontal bracing being used in the plane of the bottom chord, or all three systems of bracing may be used. In bridges with solid floors the bracing at the floor level, *i.e.*, the bracing at the bottom chord for a through bridge and at the top chord for a deck bridge, is needed only during erection or during repairs involving the removal of the floor.

In through bridges the end reactions of the top lateral truss cannot be directly transmitted to the abutments owing to the necessity of preserving a suitable opening for the traffic; hence, portal bracing is required in the plane of the end posts, the purpose of this bracing being to tie the end posts together

[1] One of the reasons for using vertical sway bracing when both top and bottom lateral systems are used is to assist in distributing unequal train loads between the trusses. In several important highway bridges constructed in recent years, such bracing is omitted at intermediate panel points.

and make thereby a rigid frame by which the end reactions can be transferred to the abutments.

Figures 180 and 181 show the lateral bracing in through and deck bridges, respectively.

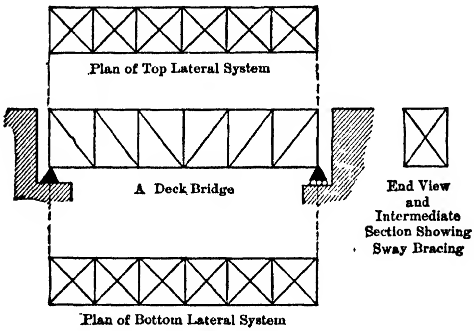


FIG. 181.

115. Lateral-bracing Trusses.—Lateral trusses are usually statically indeterminate, the diagonals consisting of riveted members capable of carrying both tension and compression. The cross struts of the top system and the floor beams in the bottom system (in a through bridge) connect the two sets of diagonals so rigidly that it is impossible to divide into separate trusses. A reasonable assumption for such a truss is to consider the shear in a panel to be divided equally between the two diagonals, one being brought into tension and the other into compression.

116. Approximate Determination of Maximum Stresses in Lateral Bracing.

Problem: Let the problem be the determination of the maximum stresses in the bottom lateral system of a through bridge with eight panels at 25 ft. and with trusses spaced 18 ft. between centers, it being assumed that the laterals are stiff members and able to carry both tension and compression. The horizontal lateral truss is shown in Fig. 182.

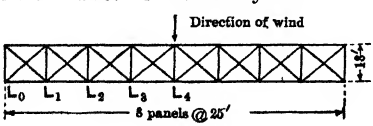


FIG. 182.

Solution: The lateral force acting at the bottom chord will be assumed as a moving force of 500 lb. per lineal foot = 12,500 lb. per panel. It is unnecessary to compute the lateral stresses in the floor beams, since the addition of a slight direct stress in these would be of no importance; hence,

it is immaterial whether this later force is assumed to be distributed between the two chords or is applied entirely to the windward chord. The latter condition will be assumed, however, for ease in computation. For convenience, the components of the diagonal stress at right angles to the axis of the truss will be spoken of hereafter as *vertical components*, and those along the truss axis as *horizontal components.*

Index Stresses.—These will be written for the full load, this being the simplest method of getting the chord stresses, and are shown in Fig. 183. The actual chord stresses will be $^{25}/_{18}$ of the index stresses.

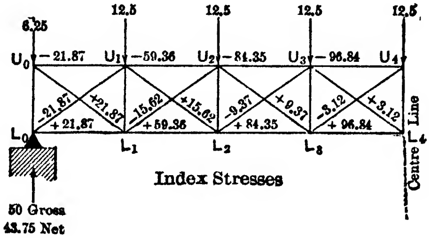


FIG. 183.

It should be noted that the lateral-truss chords are also the chords of the main truss and that the wind stresses in them are sometimes of sufficient size to require additional area in these members, although it is customary to permit higher unit stresses for the combination of live, dead, and wind loads than would be allowable for live and dead stresses only.

Maximum Diagonal Stresses.—The vertical components of the maximum diagonal stresses in 1,000-lb. units will be as follows, it being assumed that the shear in each panel is divided equally between the two diagonals:

Panel 0–1, $\frac{1}{2}(3\frac{1}{2} \times 12.5) = +21.9$
Panel 1–2, $\frac{1}{2}(^{21}/_{8} \times 12.5) = +16.4$
Panel 2–3, $\frac{1}{2}(^{15}/_{8} \times 12.5) = +11.7$
Panel 3–4, $\frac{1}{2}(^{10}/_{8} \times 12.5) = +\ 7.8$

117. Portals. Approximate Solution.—The portal bracing and end posts of a through bridge must be designed to carry to the abutments the reactions from the top lateral system and also to withstand the wind pressure on the end posts themselves, the former being the more important factor. This combination of bracing and end posts is called the *portal* and is a statically indeterminate structure. Accurate solutions of such structures may be made by the method of least work, but the approximate solution that follows is sufficiently accurate for all ordinary cases.

Figure 184 shows a common type of end portal for a through bridge. The statical indetermination is due to the condition at the bottom of the end posts and to the rigidity of the portal brac-

ing. Neither of the posts is pin-ended; *i.e.*, neither has a pin at right angles to the plane of the portal, the main truss pins being in the plane of the portal. The ends of the posts are, however, really fixed to a considerable degree, since they bear upon the foundations, although they are not usually rigidly fixed thereto, and the dead weight of the structure is sufficient to offer a very considerable resistance to overturning under the action of the wind forces.

If the weight of the bridge is sufficient, it is evident that the posts may be treated as if they were fixed at the bottoms. Moreover, if the knee braces M_0I_m and $M_0'I_m'$ are rigidly fixed to the posts, the latter may be considered as fixed at points M_0 and

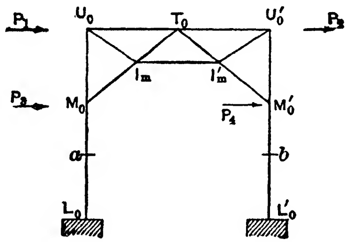


FIG. 184.

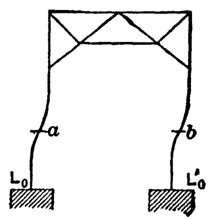


FIG. 185.

M_0'. On the assumption that such is the condition, the posts will bend under the action of the wind forces as shown in Fig. 185, and a point of inflection will occur at a point in each post between the bottom of the knee brace and the bottom of the post. a and b indicate these points of inflection. If the position of each point of inflection is known and if the horizontal reaction at the bottom of each post is also known, the stresses in the structure become determinate, since the moment at a point of inflection must equal zero.[1]

[1] This may be proved as follows:

Let R = radius of curvature at any section of a member exposed to bending.

M = bending moment at this section due to external forces.

I = the moment of inertia at this section.

E = the modulus of elasticity.

From mechanics,

$$\frac{1}{R} = \frac{M}{EI}$$

At the point of inflection the beam must be straight, since at this point the curvature changes; hence, R = infinity, and $1/R = 0$. Therefore, $M = 0$.

It is commonly assumed that each point of inflection occurs midway between the bottom of the knee brace and the bottom of the post. It is also commonly assumed that the portal bracing is so rigid that the distance apart of the posts remains unchanged under the action of the wind forces and that in consequence the horizontal reaction at the bottom of each post equals one-half the sum of the applied loads. The first of these assumptions is only approximately correct, but the latter assumption is practically correct in the ordinary bridge in which the error introduced by the approximation in the design of the end posts is small, since the wind stresses in these members are in themselves small compared with the live and dead stresses, and the percentage error in consequence is still smaller. The portal bracing itself is frequently made considerably larger

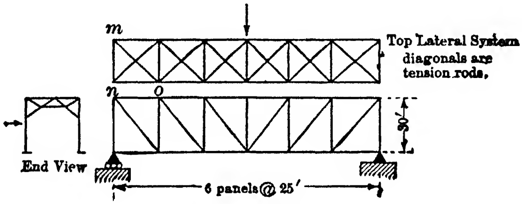


Fig. 186.

than is necessary, owing to the comparatively small magnitude of the wind forces and the difficulty in choosing members with small enough areas that are also suitable in other ways.

With the points of inflection and the distribution of the horizontal reactions between the two posts known or assumed, the computation of the stresses in the various members may be easily made. The structure, however, differs somewhat from those that have been previously treated, since it consists of a combination of columns, carrying direct stresses and bending, and a truss.

The example that follows illustrates the method of computation based upon these assumptions:

Problem: Let the problem be the determination of the stresses in the portal of the bridge shown in Fig. 186.

Solution.—The wind force on top chord at, say 200 lb. per lineal foot of bridge equals 2,500 lb. per panel point per truss. The force applied by the lateral truss to the portal at *m* equals the vertical component in diagonal *mo* plus the panel load at *m*. The sum of these two forces equals 5,000 ×

$\frac{5}{2}$ + 1,250 = 13,750 lb. There will also be a force of 1,250 lb. at *n*. In addition to the wind force acting along the top chord, there will be a uniformly distributed wind force applied directly to the end posts. This will be assumed as 100 lb. per lineal foot of the member. The outer forces acting upon the portal will then be as shown in Fig. 187, with points of inflection and distribution of horizontal forces being assumed as previously stated. The vertical forces and bending moments at the bottoms of these posts may be computed as follows: Let the moment at the bottom of each post = M and the vertical force = V. Then, since the moment at the point of inflection = zero, the moment about an axis through the point of inflection in each post of the forces below that point = $10{,}500 \times 11 - 100 \times 11 \times \frac{11}{2} - M = 0$. Therefore, $M = 109{,}450$ ft.-lb.

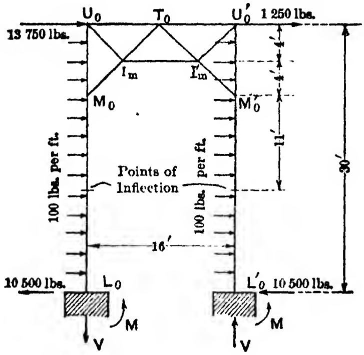


FIG. 187.

The direction of the moment in each case must be counterclockwise, as shown, to balance the clockwise moment due to the horizontal forces.

In order that equilibrium may exist, the moment of the couple due to the vertical forces must equal the moment of the horizontal forces about any axis minus $2M$. Taking the origin of moments at the bottom of either post, the following equation may therefore be written:

$$15{,}000 \times 30 + 6{,}000 \times 15 - 109{,}450 \times 2 - 16V = 0$$

Therefore,

$$V = 20{,}070 \text{ lb.}$$

The next step is the determination of the stresses in the portal members themselves, and the direct stresses, bending moments, and shears in the end posts. It is evident that each main post is a continuous member without hinges. That is, the joint at M_0 can in no sense be considered a pin joint so far as the two sections U_0M_0 and M_0L_0 of this member are concerned, since the stability of the entire structure depends upon the lateral stability of these end posts. Indeed the moment in the post at this point, according to our hypothesis, equals $109{,}450 - 10{,}500 \times 22 + 100 \times 22 \times 11 = -97{,}350$ ft.-lb.[1] The other joints may, however, be pin joints, and will

[1] This value may be verified by considering the portion of the post between M_0 and the point of inflection. The shear at the point of inflection in pounds = 10,500 − 1,100 = 9,400, and the moment is zero; therefore, the moment in foot-pounds at $M_0 = 9{,}400 \times 11 - 100 \times 11 \times \frac{11}{2} = 97{,}350$. It should be observed that it is often simpler to determine the vertical reaction directly by taking moments about one of the points of inflection of the forces above a section through the point of inflection than to do it by the method previously used.

be so considered. Moreover, the joint M_0 will also be considered a pin joint so far as the stress in M_0I_m is concerned; *i.e.*, the stress in M_0I_m will be assumed to be direct stress. To compute the stress in the portal bars, it is necessary to treat the post U_0L_0 as a beam supported at the point M_0 by a truss bar, the direction of which determines the direction of the beam reaction at this point, and at the point U_0 by a reaction which is unknown in direction and which equals the resultant of the unknown stresses in U_0T_0 and U_0I_m. This beam is loaded by a uniform load of 100 lb. per foot over its entire length and by the horizontal forces of 10,500 lb. at L_0 and of 13,750 lb. at U_0. It is also subjected at L_0 to a bending moment of 109,450 ft.-lb. and a tension of 20,070 lb. This condition is shown by Fig. 188, in which the reactions at U_0 and M_0 are represented by their horizontal and vertical components.

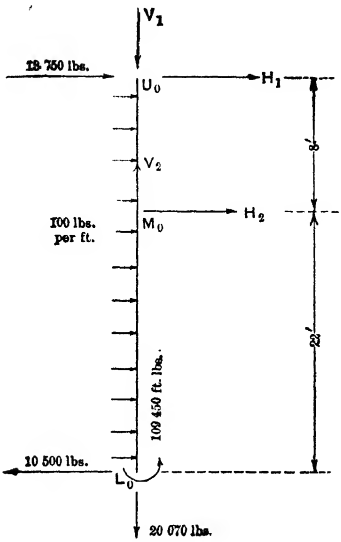


FIG. 188.

The ratio of V_2 to H_2 is determined by the slope of the portal bar M_0I_m. Since this makes an angle of 45°, these two components are equal.

The ordinary equation of statics may now be applied. Application of the equation $\Sigma M = 0$, U_0 being used as the origin of moments, gives the following equation:

$$10{,}500 \times 30 - 3{,}000 \times 15 - 109{,}450 - 8H_2 = 0$$

Therefore,

$$H_2 = +20{,}070 \text{ lb.} = V_2$$

Application of the equation $\Sigma H = 0$, gives the following equation:

$$-H_1 - 20{,}070 - 3{,}000 + 10{,}500 - 13{,}750 = 0$$

Therefore,

$$H_1 = -26{,}320 \text{ lb.}$$

Application of $\Sigma V = 0$ gives the following equation:

$$V_1 - 20{,}070 + 20{,}070 = 0$$

Therefore,

$$V_1 = 0$$

Hence, the stress in bar $U_0I_m = 0$, from which it follows that the stress in $I_mI_m{}'$* $= 0$ and that in $U_0T_0 = -26{,}320$ lb.

* This value may be checked by taking moments about T_0 of the forces to the left of a vertical section through this point. It will be found that this moment = 0; hence, the stress in $I_mI_m{}'$ equals zero.

The actual stress in M_0I_m = stress in I_mT_0 = 20,070 × 1.414 = +28,380 lb.

Proceeding in a similar manner with the other post yields the following results:

Stress in T_0I_m' and $I_m'M_0'$ = −28,380 lb.
Stress in $I_m'U_0'$ = 0
Stress in T_0U_0' = +13,820 lb.

The computations may be checked by considering the joint T_0 and applying the equations of equilibrium. The forces acting at the joint are as shown in Fig. 189 and evidently satisfy the equations of equilibrium.

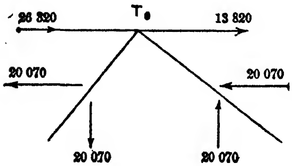


FIG. 189.

Since the stresses in U_0I_m, I_mI_m', and $I_m'U_0$ are zero, it might perhaps be thought that these bars should be omitted; but it should be remembered that the computations are approximate and that the stresses as determined by more exact methods may not be zero. Moreover the appearance of the portal is improved somewhat by the inclusion of these bars.

In addition to the determination of the stresses in the portal bars, the maximum moments and shears in the posts should also be obtained. These are alike for both posts and are shown by the curves of Fig. 190.

The maximum direct stress in each post = 20,070 lb. It is tension in L_0M_0, compression in $L_0'M_0'$, and zero in M_0U_0 and $M_0'U_0'$.

Before leaving this subject, attention should be called to the fact that the wind forces cause stresses in the main truss members. These stresses are relatively small compared with the stresses due to the vertical loads, but may attain high absolute values in large trusses. In the windward trusses, these stresses tend to cause compression in the bottom chord which in conjunction with the stresses due to longitudinal thrust caused by the tractive force may even reverse the normal tension in the end bottom-chord members, which are frequently made as columns to resist this compression.

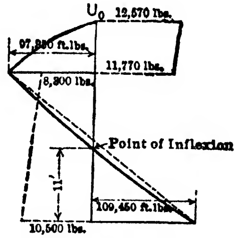


FIG. 190.—Curves of moment and shear in column. Full line shows moment.

118. Portals. Miscellaneous.—The portal treated in Art. 117 represents a common type of portal that is statically determined with respect to the inner forces. Portals are frequently built, however, that are statically undetermined with respect to the inner as well as the outer forces. For such cases the methods used in the treatment of double-system trusses may sometimes be applied. For more complicated portals, special methods

may have to be devised, but the construction of such portals should be avoided.

Portals that lie in a plane inclined to the vertical, as would be the case for a bridge with inclined end posts, may be treated in the same manner as vertical portals, care being taken to use the correct lengths along the posts and not the vertical projections of these lengths.

119. Transverse Bents in Mill Buildings. Approximate Method. A typical structure of this type is illustrated by Fig. 191. The stresses due to the vertical loads may be figured in the ordinary manner, vertical reactions being assumed at points b and i and zero stresses in knee braces ac and hk. To determine the wind stresses, an approximate method similar to that used in computing portals is commonly employed, it being assumed that the horizontal reactions at the bottoms of the columns are equal and that points of inflection occur midway between bottoms of columns and points of connection between columns and knee braces.[1] As in the portals, all joints are assumed to be pin joints except those at a and k, and these latter are also so considered with respect to the stresses in the knee braces themselves, which are assumed to act along the axes of these bars.

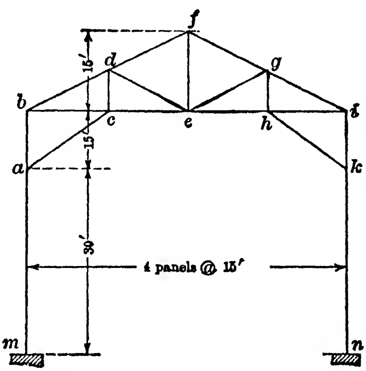


FIG. 191.

The stresses in bars ac and kh may be determined as in the portal by applying the equation $\Sigma M = 0$ to the two columns bm and in, points b and i being used as the origin of moments. The horizontal and vertical forces required at points b and i may then be obtained by the application of the equations $\Sigma V = 0$ and $\Sigma H = 0$ to the two columns. With these values determined, the roof truss may be treated as any simple truss, the outer forces

[1] This assumption should not be made unless warranted by the conditions existing at the bases of the columns. In many structures of this character the resistance to bending moments offered by the column footing is very slight; in such cases, the point of inflection may be assumed as occurring at the base of the column.

being the applied wind loads, the stresses in the knee braces, and the reactions at the column tops.

The complete determination of the stresses in such a structure by the approximate method will not be given, the problem that follows including all the essential points. An accurate determination of these stresses may be made by the theorem of least work or by other methods of solving statically indeterminate structures but will not be given here.

Problem: Compute the horizontal and vertical components of the truss reactions at points b and i and of the knee-brace stresses in the transverse bent, shown in Fig. 191, for a horizontal wind force of 600 lb. per lineal foot on bm and a normal wind force of 400 lb. per lineal foot on bf.

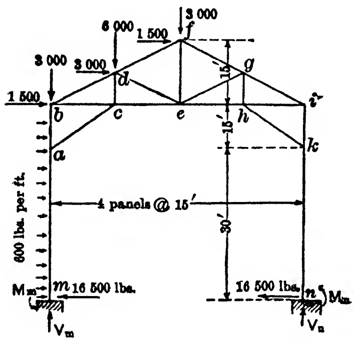


FIG. 192.

Solution: The applied loads will be as shown in Fig. 192. As in the portal the horizontal components at m and n are each assumed to equal one-half the total horizontal force on the structure, thus having a value of 16,500 lb. each. For this case the moment at point m will not equal that at point n, since no wind force is assumed to act on the leeward column. Each moment may be found, however, by applying the equation $\Sigma M = 0$ about the point of inflection of the forces below that point. The equations for these moments are as follows:

$$16{,}500 \times 15 - 600 \times 15 \times {}^{15}\!/\!_{2} - M_m = 0$$

From which

$$M_m = +180{,}000 \text{ ft.-lb.}$$

In a similar manner,

$$M_n = 16{,}500 \times 15 = 247{,}500 \text{ ft.-lb.}$$

The vertical reaction V_m may now be determined by the application of the equation, $\Sigma M = 0$, the point n being used as an origin. The resulting equation is as follows:

$$-180{,}000 + 60V_m + 600 \times 45 \times 22.5 + 6{,}000 \times 52.5 - 12{,}000 \times 45 - 274{,}500 = 0$$

from which

$$V_m = +1{,}200 \text{ lb.}$$

Application of $\Sigma V = 0$ gives $V_n = 12{,}000 - 1{,}200 = +10{,}800$ lb.

The horizontal components in bars ac and hk may next be computed by applying the equation $\Sigma M = 0$, points b and i, respectively, being used as origins of moments.

The equations thus obtained are as follows:

$$180{,}000 + 600 \times 45 \times {}^{45}\!/_{2} + HC \text{ (bar } ac) \times 15 - 16{,}500 \times 45 = 0$$

and

$$247{,}500 - HC \text{ (bar } hk) \times 15 - 16{,}500 \times 45 = 0$$

from which

$$HC \text{ (bar } ac) = -\ 3{,}000 \text{ lb.}$$

and

$$HC \text{ (bar } hk) = -33{,}000 \text{ lb.}$$

The vertical components of these forces equal the horizontal components, since the bars have a slope of 45°.

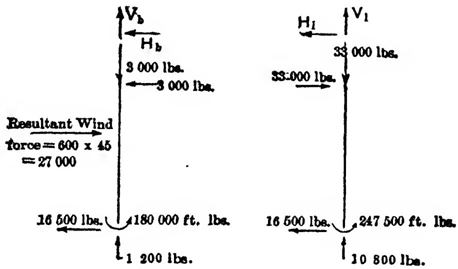


FIG. 193.

The reactions at points b and i may now be determined by applying the equations $\Sigma H = 0$ and $\Sigma V = 0$ to each column as a whole. The forces acting on the columns are shown in Fig. 193, hence,

$$\begin{aligned} H_b &= 27{,}000 - 19{,}500 = +7{,}500 \text{ lb.} \\ V_b &= +\ 1{,}800 \text{ lb.} \\ H_i &= +16{,}500 \text{ lb.} \\ V_i &= +22{,}200 \text{ lb.} \end{aligned}$$

The forces acting on the truss will therefore be as shown in Fig. 194, and the truss may now be computed in the ordinary manner. The moments, shears, and direct stresses in the columns may be determined as in the portal columns previously treated.

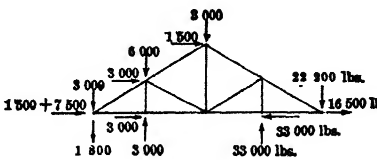
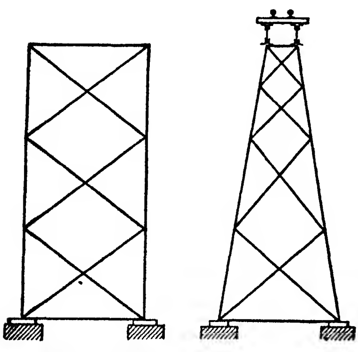


FIG. 194.

120. Viaduct Towers.—In the determination of the stresses in such towers, it is necessary to consider the vertical forces due to the weight of the structure itself and the superimposed load, the horizontal forces due to wind, centrifugal force if the structure is curved, and tractive force. Such towers are usually

composed of four columns, braced transversely and longitudinally, two of the columns being supported on planed base plates and being free to move horizontally in the direction of the longitudinal axis of the structure. To obtain sufficient width at the base to prevent excessive uplift at windward columns when the structure is either unloaded or carrying an empty train, the latter are usually built to a batter, transversely, of one horizontal to six vertical. For symmetry a double system of bracing should be used, and the structure will, therefore, be statically undetermined unless the bracing consists of rods, which is not common in railroad viaducts. The stresses due to the horizontal forces may be computed in a manner similar to that used for the wind bracing systems already computed, *i.e.*, by assuming each diagonal to carry one-half the stress it would be called upon to carry if but one system of diagonals were used. If the vertical loads are symmetrical with respect to the central axis of the tower, they will cause practically no primary stress in the bracing, and the vertical components in the columns will therefore approximately equal the vertical loads since the columns are much more rigid than the bracing. If the vertical loads are not symmetrical with respect to the tower, as may be the case with a structure built on a curve, stresses due to these loads will be caused in the diagonals as well as the columns.



Viaduct Tower

FIG. 195.

The necessary computations for the loaded structure are clearly illustrated by the problem that follows:

Problem: Compute stresses for the tower shown in Fig. 195 due to the following assumed loads:

Dead load, 600 lb. per foot per rail for girder and track and 200 lb. per foot in height of tower for each column.
Live load, 3,000 lb. per foot per rail.
Wind load, 600 lb. per lineal foot of structure applied at base of rail and 50 lb. per foot of height of tower for each column.
Tractive force, 20 per cent of live load applied at base of rail.

The spans on each side of tower are each 56 ft. in length center to center of end bearings.

Dimensions and load concentrations are shown in Fig. 196.

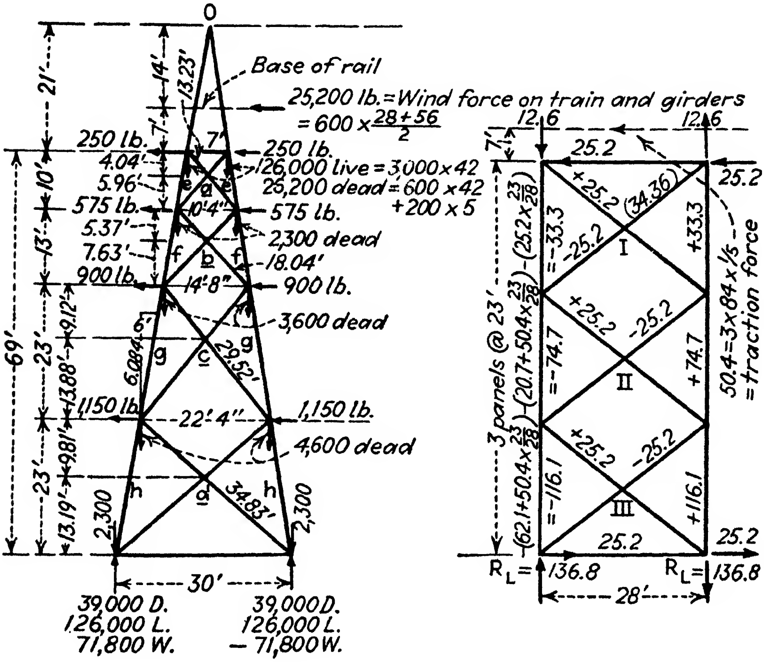


FIG. 196.

And the necessary computations follow:

COMPUTATION OF STRESSES IN VIADUCT TOWER, TRANSVERSE BRACING*

Bars	Horizontal component	Stress
a	$\dfrac{25{,}200 \times 14 + 500 \times 21}{2 \times 25.04} = \dfrac{363{,}300}{50.08} = 7{,}250$	$7{,}250 \times \dfrac{13.23}{8.67} = \pm 11{,}000$
b	$\dfrac{363{,}300 + 1{,}150 \times 31}{2 \times 36.37} = \dfrac{399{,}000}{72.74} = 5{,}500$	$5{,}500 \times \dfrac{18.04}{12.50} = \pm\ 7{,}900$
c	$\dfrac{399{,}000 + 1{,}800 \times 44}{2 \times 53.12} = \dfrac{478{,}200}{106.24} = 4{,}500$	$4{,}500 \times \dfrac{29.52}{18.50} = \pm\ 7{,}200$
d	$\dfrac{478{,}200 + 2{,}300 \times 67}{2 \times 76.81} = \dfrac{632{,}300}{153.62} = 4{,}100$	$4{,}100 \times \dfrac{34.83}{26.17} = \pm\ 5{,}500$

* The column stresses are shown on next page.

COMPUTATIONS OF STRESSES IN VIADUCT TOWER

Columns

Bars	Dead stress	Live stress	Wind stress		
			Moments about intersection of diagonals in each panel	Vertical components of stress	Stress
e	$26,200 \times 1.014 = -26,600$	$126,000 \times 1.014 = -127,700$	$25,200 \times 11.04 + 500 \times 4.04 = 280,200$	$280,200 \div [7 + \frac{1}{3}(4.04)] = 33,600$	$33,600 \times 1.014 = \pm 34,000$
f	$28,500 \times 1.014 = -28,900$	$-127,700$	$280,200 + 25,700 \times 11.33 + 1,150 \times 5.37 = 577,600$	$577,600 \div [10.33 + \frac{1}{3}(5.37)] = 47,600$	$47,600 \times 1.014 = \pm 48,200$
g	$32,100 \times 1.014 = -32,600$	$-127,700$	$577,600 + 26,850 \times 16.75 + 1,800 \times 9.12 = 1,043,800$	$1,043,800 \div [14.67 + \frac{1}{3}(9.12)] = 58,900$	$58,900 \times 1.014 = \pm 59,700$
h	$36,700 \times 1.014 = -37,200$	$-127,700$	$1,043,800 + 28,650 \times 23.69 + 2,300 \times 9.81 = 1,745,000$	$1,745,000 \div [22.33 + \frac{1}{3}(9.81)] = 68,200$	$68,200 \times 1.014 = \pm 69,200$

LONGITUDINAL BRACING

The horizontal components of the stresses in the diagonals due to reactions are shown in Fig. 196 and were obtained by the method of shear. The actual stresses may be obtained from these components in the usual manner and will not be given here.

The maximum uplift on the windward column should also be determined. For the wind load previously considered, the uplift at base of column = 71,800 lb. To this should be added the uplift due to the tractive force. The uplift of the loaded train due to tractive force = $50.4 \times {}^{76}\!/_{28}$ = 136,800 lb. The total reaction on one column due to live and dead loads =

$$39{,}000 + 136{,}800 = 175{,}800 \text{ lb.}$$

Hence, the net uplift = (71,800 + 136,800) − 165,000 = 43,600 lb. It is also common to determine the uplift on the unloaded structure due to an assumed wind force of 50 lb. per square foot on 1½ times the vertical projection of the structure.

In the design of viaduct towers, it is common to assume that the combination of dead, live, wind, and tractive forces will seldom, if ever, occur simultaneously and that in consequence a higher unit stress may be used for these combined forces than would be employed for live and dead stresses only, a common practice being to increase the unit stress 25 per cent. For example, if the allowable unit stress for dead load, and live load corrected for impact, is 16,000 lb., a value of 20,000 lb. would be used for the maximum stresses due to live, dead, wind, and traction. If centrifugal force exists, the stresses due to it should be considered as live stresses but need not be corrected for impact.

If the vertical loads are not applied to the tower symmetrically, they will also cause stresses in the diagonal bracing, since their moment about the point O will no longer equal zero. Such a condition will usually occur if the viaduct is located on a curve, in which case the eccentricity will be due not only to the eccentricity of the center line of the track, but also to the shifting laterally of the center of gravity of the train by the superelevation of the outer rail. The computations for such a case present no difficulty and will not be given.

If rods are used for the diagonal bracing, it will be necessary to use horizontal struts between panel points and only one system of rods will be in action at one time. The computations will be simplified somewhat for this case, but the same general mode of procedure may be adopted. It is frequently assumed in designing such towers that, even when rigid bracing is used, but one diagonal will be in action at any one time in a panel.

The methods given in this article are approximate but are sufficiently accurate for ordinary cases.

Problems

54. *a.* Write the index stresses for this truss for the following dead loads:

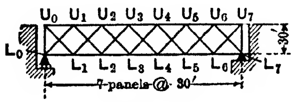
Top chord, 1,200 lb. per foot

Bottom chord, 600 lb. per foot

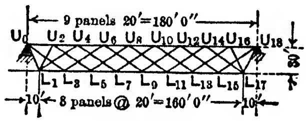
b. Compute maximum live stress in bar U_1L_2 for a uniform live load of 3,000 lb. per foot and a locomotive excess of 30,000 lb., all on top chord.

c. Draw influence line for stress in bar U_2U_3, and compute from it the maximum live stress for the loads given in *b*.

d. Compute maximum live stress in bar U_2U_3 for loads given in *b* by the use of index stresses.



Prob. 54.



Prob. 55.

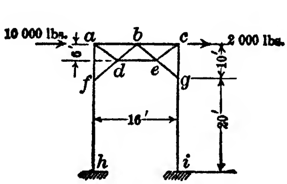
55. *a.* Make a sketch of the truss, showing by different colors the systems into which you would divide it.

b. Write the index stresses, using the dead loads of previous problem.

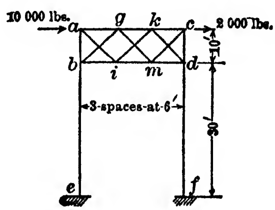
c. Draw an influence line for the stress in bar L_7L_9.

56. *a.* Compute the stress in all portal bars, and the maximum moments, shears, and direct stresses in the columns of this portal for the applied forces shown, assuming that the moment at points *h* and *i* = zero and that diagonals can carry both tension and compression.

b. Compute the stresses in the same bars, assuming that the point of inflection in each post is one-quarter of the distance up from the bottom of the column to the bottom of the knee brace.



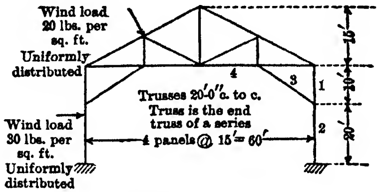
Prob. 56.



Prob. 57.

57. Determine stresses in all bars of this portal, and maximum moments, shears, and direct stresses in the columns, assuming points of inflection in each column at a point midway between the bottom of the column and the bottom of the portal and diagonals to carry either tension or compression.

58. Assuming the columns in this transverse bent to have points of inflection at bottom, compute maximum bending moment in right-hand



Prob. 58.

column and maximum shear in sections 1 and 2. Compute also maximum stresses in bars 3 and 4 and state their character.

CHAPTER IX

CANTILEVER BRIDGES

121. Types of Structures for Long-Span Bridges.—Where the expense of constructing foundations or the restrictions of navigation prohibit the use of spans shorter than 600 to 700 ft., other types of structures than the simple end-supported spans that have been previously considered are more economical and are commonly used. Four types of such structures are frequently employed, *viz.*, the steel arch, the suspension bridge, the continuous bridge, and the cantilever bridge. Of these the suspension bridge and the continuous bridge are statically indeterminate and will not now be considered. The arch and cantilever may be either determinate or indeterminate, but only the determinate types will be treated in this chapter.

122. Cantilever Bridges Described.—In the construction of cantilever bridges, advantage is taken of the fact that a span with

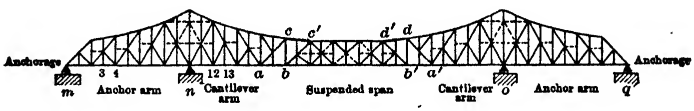


Fig. 197.

one or two projecting arms may be erected by constructing false work under the main span only, the projecting ends being gradually built out from the supporting piers, their weight being balanced by the weight of the main span. For example, the bridge shown in Fig. 197 may be built by using falsework from *m* to *n* and from *o* to *q*, the channel span *no* being sustained during erection by the weight of the anchor arms *mn* and *oq* and by anchorages, at *m* and *q*. A number of large bridges of this type have been constructed in the United States, among which may be mentioned the Red Rock Bridge, the Poughkeepsie Bridge, and the Beaver Bridge the outline of which is shown in Fig. 197. The Harvard Bridge at Boston is a plate-girder cantilever. The Quebec Bridge across the Saint Lawrence River

with a center span of 1,800 ft., has the longest clear span of any cantilever bridge in the world and its span is surpassed only by the spans of several suspension bridges. If the suspended span is omitted and the cantilever arms are connected, the bridge becomes a continuous bridge and is statically indeterminate. The Queensboro Bridge in New York, N. Y., is an important example of a bridge of this type. Such a bridge can be built along more graceful lines than if a suspended span is used, and troublesome details at the connection of the suspended span and the cantilever arm avoided.

123. Equations of Conditions.—Let Fig. 198*a* represent, diagrammatically, a cantilever bridge of three spans. As shown, there are eight unknown components of the outer forces, *viz.*, two at each point of support. Evidently this structure is statically indeterminate to a high degree. The most obvious method of reducing the degree of indetermination is to fix the

FIG. 198*a*.

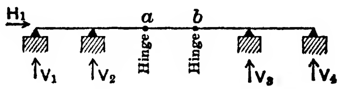


FIG. 198*b*.

direction of some of the reactions. Since one of the reactions, at least, must have a horizontal as well as a vertical component to give stability, it is possible in this manner to eliminate only three of the eight unknowns, this being insufficient to make the structure statically determined. The remaining equations necessary to secure statical determination may be established by the method of construction and are called *equations of condition.* Such equations may be obtained by introducing hinges in the middle span, as indicated in Fig. 198*b*, these hinges being so constructed as to make it impossible to transmit bending moment through them. This construction therefore gives the two following additional equations:

$$\Sigma M_a = 0 \qquad \text{and} \qquad \Sigma M_b = 0$$

These equations signify that the moment of all the forces on *either* side of either hinge about an axis passing through the hinge at right angles to the plane of the forces equals zero. These equations should not be confounded with the general equation $\Sigma M = 0$, which is also applicable at the hinges, but which means

that the moment of all the forces on *both* sides of any section about *any* axis perpendicular to the plane of the forces equals zero.

With regard to the moment about the hinges, it should be noted that although the moment about a of all the forces to the left or right thereof $= 0$, it should not be supposed that this gives two independent equations; for if the moment about a of all the forces to the left of a equals zero, the moment of all the forces to the right of a about the same point must also equal zero, such a result following at once by the subtraction of the former equation from the general equation $\Sigma M = 0$. There

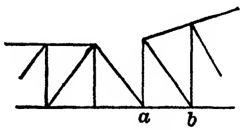


FIG. 199*a*.

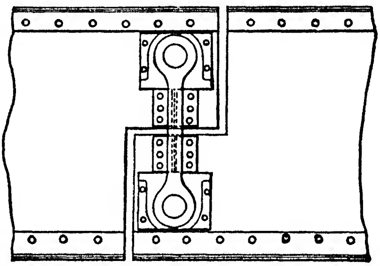


FIG. 199*b*.

are then but five entirely independent equations; hence, five and only five unknown quantities can be determined, and the structure is, therefore, determinate. Were there more than five independent equations, the structure would be unstable; were there less, it would be statically undetermined.

The simplest method of providing a hinge in a truss is to omit a chord bar in one panel as is done in Fig. 199*a*. Evidently the moment about an axis through a of all the forces on either side of a must equal zero, since the structure can by its construction offer no resistance to moment at this point.

For a plate-girder cantilever a hinge may be constructed as shown in Fig. 199*b*.

As it is uneconomical in practice to have a long cantilever bridge restrained longitudinally at one point only—such a condition involving the transmission of a longitudinal force, if applied at one end of the structure, throughout the entire length of the bridge—it is common to fix the structure longitudinally at more than one pier and to omit a bottom-chord bar at one or more

of the hinges, the transmission of longitudinal forces along the entire length of the bridge being thus avoided. For the case illustrated by Fig. 199*a*, the omission of bar *ab* accomplishes this result.

The application of these methods to the structure shown in Fig. 197 would involve the omission of bars *cc'*, *d'd*, and *ab*, and of the rollers at *o*. In practice the bars mentioned would not actually be omitted, as they may be necessary in erection and improve the appearance of the structure. They should, however, be made adjustable and incapable of resisting a longitudinal thrust.

124. Anchorage.—Since a load on the suspended span or on the cantilever arms may cause negative reactions at points such as *m* and *q*, Fig. 197, which may exceed the dead reactions at these points, it may be necessary to anchor the structure to the masonry and to provide sufficient weight in the piers to equal this uplift. The anchorage usually consists of girders embedded in the pier and fastened to the structure by eyebars. The freedom to move horizontally may be obtained by rollers or other devices.

125. Reactions. Cantilever Trusses.—The reactions of structures of this type may be determined by the application of the three equations of statics combined with the equations of condition, in the same general manner as for simple trusses and girders. The problem is, however, more complicated than for structures supported at two points, and in consequence influence lines for certain of the reactions in typical cantilevers will be given and methods of determining the reaction values stated.

Consider first the structure illustrated by Fig. 197. For this cantilever the trusses *mbc* and *db'q* evidently act like beams supported at two points only and supporting the suspended span *bb'* at their ends. That this follows from the application of the equations of equilibrium and condition may be proved in the following manner:

Assume a vertical force *P*, upon the truss *db'q*. For this condition the forces acting to the left of *b* are the same as the forces acting to the left of *b'*, *viz.*, V_1 and V_2, Fig. 200, and the moment of these forces about each of the hinges *b* and $b' = 0$.

Therefore, the following equations may be written:

$$(1)\ V_1(L_1 + L_2) + V_2L_2 = 0$$
$$(2)\ V_1(L_1 + L_2 + L_3) + V_2(L_2 + L_3) = 0$$

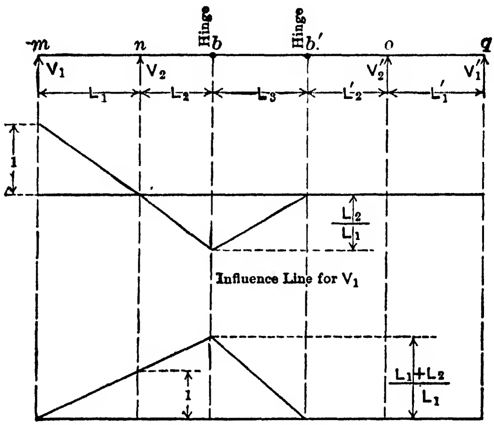
Subtracting (1) from (2) gives $V_1L_3 + V_2L_3 = 0$.
Therefore,

$$V_1 + V_2 = 0;$$

hence,

$$(V_1 + V_2)L_2 = 0.$$

Subtracting this latter equation from (1) gives $V_1L_1 = 0$. Therefore, $V_1 = 0$, $V_2 = 0$, and $V_1' + V_2' = P$; hence, the span $db'q$, when loaded, acts like a simple beam, since the moment at each end is zero, and the sum of the reactions equals the load.



Influence Line for V_2

FIG. 200.

Now consider a load P on span bb' at a distance x from b'. The following equations may be written for the moments about b and b', respectively:

$$(3)\ V_1(L_1 + L_2) + V_2L_2 = 0$$
$$(4)\ V_1(L_1 + L_2 + L_3) + V_2(L_2 + L_3) - Px = 0$$

Subtracting (3) from (4) gives $V_1 + V_2 = Px/L_3$ = positive shear at b. The moment at n equals the moment at b plus the product of the shear at b and the lever arm L_2; it therefore equals $-(Px/L_3)L_2$, the moment at b being zero by construction. This moment also equals $-V_1L_1$; hence, the following equation may be written:

$$\left(\frac{Px}{L_3}\right)L_2 + V_1L_1 = 0$$

whence

$$V_1 = -\left(\frac{Px}{L_3}\right)\left(\frac{L_2}{L_1}\right)$$

This is identical with the reaction that would be obtained if the span bb' were assumed to be supported on the ends of the two simple beams mb and qb'. In a similar manner the reaction at q may be shown to equal the reaction that would exist if a similar assumption were to be made; hence, the statement is proved that the reactions in a structure such as that shown are identical with the reactions which would occur if the structure were to be considered as composed of two independent beams mb and qb', supporting the simple span bb' at their ends.

The influence lines in Fig. 200 show clearly the reactions due to loads in the different portions of the structure. It should

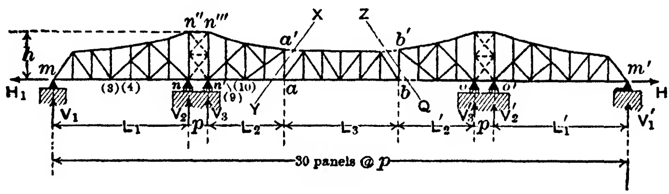


FIG. 201.

be noted that these influence lines would be unchanged and statical determination accomplished by omitting the rollers at o and bar ab in Fig. 197.

The cantilever shown in Fig. 201 differs somewhat from that of Fig. 197, some of the equations of condition for the structure being established by the omission of diagonals over the central piers. The structure as shown has eight unknown reactions, six vertical and two horizontal. To determine these unknowns, there are available, in addition to the three general equations of statics, five equations of condition. Two of these condition equations are obtained by the insertion of hinges at a and b, *i.e.*, by the omission of upper chord bars at these points; two by the omission of diagonals over the center piers,[1] and one by the omission of the bottom-chord bar in the cantilever arm adjoining

[1] Diagonals may be used in the towers for purposes of bracing, but they should be of small size and offer slight resistance to distortion. This same device is frequently employed in partially continuous drawspans.

the hinge at b. Of these condition equations the latter is of use in determining reactions due to horizontal forces only (wind or longitudinal thrust of train) and in combination with the equation $\Sigma H = 0$ is sufficient for this purpose. The remaining four equations of condition and two of statics may be used in the following manner to determine the six vertical reactions:

Let S = shear in panel nn' = 0 by construction.
M_n = moment at n.
M_n' = moment at n'.
M_a = moment at a = 0 by construction.
M_b = moment at b = 0 by construction.
S_a = tension in hanger aa'.

Case 1. Load on suspended span ab. Consider the load P at distance x from b and the portion of the structure between sections XY and ZQ. The following equation may be written:

$$M_b = Px - S_a(L_3) = 0$$

Therefore,

$$S_a = \frac{Px}{L_3}$$

But S_a is the supporting force at the left end of the suspended span and equals the reaction at the corresponding end of a simple end-supported span. It is evident, therefore, that the suspended span ab acts like a simple end-supported truss, since the moment at each end equals zero and the reactions are inversely proportional to the distance of the load from either end. It should be observed that no stress is caused in the hangers at a and b by a load unless it is applied to the suspended span.

Case 2. Load P on cantilever arm $n'a$ at distance x from n'. For this case,

$$\begin{aligned} M_a &= M_n' + (S + V_3)L_2 - P(L_2 - x) \\ M_n' &= M_n + SP \\ V_1 + V_2 &= S \\ M_n &= V_1L_1 \\ S_a &= 0 \end{aligned}$$

By construction, $M_a = 0$, and $S = 0$.
Therefore,

$$M_n' = M_n \qquad \text{and} \qquad V_1 = -V_2$$

and

$$M_n' + V_3L_2 - P(L_2 - x) = 0$$

Hence,

$$V_1L_1 + (V_3 - P)L_2 + Px = 0$$

But

$$V_1 + V_2 = S = 0$$

and

$$V_3 - P = S,$$

since

$$S_a = 0$$

Hence,

$$V_3 - P = 0 \qquad \text{and} \qquad V_3 = P$$

Therefore,

$$V_1L_1 = -Px$$

Hence

$$V_1 = -\frac{Px}{L_1} = -V_2$$

It follows that, for a load on the cantilever arm an', V_1 equals the reaction that would occur on the truss ma if points n and n' coincided, $V_2 = -V_1$, and $V_3 = P$.

Case 3. Load P on anchor arm mn at distance x from n.

For this case,

$$Sa = 0$$
$$M_n = M_n' = 0$$
$$V_3 = 0$$

Also,

$$V_1L_1 - Px = M_n = 0$$

and

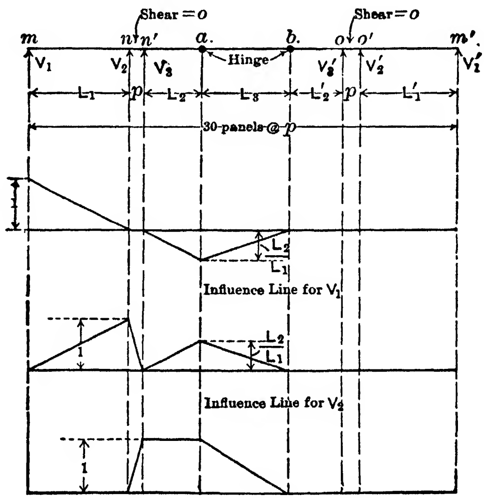
$$V_1 = \frac{Px}{L_1}$$

Hence, for this case, the anchor arm mn acts like a simple span supported at m and n.

Figure 202 shows influence lines for reactions V_1, V_2, and V_3 and should offer no difficulty to the student.

From the influence lines of Fig. 202, it is evident that for a uniform live load in a cantilever like that shown in Fig. 201 the maximum upward reaction at m occurs with mn fully loaded and maximum negative reaction with $n'b$ fully loaded. Also V_2 is always upward and has its maximum value when the load extends from m to b, while V_3 is also upward for any loading and has a maximum value with load from n to b.

126. Shears and Moments. Cantilever Trusses.—With the reactions known, the shears and moments at any section of a cantilever truss may be readily determined. The influence lines in Figs. 203 and 204 show clearly the variations in these functions for certain typical portions of the anchor and cantilever arms of the trusses shown in Figs. 197 and 201. No influence lines are drawn for the suspended spans since these, as has been shown, may be treated like any simple span.



Influence Line for V_3

FIG. 202.

127. Bar Stresses. Cantilever Trusses.—The determination of the bar stresses in cantilever trusses involves no special difficulties and may be accomplished by the use of the methods employed for simple trusses. Influence lines may be employed for determining the position of the loads, if concentrated load systems are to be used. For structures of the magnitude and weight of such bridges, however, the actual use of concentrated load systems for the stresses in the main truss members is generally unnecessary, an equivalent uniform load giving nearly if not quite as accurate results.

The determination of the position of a uniform live load for maximum stress in each bar and the computation of that stress may be accomplished by the use of influence lines if desired.

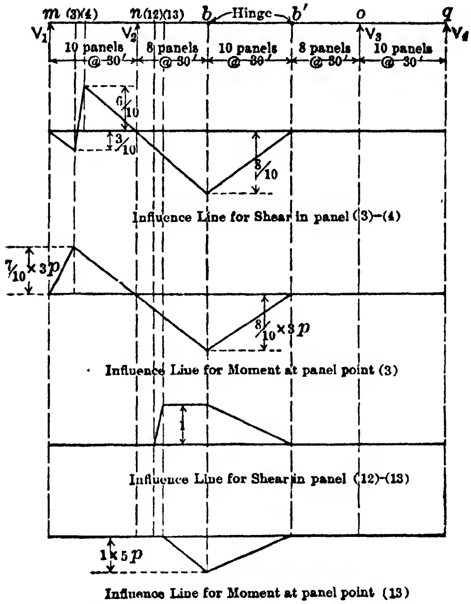


FIG. 203.—(Refers to truss on page 258.)

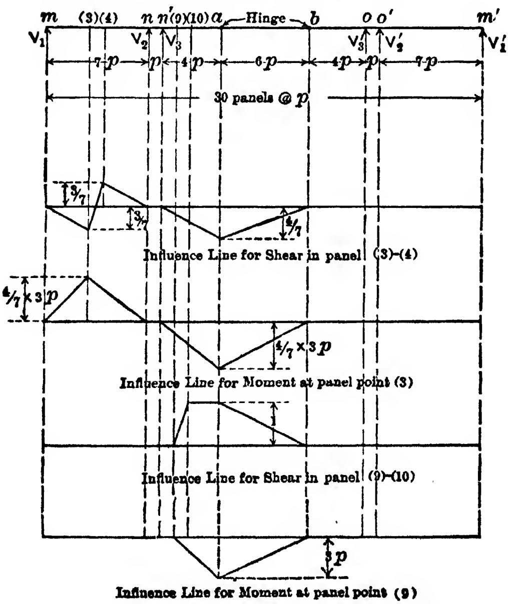


FIG. 204.—(Refers to truss on page 263.)

An influence table similar to that prepared for the three-hinged arch given later showing the stress in each bar for a load at every panel point should, however, generally be prepared to facilitate the computation of the stress due to the dead load, which in a large structure should not be taken as uniformly distributed, and with this table once prepared no advantage would be gained by using influence lines. The influence line in Fig. 205 is given to illustrate the variation in stress in a particular bar rather than for its aid in computing the stress. This statement also applies to the influence lines of the previous articles.

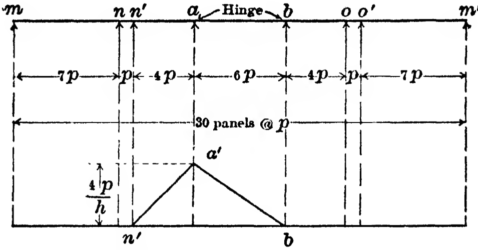


FIG. 205.—Influence line for stress in bar $n''\ n'''$, truss shown in Fig. 201.

If it is desired to use influence lines to check the tabular results, the actual stress may be determined most readily for uniform loads by multiplying the areas between the influence line and the horizontal axis by the proper load.

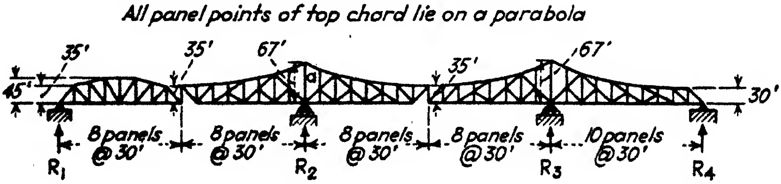
On referring to Fig. 205, it is evident that the maximum stress in bar $n''\ n'''$ will occur with the truss loaded with the uniform live load from n' to b and that its value equals the product of the area $n'a'b$ and the combined live and dead loads per foot, provided that these are uniformly distributed.

Problems

59. *a.* Show that this structure is statically determined with respect to the outer forces.

b. Draw influence line for reaction at R_2.

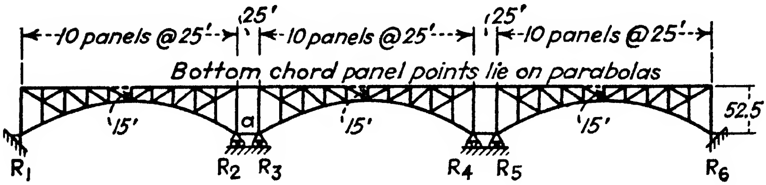
All panel points of top chord lie on a parabola



PROB. 59.

c. Draw influence line for stress in bar *a* (vertical at R_2), and compute its maximum value for a uniform live load of 3,000 lb. per foot.

60. Same as Prob. 59 except for figure.



PROB. 60.

CHAPTER X

THREE-HINGED ARCHES

128. Characteristics of the Arch.—The essential difference between the ordinary arch and the girders and trusses that have hitherto been investigated is that the stresses in an arch may be confined to compression and shear, whereas in trusses and girders large tensile stresses are also developed. This possible elimination of tensile stress in the ordinary arch rib is due to the fact that both ends of the arch are fixed in position by construction. Hence, each reaction has a horizontal component even under vertical loads; in consequence the reactions converge, and if the shape and thickness of the arch rib are properly chosen, the resultant force at each section for any given position of the loads may be made to pass through the centroid of the section and therefore cause no bending moment, or so near the centroid that the tensile fiber stress due to the bending moment caused by the eccentricity is insufficient to overcome the compression due to the thrust.

The advantage of the arch form was well known to the ancients, as is shown by the many stone arches constructed by the Romans and even by older races, and the arch remains to the present day one of the most useful and graceful of structures, its employment being frequently dictated both by aesthetic and utilitarian considerations.

129. Types of Arch.—Up to a comparatively recent period the arch was always constructed as a statically undetermined structure, similar to that shown in Fig. 206, which represents the conventional masonry arch with neither reaction fixed in direction, magnitude, or point of application, the arch being in consequence statically undetermined in a threefold degree, having six unknowns.

With the application of iron and steel to bridge construction came a recognition of the advantage of statical determination, and metal arches began to be constructed in which some of the unknowns were eliminated by the insertion of hinges. Such

arches are shown in Figs. 207 and 208. If in the arch shown in Fig. 208 a hinge is inserted at the center similar to that of the arch shown in Fig. 207, the arch becomes a three-hinged arch and is statically determined. The ribs of metal arches may be formed

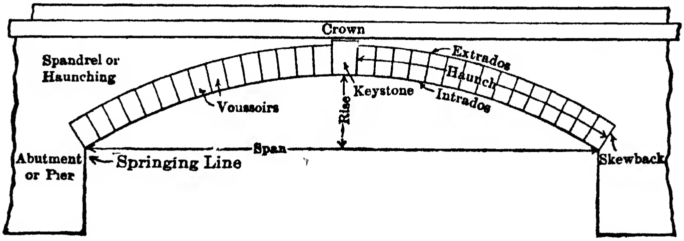


Fig. 206.—Masonry arch.

of plates and angles as in plate girders; of cast-iron or cast-steel segments riveted together; or of riveted trusses.

In recent years a considerable number of two-hinged and three-hinged reinforced-concrete arch bridges have been con-

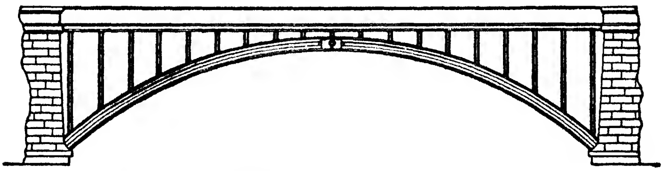


Fig. 207.—Metal arch with one hinge.

structed with separate ribs replacing the arch barrel and with the loads applied through columns or transverse walls, many of the advantages of the metal arch being thus secured. Figure 209 illustrates such an arch. It should be said, however, that

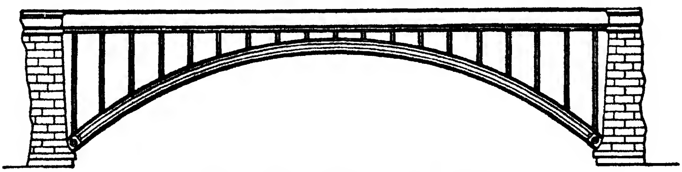


Fig. 208.—Two-hinged metal arch.

the common type of reinforced-concrete arch is the barrel arch without hinges.

One other important type of metal arch, the spandrel-braced arch, is shown by Fig. 210. This structure is in reality a combination of a truss and an arch rib. As will be shown later, if

the arch rib in a three-hinged spandrel-braced arch structure is constructed to a parabolic curve, the diagonals and top chord will not be in action under a full uniform load, the arch rib in that case acting like the arches previously described, the loads being applied through the vertical posts.

The establishment of a center hinge involves either the omission of a top-chord bar in one of the panels adjoining the center

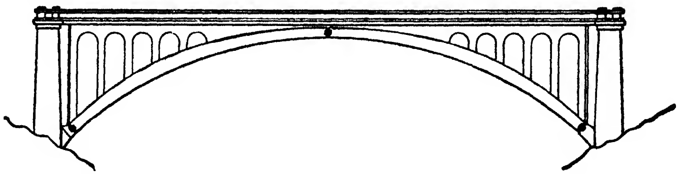


FIG. 209.—Three-hinged reinforced-concrete arch.

panel point or the making of one of these an idle bar incapable of carrying stress.

Like the other arches the spandrel-braced type is frequently constructed as a two-hinged arch.

The three-hinged arch is the only type that will be considered in this chapter, the statical indetermination of the other forms requiring the development of other than statical methods as a preliminary to their investigation.[1]

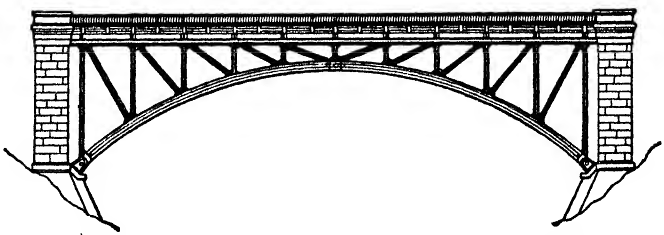


FIG. 210.—Three-hinged spandrel-braced metal arch.

130. Reactions. Three-hinged Metal Arches.—These may be computed for any position of the load by the application of the three general equations of statics combined with the equations of condition established by the hinges.

If the end supports are at the same elevation, as is generally the case, the horizontal components of the reactions balance

[1] Such arches are fully considered in Spofford's *Theory of Continuous Structures and Arches*, McGraw-Hill Book Company, Inc., New York, 1937.

and hence have no effect upon the vertical reactions, which would be the same as for a simple truss or girder of the same span. To obtain the horizontal reactions, it is necessary to make use of the equation of condition, *viz.:* the moment about the center hinge of all the forces on *either* side of that hinge equals zero. The application of this equation is so simple as to need no expla-

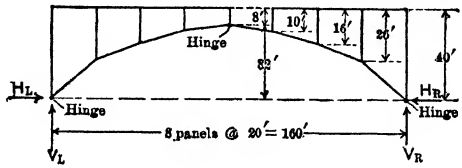
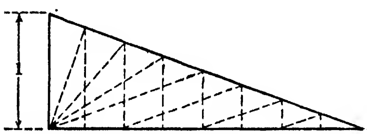


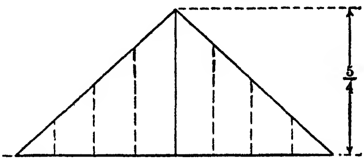
FIG. 211.

nation. The influence lines for the vertical and horizontal components of the reactions are given in Figs. 212 and 213 for the arch shown in Fig. 211 in order to show clearly the variations in the reactions as the load crosses the structure. These lines are also correct for the spandrel-braced arch shown in Fig. 210, provided that the arch rib has the same dimensions, since the construction above the arch rib has no influence upon the value of the



Influence Line for V_L

FIG. 212.



Influence Line for H_R and H_L

FIG. 213.

reactions. For a uniform load the maximum value of both the horizontal and the vertical component, and hence of the actual reaction, evidently occurs when the entire structure is loaded. For the arch under consideration the maximum reaction for a concentrated load occurs when the load is placed at the center of the span.

In designing the piers, it is as important to know the direction of the reaction as its magnitude. Both may be determined graphically for a load at any panel point by the methods shown in Fig. 212, in which the sloping dotted lines showing the direction and magnitude of the reactions are determined by laying off at the foot of each vertical the corresponding horizontal ordinate

as obtained by scale from Fig. 213. It will be observed that the direction of the left reaction is constant for loads on the right half of the structure. This is not accidental but is due to the effect of the center hinge. Since with a load on the right half of the structure the only force to the left of the center hinge is the left reaction and since the moment about the center hinge equals zero, the left reaction must pass through it. This principle may be stated as follows: For a load to the *right* of the center hinge the direction of the *left* reaction coincides with a line drawn through the left and center hinges, and vice versa. It is evident that, although the reaction at one end due to the live load on the other half of the arch may pass through the end and center hinges, the actual reaction will not have this direction, since such a condition would involve the entire absence

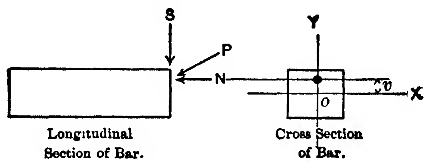


FIG. 214.

of dead load in the half of the structure adjoining the reaction in question.

With a concentrated load system the maximum vertical reactions may evidently be determined as for any simple beam, and the shape of the influence line shows that the maximum horizontal component will occur for that position of the live loads which would give a maximum moment at the center of a span of the length of the arch and hence may be determined by influence lines as illustrated in Art. 109. The exact position for the maximum value of the reaction itself is less easily determined, but an equivalent uniform load may be used with safety to determine the actual maximum reaction.

131. Maximum Stresses in Elastic Arch Ribs.—The maximum fiber stress at any section of the arch rib of a structure like that shown in Fig. 208 may be determined if the direction, point of application, and magnitude of the resultant force at the section are known. It is a well-known principle of mechanics that a force P applied at one of the principal axes OY of a cross section of a straight elastic bar, in the manner shown in Fig. 214, causes a

direct fiber stress at a distance c from the other principal axis OX, which may be expressed by the equation

$$s = \frac{N}{A} \pm Nv\frac{c}{I}$$

in which N = normal component of the force P
S = transverse component
A = area of cross section
v = distance of point of application of force from the axis OX
I = moment of inertia of cross section about axis OX

The shearing stress due to the transverse component S may for such a case be computed in the same manner as in an ordinary beam.

For a curved bar, such as an arch rib, the formula just given for the value of s is not strictly correct but for exactness should be replaced by the following equation:

$$s = \frac{N}{A} \pm \frac{Nv}{AR} \pm \frac{NvRc}{(R + c)\int \frac{Ry^2dA}{R + y}}$$

in which R = radius of curvature of the axis of the arch
y = distance of any fiber from axis OX

For bridge arches the radius of curvature R is usually very large, compared with the dimensions of the cross section; hence,

$$\int \frac{Ry^2dA}{R + y} \text{ equals } I \text{ very nearly and } \frac{R + c}{Rc} = \frac{1}{c}$$

v is also small compared with R in any well-proportioned arch; hence, the second term of the expression for s, above, may be neglected with but little error, giving for a closely approximate value

$$s = \frac{N}{A} \pm Nv\frac{c}{I}$$

the same expression as for a straight bar.[1] In this formula

[1] The error made by these approximations is extremely small, even for arches with so sharp a radius of curvature as an ordinary sewer arch. The general formula should, however, be employed in determining the stress in a curved bar such as a crane hook.

Nv = external bending moment on the section; hence, the formula may be written

$$s = \frac{N}{A} \pm \frac{Mc}{I}$$

in which $M = Nv$.

In order to determine the maximum compression at the cross section of any arch rib, it is necessary to determine the position of the loads that will produce the maximum value of the expression $\frac{N}{A} + \frac{Mc}{I}$, and to determine the maximum tension (or minimum compression), the position of loads giving the maximum negative value or minimum positive value of $\frac{N}{A} - \frac{Mc}{I}$ must be determined.

These equations are applicable for arches that can carry both tension and compression. If the arch can carry compression only, as in the case of the ordinary stone arch, they are correct only when the value of $\frac{N}{A} - \frac{Mc}{I}$ is positive. Stone arches should, however, be so proportioned that this condition will always exist.

For uniform loads the position for maximum direct fiber stresses may be determined by an influence table, in which the maximum values of the direct tension and compression at various sections for load unity at each panel point are given, a sufficient number of sections being chosen to ensure economy and safety in the design.

For arches carrying concentrated load systems, influence lines may be drawn for maximum stresses of both kinds at as many sections as may be desired, and the position of the loads determined in the manner previously used for trusses, or an influence table may be employed and the maximum stresses determined by trial, the value of the panel loads for probable positions being first tabulated. The properties of the kernel of the cross section may also be advantageously used for this case (see standard books on mechanics).

Examples of the computations for such an arch will not be given, as it involves nothing but the application of the principles already thoroughly illustrated for other structures, and the

student who is familiar with these principles should have no difficulty in applying them to such a structure.

132. Parabolic Three-hinged Arches.—In practice, three-hinged arches are frequently constructed either with a parabolic axis or with panel points lying on a parabola. If the end pins of such an arch are at the same elevation and if the load is vertical, uniformly distributed, and applied to the arch by vertical posts,

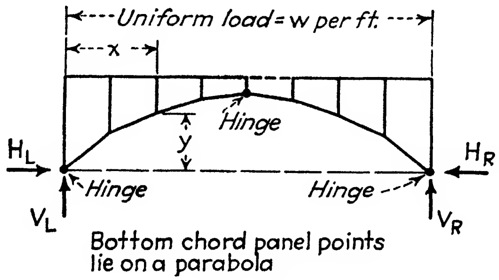


FIG. 215.

the moment at any panel point equals zero. This proposition may be proved as follows:

Let H_L and V_L, Fig. 215, = the horizontal and vertical components, respectively, of the left reaction.

x and y = the abscissa and ordinate, respectively, of any panel point on the arch axis referred to the left hinge.

M = moment at this point due to a uniform vertical load over the entire span.

M_v = moment at same point due to the vertical loads and vertical reactions only.

M_h = moment at the same point due to the horizontal reaction, H_L.

The vertical reactions in this structure are the same as for an end-supported beam since the supports are at the same level. Therefore M_v equals the moment on such a beam; hence, it varies as the ordinates to a parabola (see Art. 43, Case 8). Since H_L is constant for the loading under consideration and y is the ordinate of a parabola, $M_h = (H_L)(y)$; therefore, it

also varies as the ordinates to a parabola. But $M = M_v - M_h$. Therefore, it also varies as the ordinates to a parabola, and therefore M/y is constant for every panel point. At the center hinge, $M = 0$. Therefore, $M/y = 0$ at this point and consequently at every panel point of the arch. For sections between panel points, M_v varies as a straight line (see Art. 36). If the arch itself is straight between panel points, $(H_L)(y)$ also varies as a straight line. Hence, M/y varies as a straight line between panel points and in consequence equals zero, and hence the arch rib carries direct compression only. This is the ordinary condition for spandrel-braced arches. Hence, under a full uniform load the stresses equal zero in top chord and diagonals of such an arch, *i.e.*, an arch conforming to the conditions stated at the beginning of this article; the stress in each vertical equals the panel load, and the stress in the bottom chord is direct compression throughout and has a horizontal component equal to the horizontal component of the reactions. If the arch rib is curved between panel points, the bending moment in it will be zero at the panel points only.

For partial loads the moments at the panel points will not equal zero and the arch rib will be subjected to bending moments throughout its length. It should be observed, however, that the maximum positive moment at any panel point due to a uniform live load will equal the maximum negative moment at the same point due to the same load. This is due to the fact that the portion of the structure which should be loaded with a uniform load for maximum positive live moment at any section should be unloaded for a maximum negative live moment at the same section, and vice versa. Hence, the combined loading for maximum positive and maximum negative moments is equivalent to a full uniform load, and therefore the maximum positive live moment plus the maximum negative live moment equals zero.

For spandrel-braced arches a partial load causes stress in the diagonals and in all the top-chord bars except adjustable bars similar to the bar shown dotted in Fig. 215, and the maximum tension in these members under uniform live load equals the maximum compression for the reasons already given. For this type of arch the bottom chord, or arch rib, carries only direct stress if straight between panel points, the structure acting like any other framed structure. With a concentrated load system the

maximum positive bending moment will not equal the maximum negative moment, nor will they be equal for a uniform load with locomotive excess.

These conclusions for a spandrel-braced arch are confirmed by the problem that follows:

Problem: Compute the maximum stresses in all members of the spandrel-braced three-hinged parabolic arch shown in Fig. 216.

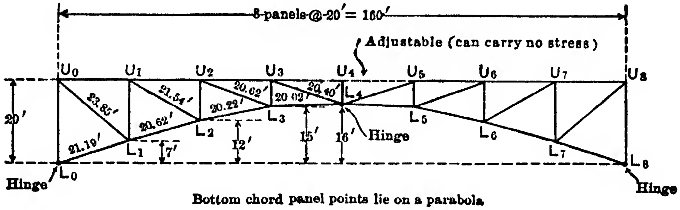


Fig. 216.

Dead weight of bridge.

1,000 lb. per foot per truss, top chord = 20,000 lb. per panel

400 lb. per foot per truss, bottom chord = 8,000 lb. per panel

Uniform live load.

2,000 lb. per foot per truss, top chord = 40,000 lb. per panel

Locomotive excess = 25,000 lb.

This problem may be solved by use either of influence lines or of an influence table. The latter will be employed here in order to illustrate its use.

The following laws concerning the magnitude and direction of the *left* reaction when the load is to the *right* of the center hinge are of material asistance in preparing such a table:

a. The line of action of the left reaction passes through L_0 and L_4.

b. Its vertical and horizontal components and hence its magnitude vary directly with the distance of the load from U_8.

c. The moment about each of the bottom panel points to the left of the center is counterclockwise; hence, the stress in the top chord is tension and that in the lower chord compression.

It follows from the foregoing rules that the magnitude of the stress in all the lower chord bars in the left half of the arch varies uniformly as the load moves from U_8 to U_4; hence, the magnitude of the stress in the web members of the left half of the arch also varies uniformly, since the stress in each of these members is a function of the combined vertical components of the left reaction and the stress in one of the bottom-chord members. With the load on the *left* half of the arch, the stresses in the bars on the left half of the structure will not vary uniformly and may be either tension or compression, since the left reaction varies in magnitude and direction. The influence table will now be given, and a table giving maximum stresses in all bars follows.

INFLUENCE TABLE FOR REACTIONS AND HORIZONTAL COMPONENTS IN CHORDS

Load at	Reaction		Horizontal components of stress in chord bars					
	V_L	H_L	Bar	Horizontal-component stress	Bar	Horizontal-component stress	Bar	Horizontal-component stress
U_1	$\frac{7}{8}$	$\frac{1}{8} \times \frac{80}{16} = \frac{5}{8}$	U_0U_1	$-\frac{7}{8} \times \frac{20}{13} + \frac{5}{8} \times \frac{7}{13} = -1.010$	U_1U_2	$-\frac{7}{8} \times \frac{40}{8} + \frac{5}{8} \times \frac{12}{8} + 1 \times \frac{20}{8} = -0.938$	U_2U_3	$-\frac{7}{8} \times \frac{60}{5} + \frac{5}{8} \times \frac{15}{5} + 1 \times \frac{40}{5} = -0.625$
			L_1L_2	$+\frac{7}{8} \times \frac{20}{13} - \frac{5}{8} \times \frac{20}{13} = +0.385$	L_2L_3	$+\frac{7}{8} \times \frac{40}{8} - \frac{5}{8} \times \frac{20}{8} - 1 \times \frac{20}{8} = +0.313$	L_3L_4*	$+\frac{7}{8} \times \frac{60}{5} - \frac{5}{8} \times \frac{20}{5} - 1 \times \frac{40}{5} = 0.000$
U_2	$\frac{3}{4}$	$\frac{1}{4} \times \frac{80}{16} = \frac{5}{4}$	U_0U_1	$-\frac{3}{4} \times \frac{20}{13} + \frac{5}{4} \times \frac{7}{13} = -0.481$	U_1U_2	$-\frac{3}{4} \times \frac{40}{8} + \frac{5}{4} \times \frac{12}{8} = -1.875$	U_2U_3	$-\frac{3}{4} \times \frac{60}{5} + \frac{5}{4} \times \frac{15}{5} + 1 \times \frac{20}{5} = -1.250$
			L_1L_2	$+\frac{3}{4} \times \frac{20}{13} - \frac{5}{4} \times \frac{20}{13} = -0.769$	L_2L_3	$+\frac{3}{4} \times \frac{40}{8} - \frac{5}{4} \times \frac{20}{8} = +0.625$	L_3L_4	$+\frac{3}{4} \times \frac{60}{5} - \frac{5}{4} \times \frac{20}{5} - 1 \times \frac{20}{5} = 0.000$
U_3	$\frac{5}{8}$	$\frac{3}{8} \times \frac{80}{16} = \frac{15}{8}$	U_0U_1	$-\frac{5}{8} \times \frac{20}{13} + \frac{15}{8} \times \frac{7}{13} = +0.048$	U_1U_2	$-\frac{5}{8} \times \frac{40}{8} + \frac{15}{8} \times \frac{12}{8} = -0.312$	U_2U_3	$-\frac{5}{8} \times \frac{60}{5} + \frac{15}{8} \times \frac{15}{5} = -1.875$
			L_1L_2	$+\frac{5}{8} \times \frac{20}{13} - \frac{15}{8} \times \frac{20}{13} = -1.923$	L_2L_3	$+\frac{5}{8} \times \frac{40}{8} - \frac{15}{8} \times \frac{20}{8} = -1.562$	L_3L_4	$+\frac{5}{8} \times \frac{60}{5} - \frac{15}{8} \times \frac{20}{5} = 0.000$
U_4	$\frac{1}{2}$	$\frac{1}{2} \times \frac{80}{16} = \frac{5}{2}$	U_0U_1	$-\frac{1}{2} \times \frac{20}{13} + \frac{5}{2} \times \frac{7}{13} = +0.577$	U_1U_2	$-\frac{1}{2} \times \frac{40}{8} + \frac{5}{2} \times \frac{12}{8} = +1.250$	U_2U_3	$-\frac{1}{2} \times \frac{60}{5} + \frac{5}{2} \times \frac{15}{5} = +1.500$
			L_1L_2	$+\frac{1}{2} \times \frac{20}{13} - \frac{5}{2} \times \frac{20}{13} = -3.077$	L_2L_3	$+\frac{1}{2} \times \frac{40}{8} - \frac{5}{2} \times \frac{20}{8} = -3.750$	L_3L_4	$+\frac{1}{2} \times \frac{60}{5} - \frac{5}{2} \times \frac{20}{5} = -4.000$

INFLUENCE TABLE FOR REACTIONS AND HORIZONTAL COMPONENTS IN CHORDS.—(*Continued*)

Load at	Reaction		Horizontal components of stress in chord bars					
	V_L	H_L	Bar	Horizontal-component stress	Bar	Horizontal-component stress	Bar	Horizontal-component stress
U_5	3/8	15/8	U_0U_1	$+3/4 \times 0.577 = +0.433$	U_1U_2	$+3/4 \times 1.25 = +0.937$	U_2U_3	$+3/4 \times 1.500 = +1.125$
			L_1L_2	$-3/4 \times 3.077 = -2.308$	L_2L_3	$-3/4 \times 3.750 = -2.812$	L_3L_4	$-3/4 \times 4.000 = -3.000$
U_6	1/4	5/4	U_0U_1	$+1/2 \times 0.577 = +0.289$	U_1U_2	$+1/2 \times 1.25 = +0.625$	U_2U_3	$+1/2 \times 1.500 = +0.750$
			L_1L_2	$-1/2 \times 3.077 = -1.538$	L_2L_3	$-1/2 \times 3.750 = -1.875$	L_3L_4	$-1/2 \times 4.000 = -2.000$
U_7	1/8	5/8	U_0U_1	$+1/4 \times 0.577 = +0.144$	U_1U_2	$+1/4 \times 1.25 = +0.312$	U_2U_3	$+1/4 \times 1.500 = +0.375$
			L_1L_2	$-1/4 \times 3.077 = -0.769$	L_2L_3	$-1/4 \times 3.750 = -0.938$	L_3L_4	$-1/4 \times 4.000 = -1.000$

H. C. stress in $L_0L_1 = -H_L$ for each position of load.

Under full load the stress in any member equals the algebraic sum of the tabular stresses. For the top chord, this should equal zero, and for the bottom chord it should equal the sum of the tabular values of H_L and should have a negative sign. The application of these tests shows the accuracy of the tabular values.

* Origin of moments for L_3L_4 is at U_3 and right reaction with load at U_1, U_2, or U_3 acts through U_3; hence, in each of these cases, M at $U_3 = 0$ (note that arch is not drawn to scale).

INFLUENCE TABLE FOR VERTICAL COMPONENTS IN DIAGONALS

V_1 = shear in panel containing diagonal.

V_2 = vertical component in bottom-chord bar in panel as determined from previous table.

$V_3 = V_1 \pm V_2$ = vertical component in diagonal.

Load at	Bar U_0L_1	Bar U_1L_2	Bar U_2L_3	Bar U_3L_4
U_1	$V_1 = +0.875$ $V_2 = -0.219$ $V_3 = +0.656$	-0.125 $+0.096$ -0.029	-0.125 $+0.047$ -0.078	-0.125 0.000 -0.125
U_2	$V_1 = +0.750$ $V_2 = -0.437$ $V_3 = +0.313$	$+0.750$ -0.192 $+0.558$	-0.250 $+0.094$ -0.156	-0.250 0.000 -0.250
U_3	$V_1 = +0.625$ $V_2 = -0.657$ $V_3 = -0.032$	$+0.625$ -0.481 $+0.144$	$+0.625$ -0.234 $+0.391$	-0.375 -0.000 -0.375
U_4	$V_1 = +0.500$ $V_2 = -0.875$ $V_3 = -0.375$	$+0.500$ -0.769 -0.269	$+0.500$ -0.563 -0.063	$+0.500$ -0.200 $+0.300$
U_5	$V_3 = -0.281$	-0.202	-0.047	$+0.225$
U_6	$V_3 = -0.187$	-0.134	-0.031	$+0.150$
U_7	$V_3 = -0.094$	-0.067	-0.016	$+0.075$

Under full load the vertical component in each diagonal equals the algebraic sum of the tabular values. This sum should and does equal 0, thus checking all the tabular values. The horizontal component in any diagonal can be readily determined from the chord stresses by application of the equation $\Sigma H = 0$.

INFLUENCE TABLE FOR STRESSES IN VERTICALS

V_3 = vertical component in diagonal running to joint at top of vertical (see previous table).

V_4 = panel load at top of vertical.

V_5 = stress in bar.

Load at	Bar U_0L_0	Bar U_1L_1	Bar U_2L_2	Bar U_3L_3	Bar U_4L_4
U_0	V_3 = 0.000 V_4 = −1.000 V_5 = −1.000	0.000	0.000	0.000	0.000
U_1	V_3 = +0.656 V_4 = 0.000 V_5 = −0.656*	−0.029 −1.000 −0.971	−0.078 0.000 +0.078	−0.125 0.000 +0.125	0.000
U_2	V_3 = +0.313 V_4 = 0.000 V_5 = −0.313	+0.558 0.000 −0.558	−0.156 −1.000 −0.844	−0.250 0.000 +0.250	0.000
U_3	V_3 = −0.032 V_4 = 0.000 V_5 = +0.032	+0.144 0.000 −0.144	+0.391 0.000 −0.391	−0.375 −1.000 −0.625	0.000
U_4	V_3 = −0.375 V_4 = 0.000 V_5 = +0.375	−0.269 0.000 +0.269	−0.063 0.000 +0.063	+0.300 0.000 −0.300	−1.000
U_5	V_5 = +0.281	+0.202	+0.047	−0.225	0.000
U_6	V_5 = +0.187	+0.134	+0.031	−0.150	0.000
U_7	V_5 = +0.094	+0.067	+0.016	−0.075	0.000

* NOTE: (+) stress in diagonal gives (−) stress in vertical, and for full load the stress in each vertical equals unity.

INFLUENCE TABLE FOR STRESS IN EACH BAR FOR LOAD AT

Bar	U_0	U_1	U_2	U_3	U_4	U_5	U_6	U_7
U_0U_1	0.000	−1.010	−0.481	+0.048	+0.577	+0.433	+0.289	+0.144
U_1U_2	0.000	−0.938	−1.875	−0.312	+1.250	+0.937	+0.625	+0.312
U_2U_3	0.000	−0.625	−1.250	−1.875	+1.500	+1.125	+0.750	+0.375
U_3U_4	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
L_0L_1	0.000	−0.662	−1.325	−1.990	−2.650	−1.990	−1.325	−0.662
L_1L_2	0.000	+0.397	−0.793	−1.980	−3.170	−2.380	−1.590	−0.793
L_2L_3	0.000	+0.317	+0.632	−1.580	−3.790	−2.840	−1.900	−0.950
L_3L_4	0.000	0.000	0.000	0.000	−4.000	−3.000	−2.000	−1.000
U_0L_1	0.000	+1.204	+0.575	−0.059	−0.688	−0.515	−0.343	−0.172
U_1L_2	0.000	−0.078	+1.500	+0.388	−0.725	−0.545	−0.362	−0.180
U_2L_3	0.000	−0.322	−0.644	+1.614	−0.260	−0.194	−0.128	−0.066
U_3L_4	0.000	−0.637	−1.275	−1.913	+1.530	+1.148	+0.765	+0.382
U_0L_0	−1.000	−0.656	−0.313	+0.032	+0.375	+0.281	+0.187	+0.094
U_1L_1	0.000	−0.971	−0.558	−0.144	+0.269	+0.202	+0.134	+0.067
U_2L_2	0.000	+0.078	−0.844	−0.391	+0.063	+0.047	+0.031	+0.016
U_3L_3	0.000	+0.125	+0.250	−0.625	−0.300	−0.225	−0.150	−0.075
U_4L_4	0.000	0.000	0.000	0.000	−1.000	0.000	0.000	0.000

The values in this table may be verified by the same methods used for preceding tables.

From the influence table for stress in each bar, the maximum live stresses, due to uniform load, may be easily obtained for any given bar by summing up the total positive and negative values for the bar and multiplying each sum by the live panel load. The stress due to the locomotive excess may be computed by multiplying the maximum value for each bar by the excess load.

The table that follows shows the stresses thus obtained:

TABLE FOR MAXIMUM LIVE STRESSES IN ALL BARS

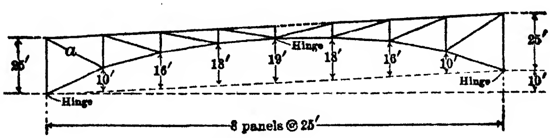
Bar	Tension			Compression		
	Uniform live load	E at	Stress, pounds	Uniform live load	E at	Stress, pounds
U_0U_1	U_3 to U_7 incl.	U_4	74,100	U_1 and U_2	U_1	84,900
U_1U_2	U_4 to U_7 incl.	U_4	156,200	U_1 to U_3 incl.	U_2	171,900
U_2U_3	U_4 to U_7 incl.	U_4	187,500	U_1 to U_3 incl.	U_3	196,900
U_3U_4			0,000			0,000
L_0L_1				U_1 to U_7 incl.	U_4	490,400
L_1L_2	U_1	U_1	25,800	U_2 to U_7 incl.	U_4	507,500
L_2L_3	U_1 and U_2	U_2	53,800	U_3 to U_7 incl.	U_4	537,200
L_3L_4				U_4 to U_7 incl.	U_4	500,000
U_0L_1	U_1 and U_2	U_1	101,300	U_3 to U_7 incl.	U_4	88,300
U_1L_2	U_2 and U_3	U_2	113,100	U_1 and U_4 to U_7 incl.	U_4	93,700
U_2L_3	U_3	U_3	104,900	U_1 and U_2 and U_4 to U_7 incl.	U_2	80,700
U_3L_4	U_4 to U_7 incl.	U_4	191,300	U_1 to U_3 incl.	U_3	200,600
U_0L_0	U_3 to U_7 incl.	U_4	48,200	U_0 to U_2 incl.	U_0	83,800
U_1L_1	U_4 to U_7 incl.	U_4	33,600	U_1 to U_3 incl.	U_1	91,200
U_2L_2	U_1 and U_4 to U_7 incl.	U_1	11,400	U_2 and U_3	U_2	70,500
U_3L_3	U_1 and U_2	U_2	21,300	U_3 to U_7 incl.	U_3	70,700
U_4L_4				U_4	U_4	65,000

The dead stresses are as follows:

Top-chord bars and diagonals.................	0
End verticals................................	10,000 lb.
Intermediate verticals........................	20,000 lb.
Bottom-chord bars (horizontal component)......	280,000 lb.

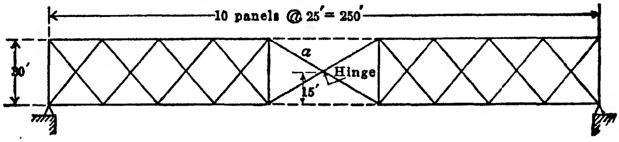
Problems

61. *a.* Draw influence line for horizontal components of reactions.
b. Draw influence line for vertical component of stress in bar *a*.



PROB. 61.

62. Same as Prob. 61 except for figure.



PROB. 62.

63. *a.* Draw influence line for moment at quarter point of a three-hinged plate-girder parabolic arch rib with load applied through vertical columns at each panel point, and compute maximum positive and negative moment at each panel point due to a uniform load of 1,000 lb. per foot and a concentrated load of 32,000 lb. Arch has 24 panels at 15 ft. and rise of 72 ft.

b. Compute direct thrust and shear due to above loading, neglecting effect of deflection.

CHAPTER XI

DESIGN OF COLUMNS AND TENSION MEMBERS

133. Columns. General Considerations.—A column is a member designed primarily to resist compression, although it may also be subjected to transverse loads causing flexure. Compressive tests of blocks of plastic material of such proportions that the length does not greatly exceed the minimum lateral dimension show that failure occurs by lateral flowing of the material with no well-defined ultimate strength; a definite elastic limit, however, exists beyond which the material simply expands laterally and contracts longitudinally under increasing loads. On the other hand, compressive pieces in which the ratio of length to least lateral dimension is high fail by lateral bending, when subjected to compression, even when the load is applied along the longitudinal axis passing through the center of gravity of the bar and the bar is originally straight and of homogeneous material without initial stress.

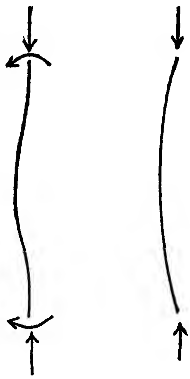


FIG. 217.

The ultimate load for the latter class of columns may be much less than the product of the elastic limit and the cross-section area. A good illustration of such a condition is presented by a straight bar of tempered steel of small cross section. A short piece of such a bar would sustain a high load per square inch without showing signs of failure, whereas a long piece would collapse by bending laterally under a comparatively light load, the column bending in one of the ways indicated by Fig. 217. The columns used in engineering structures generally have a slenderness ratio midway between these two extremes; hence, failure may be expected either by crushing or bending, or by both together, even if the columns are originally in an ideal condition so far as material, shape, and loading are concerned.

The ideal column does not exist in practice. The load is seldom if ever applied exactly at the center of gravity or along the column axis; the process of fabrication in a metal column

is sure to leave the column with some distortion and with the material in a condition of initial stress, and columns of timber or concrete are equally sure to be imperfect. Moreover, the material is never homogeneous, and in a built-up steel column, such as is generally used in important structures, the behavior of the column as a whole is dependent upon the integrity of its cross section, which may or may not be preserved by the rivets, tie plates, lattice bars, and other devices required to hold together the main pieces.

In view of these many uncertainties, the economical and efficient design of columns has always been one of the most serious problems that the engineer has had to confront, especially when dealing with unusual cases, the difficulties being increased by the lack of sufficient experimental data. Appreciating the necessity of further investigations of the strength of columns as used in engineering structures, the American Society of Civil Engineers in 1923 established a special committee on steel-column research. This committee made an exhaustive series of full-sized tests to determine the strength of columns as a whole and the effect of various details, and its final report was published in the *Transactions* of the society, Vol. 98, 1933, pages 1376 *et seq.*

134. Condition of Ends.—If the ends of a column are unrestrained against turning, it is said to have *hinged* ends; this condition, however, seldom exists. Columns in which the loads are applied by pins at the ends, as in many American bridges, are said to have *pin* ends. Columns in which the ends are subject to such restraint that the tangents to the elastic curve at the ends remain parallel to the column axis when the column deflects laterally are said to have *fixed* ends. If the ends of the column are square and bear upon flat surfaces, they are said to be square or flat-ended; this condition closely approximates the condition of fixed-ended columns when the columns are short and pin-ended columns when the columns are long.

That the condition of the end may affect the strength of the column is apparent from a study of Fig. 218, which shows the curves that both round-ended and fixed-ended columns would take under vertical loads before failure by bending.

These two cases are somewhat analogous to free-ended and fixed-ended beams. A fixed-ended beam is materially stronger

that one with ends simply supported; and the same is true of columns. The portion *cd* of the fixed-ended column corresponds to the entire length of the round-ended column, the points *c* and *d* being points of contraflexure. The distance between *c* and *d* equals one-half of *ab*, since the portion *ce* of the column is in the same condition as *ac*; *i.e.*, the tangent to the elastic curve at *e* is parallel to the original axis of the column, and so also is the tangent at *a*. It follows that in comparing fixed-ended with round-ended columns it may be considered that the unsupported length in one case is half that of the other.

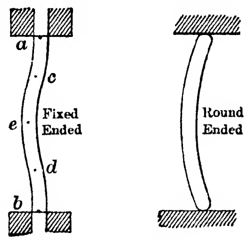


FIG. 218.

Columns with ends entirely free to turn do not exist in actual structures. The nearest approach to this condition probably occurs in the ordinary pin-ended column, but such pins are by no means frictionless; indeed, in some cases after exposure to weather, with the consequent rusting that takes place, the pins are so restrained that it is with great difficulty that the members can be turned about them. It is also seldom that structural columns are rigidly fixed at the ends, since the piece to which the column is riveted is seldom so rigid that it will not yield somewhat under the influence of the bending tendency. The formulas recommended by the committee referred to in the previous article and given in Art. 18 allow for different end conditions by providing for both pin ends and riveted ends.

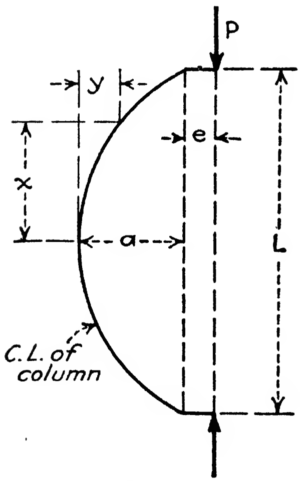


FIG. 219.

135. Column Formulas.—A formula for the allowable stress in a column may be developed as follows:

Consider the column shown in Fig. 219 with moment of inertia and modulus of elasticity equal to I and E, respectively, and apply the well-known equation for the elastic curve of a member subjected to bending, *viz.:*

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

This gives the following equation:

$$\frac{d^2y}{dx^2} = \frac{P}{EI}(a + e - y)$$

in which e = eccentricity of the applied force P

a = deflection of column due to bending caused by P

Substituting in the equation above K^2 for P/EI gives the following equation:

$$\frac{d^2y}{dx^2} + K^2y = K^2(a + e)$$

This is a well-known differential equation, the solution of which gives the following result:[1]

$$y = C_1 \cos Kx + C_2 \sin Kx + (a + e)$$

when $x = 0$, $y = 0$, and $dy/dx = 0$.

Therefore,

$$C_1 = -(a + e) \qquad \text{and} \qquad C_2 = 0$$

Hence,

$$y = -(a + e) \cos Kx + a + e = (a + e)(1 - \cos Kx)$$

when $x = L/2$, $y = a$.

Therefore,

$$a = a - a \cos K\frac{L}{2} + e - e \cos K\frac{L}{2}$$

whence

$$a = e\left(\frac{1 - \cos K\frac{L}{2}}{\cos K\frac{L}{2}}\right) = e\left(\sec K\frac{L}{2} - 1\right)$$

and the maximum value of $M = P(a + e) = P\left(e \sec K\frac{L}{2}\right)$.

Substituting for K its value, *viz.*, $\sqrt{P/EI}$, gives the following expression for the maximum fiber stress s at the extreme fiber distant c from the neutral axis:

$$s = \frac{P}{A} + Pe\frac{c}{I} \sec\left(\frac{L}{2}\sqrt{\frac{P}{EI}}\right)$$

[1] See Hudson and Lipka, A Manual of Mathematics, p. 58, John Wiley & Sons, Inc., New York, 1917.

If r = radius of gyration about neutral axis, $I = Ar^2$ which when substituted in the preceding equation gives the following:

$$s = \frac{P}{A}\left(1 + \frac{ec}{r^2}\sec\frac{L}{2r}\sqrt{\frac{P}{EA}}\right) \tag{21}$$

Formula (21) is the so-called *secant formula* and is applicable to eccentrically loaded columns of any length.

The committee referred to in Art. 133 states that as a result of its tests, it has found that 0.25 may be substituted for the eccentric ratio ec/r^2 in Eq. (21) for columns loaded without intentional eccentricity and having an ordinary amount of secondary stress due to crookedness and other irregularities.

For columns loaded with a definite known eccentricity Eq. (21) should be replaced by the following:

$$s = \frac{P}{A}\left[1 + \left(\frac{ec}{r^2} + 0.001\frac{L}{r}\right)\sec\frac{L}{2r}\sqrt{\frac{P}{EA}}\right] \tag{22}$$

In formulas (21) and (22) the value of L represents the length of the column between points of inflexion. Inasmuch as the columns in actual structures are restrained to some extent at each end, either by being riveted to other members or, if pin-ended, by the friction on the pin, the total length of the column should not be used in the equation. It is recommended by the committee referred to in Art. 133 that the value of L in formulas (21) and (22) should be modified as follows:

For pin-ended columns by substituting $\frac{7}{8}L$ for L.

For columns with riveted ends by substituting $\frac{3}{4}L$ for L.

Because of the fact that the secant formulas cannot be solved directly, a simpler formula is desirable. Formulas of the parabolic type are found to give results agreeing very closely with the secant formulas and somewhat on the safe side for ratios of L/r less than 140, and formulas of this type have been generally adopted in the United States and are given in Art. 18. For columns having a ratio of L/r greater than 140 the secant formulas should be used. Comparisons of the results obtained from the parabolic and the secant formulas for columns stressed to the yield point of the material are given on page 1455 of the 1933 *Transactions* of the American Society of Civil Engineers.

The preceding formulas all apply to the structural steel for bridges specified by the American Society of Testing Materials in its 1936 specifications.

For columns of structural silicon and other alloy steels, the yield point is considerably higher and the column formulas must be modified, thus giving the value for silicon-steel columns referred to in Art. 18.

136. Value of Ratio of Length to Radius of Gyration.—The term L/r in column formulas equals the maximum ratio of unrestrained length to radius of gyration. If the column is restrained against lateral deflection in all directions at the two ends and at no intermediate point, then L is the total length of the column between lateral supports, and r the least radius of gyration of its transverse section, provided that the column is of constant cross section between supports, as is usually the case. If the column is held against lateral deflection in all directions at one or several intermediate points, the value of L/r to be used should be the largest possible value for any portion of the column between any two points of support. If the column is held at an intermediate point in one direction only, then the value of L/r to be used in the formula should be the maximum obtained by using for L either the length of any section or the total length of the column, and for r in each case the radius of gyration referred to the axis about which the column is free to bend. For example, the maximum value of L/r for the column shown in Fig. 220 may be either $L/2r$ or L/r_1.

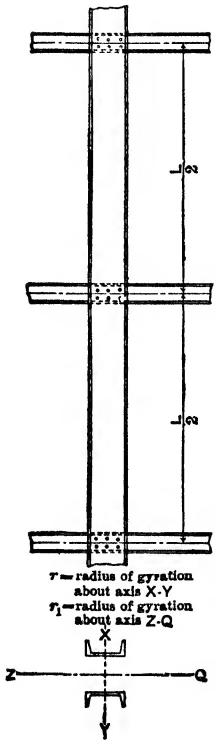


Fig. 220.

If the column is of variable section, the designer must use his judgment in selecting the proper value of r to use.

137. Formulas for Long Columns.—The term *long column*, as used in this article, refers to columns of length such that failure will occur by lateral bending before the material has reached its elastic limit. The *collapsing* load that such columns will carry without yielding, when centrally loaded, can be closely determined by mathematical investigations,[1] provided that the

[1] See Applied Mechanics, Vol. II, Fuller and Johnston, John Wiley & Sons, Inc., New York, 1919.

columns are initially straight, free from stress, and homogeneous, and the load is axially applied and dependent upon the elasticity rather than the crushing strength of the material. The formula commonly used for such columns is known as the *Euler formula* and is given in treatises on mechanics, as follows:

$$\text{Columns with fixed ends, } \frac{P}{A} = 4\pi^2 E\left(\frac{r}{L}\right)^2 \qquad (23)$$

$$\text{Columns with round ends, } \frac{P}{A} = \pi^2 E\left(\frac{r}{L}\right)^2 \qquad (24)$$

$$\text{Columns with one free end, } \frac{P}{A} = \frac{\pi^2 E}{4}\left(\frac{r}{L}\right)^2 \qquad (25)$$

In these formulas, E equals modulus of elasticity and P/A is the axial stress required to hold the column in equilibrium if slightly deflected laterally. If a load greater than P is applied to such a column, the column will collapse; if a smaller load, the column will spring back to its original condition when the lateral forces are removed. If Euler's formula is applied to columns composed of material with an elastic limit of 33,000 lb. and a modulus of elasticity of 29,000,000, both of which are reasonable values for structural steel, the value of P/A for round-ended columns would exceed the elastic limit whenever $L/r < 94$; hence, Euler's formula should not be used for such columns when the ratio of length to radius of gyration is less than this limit. For fixed-ended columns, on the other hand, the use of the same constants would give for L/r the value of 186.

In using these formulas a suitable factor of safety should be employed. For columns of the character generally used in bridge and building structures with a ratio of $L/r > 140$, the secant formulas given in Art. 135 may be used.

138. Cast-iron Columns.—Cast iron is unsuitable for structural members exposed to tension or bending because of its low tensile strength and brittleness. It may, however, be used for compression pieces if these are properly designed, cast-iron columns being frequently used for interior columns in buildings. Such columns cannot be made in long lengths, and the different sections cannot be fastened so rigidly together as steel columns; hence, they are inferior in rigidity to the latter. Moreover, the fact that it is difficult to secure uniform thickness of shell and that the material is often variable in composition, may contain

flaws, and is frequently in a state of initial stress, is against their use. It is also difficult to obtain good connections of transverse beams and girders.

A set of valuable tests was made upon cast-iron columns at the works of the Phœnix Iron Company in 1896–1897, and a formula based upon these tests is probably as reliable as anything that can be obtained, although the results of the tests were so variable that a large factor of safety should be used in applying the formula. A study of the tests shows that a straight-line formula conforms to the results, as well as any other type of formula. Such a formula is derived by Burr.[1]

Ultimate strength per square inch for circular flat-ended columns,

$$\frac{P}{A} = 30{,}500 - 160\frac{L}{d} \tag{26}$$

In this formula, L = unsupported length and d = diameter.

A formula based upon this same set of tests, as given by Johnson[2] differs somewhat from the foregoing and is as follows:

$$\frac{P}{A} = 34{,}000 - 88\frac{L}{r} \tag{27}$$

In this formula, r = least radius of gyration.

A factor of safety of 5, which is none too much for a material as uncertain as cast iron, reduces Burr's formula to the following form:

$$\frac{P}{A} = 6{,}100 - 32\frac{L}{d} \tag{28}$$

This value is much less than the corresponding value for steel columns in spite of the high compressive strength of cast iron; and in consequence, though cast iron is cheaper per pound than steel, there is little if any economy in the use of properly designed cast-iron columns except for light loads for which it may be difficult to obtain steel columns of sufficiently small cross section. It should be noted, however, that the building laws of the

[1] See The Elasticity and Resistance of the Materials of Engineering, Burr, 7th ed., John Wiley & Sons, Inc., New York, 1915.

[2] Materials of Construction, Johnson, John Wiley & Sons, Inc., New York, 1930.

large cities permit, in general, the use of higher unit stresses[1] than those given by either of the above formulas when properly reduced by a factor of safety. The employment of cast iron in bridges was abandoned many years ago both because of its treacherous character and because of the difficulty of making satisfactory connections between members.

The limiting lengths to which these formulas are applicable are stated to be as follows:

Johnson:

$$\frac{L}{r} \leq 120$$

New York Building Laws:

$$\frac{L}{r} \leq 70$$

Burr:

$$\frac{L}{d} \leq 40$$

d = diameter of circular column or shorter side of rectangular column

r = radius of gyration

139. Timber and Concrete Columns.—Timber columns resemble cast-iron columns in being variable in strength. This is largely due to the presence of knots and other defects. Formulas for timber columns are given in Art. 18.

It is important to note that timber columns made by bolting a number of sticks together are no stronger than if each stick is separate and loaded by its share of the total load. This has been shown by tests and may be readily understood, since the bolts cannot be counted upon to hold the individual sticks in place, owing to the small bearing value of wood across the grain and the difficulty of keeping nuts tight.

Concrete columns will not be considered here. The student is referred to standard books upon concrete structures for full treatment of such columns.

140. Typical Column Sections.—Figure 221 represents the cross sections of a number of types of columns. A and B are

[1] The New York Building Laws, 1937, give the following formula for cast-iron columns:

$$\frac{P}{A} = 9{,}000 - 40\frac{L}{r}$$

columns frequently used in bridge construction, the latter set representing the common type for upper chords of pin bridges, the horizontal plate on the top flange being used to give lateral rigidity. *C* shows some very heavy column sections, used

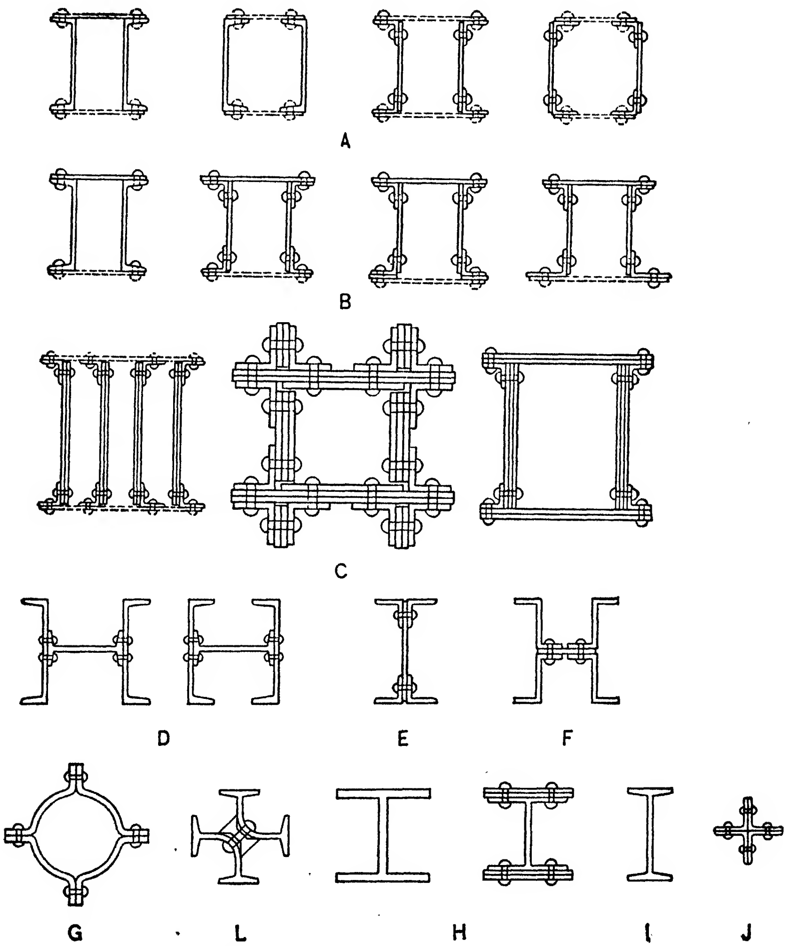


FIG. 221.—Column types. Dotted lines represent latticing.

in the Queensboro Bridge, the Metropolitan Tower, and the Bankers' Trust Building, all in New York, N. Y. *D* shows the columns sometimes used in elevated-railroad construction in which the central diaphragm is useful both in adding to the cross

section and in preserving the integrity of the column. *E* is a type of column frequently used for verticals of riveted trusses. *F* is a Z-bar column sometimes used in building construction. *G* is the well-known Phœnix column, made by the Phœnix Iron Company and once widely used for bridges and elevated railroads. *L* is the Larimer column made by Jones and Laughlins, Ltd. *H* shows wide-flange I-beam columns. *I* is an ordinary I-beam column; and *J* is an angle column. The towers of the Golden Gate Bridge are composed of a considerable number of rectangular cells arranged to give a variable section and a pleasing appearance.

141. General Dimensions and Limiting Conditions.—In designing a column the first thing to be determined is the type and general dimensions of the member, *i.e.*, the width and depth, provided that these are limited by other considerations than

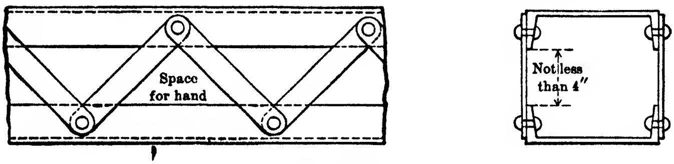


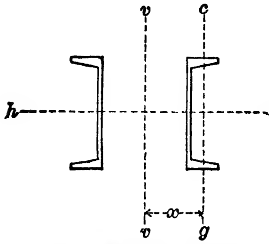
FIG. 222.

those of strength, as is often the case. For example, the compression chords in bridges must be of sufficient width and depth to permit of proper connections to the web members, and the verticals must be of such a size as to give suitable floor-beam connections; in buildings the columns are frequently limited in size because of the space available or the character of the necessary connections. Ease of construction must also be considered, and this is frequently a ruling factor, as in the case of channel columns with flanges turned toward each other, where the space between the flanges should not be less than 4 in. and lattice bars[1] must be far enough apart to permit insertion of the hand as illustrated in Fig. 222.

In determining the dimensions of the individual pieces, there are also restrictions due to practical considerations. If the web or cover plates are too thin, they may wrinkle under com-

[1] Lattice bars are diagonal members, such as are shown in Fig. 222, intended to hold the two column halves in line and make the column act as a solid piece. These bars are of great importance and will be fully treated later.

pression; hence, it is common practice to limit the thickness of webs and cover plates to not less than $1/32$ to $1/40$, respectively, of the distance between connecting rivets. It is also desirable to proportion the column so that the center of gravity will be near the center of section. If a cover plate is used on the top flange as in the chords of Fig. 221 *B*, unequal-legged angles with wide-leg horizontal or narrow-flange plates, vertical or horizontal, are often used on the bottom flange to lower the center of gravity. Wide-flange I beams have been used frequently in recent years for columns in buildings and in small bridge and building trusses and also for tension members.



Axis *c.g.* passes through centre of gravity of one channel and axes *hh* and *vv* through centre of gravity of column.

FIG. 223.

142. Method of Design.—With these restrictions in mind, approximate design of the column may be made, by assuming the value either of the minimum radius of gyration or of the allowable unit stress. The actual allowable unit stress for the section thus obtained may then be computed, and the column redesigned if this stress varies too widely from the stress that the column actually carries. It is usually economical to place the webs such a distance apart that the radius of gyration about the principal axis parallel to the webs will equal that about the principal axis perpendicular to the webs, *i.e.*, if the unsupported length of the column is the same with respect to both axes. This involves the condition that the moments of inertia about each of these axes should be equal, since the radii of gyration will then be equal, and the value of L/r and consequently of the allowable unit stress will be the same in both directions. The following method of determining this spacing of a column composed of two channels as shown in Fig. 223 is simple and illustrates the problem sufficiently.

Let r_h = radius of gyration about axis *hh* (this is unalterable for any given channel).

r_v = radius of gyration about axis *vv*.

I_v = moment of inertia of cross section about axis *vv*.

I_h = moment of inertia of cross section about axis *hh*.

I_{cg} = moment of inertia of each channel about its axis *cg*.

A = area of one channel.

To determine the moment of inertia about axis *vv*, the following principle of mechanics may be used:

The moment of inertia of a section about any axis equals the moment of inertia of that section about a parallel axis passing through its centroid, plus its area multiplied by the square of the distance between the two axes.

From the application of this principle the following equation results:

$$I_v = 2(I_{cg} + Ax^2)$$

Hence, I_h should equal $2(I_{cg} + Ax^2)$.
But

$$I_h = 2(Ar_h^2)$$

Hence,

$$2Ar_h^2 = 2I_{cg} + 2Ax^2$$

Hence

$$x^2 = \frac{2Ar_h^2 - 2I_{cg}}{2A} = r_h^2 - \frac{I_{cg}}{A}$$

In the case of channel columns the value of I_{cg} is usually small compared with A. Hence, the error involved in omitting the last term of the equation is small and is on the safe side; therefore it may be neglected and the value of x made equal to r_h.

The proper distance between channels to secure equal rigidity about either axis is given in some of the steel manufacturers' handbooks and need not be computed; but for more complicated sections, such as plate and angle columns, it must usually be determined in the manner indicated, although the approximation mentioned is not always allowable and, in the case of top chords with cover plates, would be considerably in error and should not be made.

143. Determination of Cross Section of Typical Steel Columns.

Problem: Design a channel column for a riveted-truss bridge for the following assumed conditions:

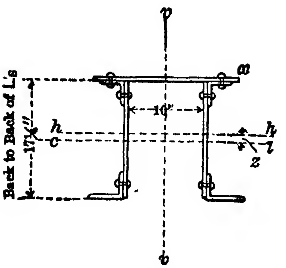
Total applied axial force (live, dead, and impact)	= 250,000 lb.
Unsupported length	= 25 ft.
Allowable stress	$P/A = 15{,}000 - \frac{1}{4}(L/R)^2$

Solution: Determine trial section by assuming $P/A = 14{,}000$ lb. This gives a trial area of 250,000/14,000 = 17.9 sq. in. which can be obtained by the use of two 15-in. channels at 33.9 lb., having a total area of 19.8 sq. in.

The radius of gyration for this section about an axis perpendicular to the web = 5.62 in.; hence, the allowable value of

$$\frac{P}{A} = 15{,}000 - \frac{1}{4}\frac{(25 \times 12)^2}{(5.62)^2} = 14{,}420 \text{ lb.}$$

The actual unit stress if the proposed section should be used equals 250,000/19.8 = 12,600 lb. It follows that the column can carry considerably more than the applied load; hence, it is safe and may possibly be decreased in size.



1 Cover Plate-18" x 3/8" @ 6.75 = 6.75 □"
2 Web Plate-17" x 7/16" @ 7.44 = 14.88 □"
2 Top Ls -3 1/2" x 3 1/2" x 3/8" @ 2.49 = 4.98 □"
2 Bottom Ls-6" x 3 1/2" x 3/8" @ 3.43 = 6.86 □"
33.47 □"

FIG. 224.

The next smaller channel is 12 in., 30 lb.; hence, try this. The allowable value of P/A for a column composed of these channels equals

$$15{,}000 - \frac{1}{4}\frac{(25 \times 12)^2}{(4.28)^2} = 13{,}770$$

The actual unit stress, if these channels are used, equals 250,000/17.58 = 14,200 lb.; hence, these channels are not large enough, and the 15-in. channels should be chosen. They should be placed so that the distance x in Fig. 223 = 5.62 in. The Cambria Steel Company's Handbook[1] gives the proper distance between the webs as 9.51 in., which agrees closely with the corresponding values when x = 5.62 in.

Problem: Design a top-chord member of a riveted-truss bridge using a top cover plate:

Minimum clear distance between web plates	= 10 in.
Minimum clear depth	= 17 in.
Total applied axial load (live, dead, and impact)	= 430,000 lb.
Unsupported length	= 25 ft.
Column formula,	$\frac{P}{A} = 15{,}000 - \frac{1}{4}\frac{L^2}{r^2}$

Thickness of web to be not less than one-thirty-second distance between horizontal flange rivets.

Thickness of cover plates to be not less than one-fortieth distance between vertical flange rivets.

Solution: In a compression piece of this type, r, with respect to the horizontal axis, is usually about four-tenths the depth of the member; therefore, this value for the radius of gyration will be tried instead of assuming the allowable unit stress, as was done in the previous example. Making this assumption and assuming also a minimum depth of 17 5/8 in. and a distance apart of webs of 10 in. give 7.0 in. as a trial value of r. Substituting this value in the column formula gives

[1] Cambria Steel Company, Johnstown, Pa., 1904.

$$\frac{P}{A} = 15{,}000 - \frac{1}{4}\frac{(25 \times 12)^2}{(7.0)^2} = 14{,}500 \text{ lb.}$$

Using this value gives 430,000/14,500 = 29.6 sq. in. as the necessary area for a preliminary trial.

The section shown in Fig. 224 complies with all the restrictions stated and has an area somewhat greater than that just determined.

Bottom angles with a wider horizontal leg than the top angles are chosen in order partly to offset the effect of the top cover plate and thus lower the center of gravity of the cross section. The exact value of r must now be computed, both about the axis vv and the axis hh, and the smaller value used to determine the allowable unit stress. The computations may conveniently be arranged in the tabular form that follows and require no explanation. It should be observed that the position of the center of gravity is determined as a step in the process of finding the moment of inertia. It is necessary to locate its position in order to detail the structure properly, since the center of gravity lines of the various members meeting at any joint should intersect at a point.

COMPUTATION OF I_{hh}

Piece	Area, sq. in.	Lever arm, in.	Moment about cl, in.^{3}	Moment about cl, in.^{3}	I about cl, in.^{4}
Cover plate...	6.75	+8.80	+59.4	......	$59.4 \times 8.80 = 523$
Webs........	14.88	0	0.0	......	$\frac{1}{12} \times \frac{7}{8} \times 17^3 = 358$
Top angles...	4.96	+7.61	+37.7	......	$37.7 \times 7.61 = 287$
Bottom angles	6.84	−7.83	......	−53.6	$53.6 \times 7.83 = 420$
Total......	33.43	......	+97.1	+53.6	1,588

$z = \dfrac{97.1 - 53.6}{33.43} = 1.30$ in. $\quad I_{hh} = 1{,}588 - 33.43 \times 1.30^2 = 1.532.$ in.^{4}

$r_{hh} = \sqrt{1{,}532/33.43} = 6.77$ in.

COMPUTATION OF I_{vv}

Piece	Area, sq. in.	Lever arm, in.	I about vv, in.^{4}
Cover........................	6.75	....	$\frac{1}{12} \times \frac{3}{8} \times 18^3 = 182$
Webs.........................	14.88	5.22	$14.88 \times 5.22^2 = 405$
Top angles...................	4.96	6.45	$4.96 \times 6.45^2 = 206$
Bottom angles................	6.84	7.47	$6.84 \times 7.47^2 = 382$
Total........................	33.43	....	$I_{vv} = 1{,}175$

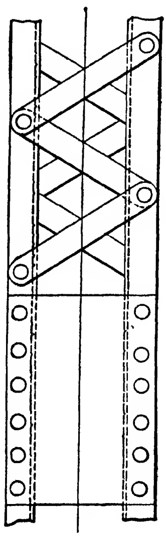
$r_{vv} = \sqrt{1{,}175/33.43} = 5.93$ in.

The minimum value of r for the assumed section is evidently that about axis vv. The allowable value of P/A for this case equals 14,360 lb. The

actual stress on the section would equal 430,000/33.43 = 12,850 lb. per square inch; hence, the area is ample and could be decreased slightly by using smaller angles.

These examples serve to illustrate the computations necessary for any form of "built-up" steel column.

144. Lattice Bars and Batten Plates.—If the two ribs of a column such as that shown in cross section by Fig. 223 are not connected, each rib would have to be proportioned as a separate column subjected to one-half the total load. The least radius of gyration for such a case would be that for one rib about the axis *cg*, which would ordinarily be much less than the value about axis *hh* and consequently much smaller than the maximum value attainable for the sections used. Such a design would require a much larger amount of material for the main section than would be necessary if the two ribs should be rigidly connected so that they would act together, and the extra amount of material required would be much in excess of that



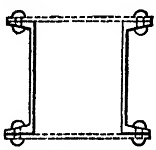


FIG. 225.

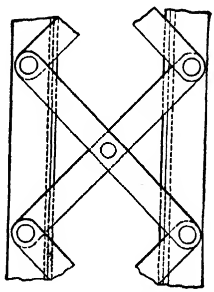


FIG. 226.

needed for the details necessary to connect the two ribs. Several conventional methods exist of connecting the ribs, the use of side plates or diaphragms, as illustrated by several of the cases of Fig. 221, being the most obvious. Either of these methods has the advantage of using for this purpose material that can also carry a portion of the stress. The use of a diaphragm is frequently impracticable for bridge members, owing to difficulties of designing proper details. Moreover, it is desirable to have

as much of the material as possible concentrated in the ribs, since the distribution of the stress over the cross section is thereby simplified. For such columns, it is therefore common to connect the two ribs by short plates, usually called *batten plates* or *tie plates*, at each end and to use diagonal bars, called lattice bars, throughout the remainder of the column, thus connecting the two ribs by a form of trussing. Batten plates are also used at points where the continuity of the latticing is interrupted. Such a column is shown in Fig. 225, in which latticing is used on both sides. It is frequently possible to use plates on one side of the column, as in the top chords shown in Fig. 221, *B*, in which case latticing may be employed on the other side. The latticing may be composed of flat bars, angles, or even small channels for unusually heavy columns and may be single on each side, as shown in Fig. 225, or double, with rivets at the points of intersection, as shown by Fig. 226.

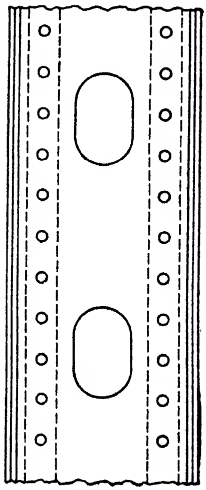


FIG. 227.

Instead of latticing, plates with handholes to permit inspection, painting, and the insertion of rivets if the flanges are turned toward each other, as shown in Fig. 227, may prove economical since such a plate is able to carry considerable stress, which is not the case with latticing. It is also pleasing in appearance.

The fact that the strength of latticed columns is largely dependent upon the proper design of the latticing requires that the proportioning of the latticing should be as carefully studied as the design of the main cross section. Unfortunately the theoretical treatment of such details is more obscure than that of the columns themselves. It is evident, however, that if the column were to remain absolutely straight under loading, no latticing would be needed, and the stress in such lattice bars as might be used would be merely the secondary stress due to the shortening of the column as a whole and the consequent distortion of the lattice bars. On the other hand, since the column bends somewhat under loading, transverse shear occurs, which causes stresses in the lattice bars, and if the value of this shear can be determined the lattice bars may be easily proportioned.

145. Stress in Lattice Bars.—An elaborate investigation of the shearing stresses in columns consistent with the results obtained by applying formula (21) was made by Shortridge Hardesty, Esq., and reported in the *Proceedings* of the American Railway Engineering Association, Vol. 36, 1935. Mr. Hardesty plotted curves covering all probable conditions and arrived at the following rule which has been adopted by the American Railway Engineering Association and the American Association of State Highway Officials:

In compression members the shearing stress normal to the member in the plane of the lacing shall be obtained by the following formulas:

For structural carbon steel—

$$V = \frac{P}{100}\left[\frac{100}{\frac{L}{r} + 10} + \frac{L/r}{100}\right]$$

For silicon steel—

$$V = \frac{P}{100}\left[\frac{100}{\frac{L}{r} + 10} + \frac{L/r}{70}\right]$$

In above formulas

V = normal shearing stress

P = allowable compressive axial load on member

L = length of member in inches

r = radius of gyration of section about the axis perpendicular to plane of lacing in inches

To the shear so determined shall be added any shear due to the weight of the member or to other forces and the lacing shall be proportioned for the combined shear.

The shear thus found should be divided equally among all parallel planes in which there are shear-resisting elements whether continuous plates or double latticing.

The total stress in a lattice bar, if single latticing is used on each side, may be taken as equal to one-half the product of V and the cosecant of the angle that it makes with the longitudinal axis of the column. This method gives the stress in the end lattice bars, but it is common to use the same size bars throughout the column. The following problem illustrates this method:

Problem: Determine stress in lattice bars for the 15-in. carbon-steel 33.9 lb. channel column designed in Art. 143.

Solution: For this column,

$$A = 19.8 \text{ sq. in.}$$
$$r = 5.62 \text{ in.}$$
$$\frac{L}{r} = \frac{25 \times 12}{5.62} = 53.5$$

Hence,

$$V = \frac{250{,}000}{100}\left(\frac{100}{53.5 + 10} + \frac{53.5}{100}\right) = 5{,}300$$

If the column is single latticed, as shown in Fig. 228, this shear will be equally divided between two bars, and the actual stress in each bar will be $5{,}300/2 \times 15.22/13.25 = 3{,}040$ lb. A thickness of one-fortieth the distance between rivets would require these bars to be 0.38 in. thick; hence, $\frac{3}{8}$-in. bars would be sufficient to comply with this condition. A width of $2\frac{1}{2}$ in. would commonly be adopted for the lattice bars, for such a column; hence, the stress in the bar would equal $3{,}040 \div (2\frac{1}{2} \times \frac{3}{8}) = 3{,}240$ lb. per square inch.

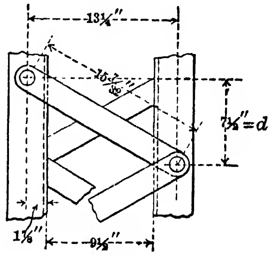


FIG. 228.

The value of the radius of gyration for a rectangular bar of width b and thickness $t = \sqrt{bt^3/12bt} = t\sqrt{1/12} = 0.288t$; hence, L/r for such a bar $= \dfrac{15.22 \times 8}{0.288 \times 3} = 140$ approximately. The allowable unit stress is $15{,}000 - \frac{1}{4}(140)^2 = 10{,}100$ lb. The secondary stress in bars of such flat slopes would not be large but will be computed in order that its effect may be seen.

The direct stress in the channels of the previous problem = 12,600 lb. per square inch; hence, the reduction in the distance d under load would be $7.5 \times 12{,}600/29{,}400{,}000 = 0.00321$ in. Therefore, the length of the lattice bar would be decreased by the following amount:

$$\sqrt{(13\tfrac{1}{4})^2 + (7\tfrac{1}{2})^2} - \sqrt{(13\tfrac{1}{4})^2 + 7.49679^2} = 0.0016 \text{ in.}$$

which corresponds to a stress of 3,080 lb. per square inch. The maximum stress, including secondary stress, therefore equals 6,620 lb. per square inch, which is considerably less than the allowable stress. If an allowable unit stress in the lattice bars somewhat higher than that for the main section is considered permissible, the stress in the bars will be still more on the safe side.

If the load is intentionally eccentric, as in the columns treated later, the same general method may be adopted, the excess fiber

stress and shear corresponding to it being obtained from the formula given later for such columns.

146. Width of Lattice Bars and Tie Plates.—In addition to designing the lattice bars to carry the stress determined by the preceding method, it is common to impose certain arbitrary conditions as to size of bars, tie plates, and rivets. The following clauses based upon the "General Specifications for Steel Railway Bridges," published by the American Railway Engineering Association in 1935, are typical of such restrictions:

The minimum width of lattice bars shall be 2½ inches for ⅞-inch rivets, 2¼ inches for ¾-inch rivets, and 2 inches if ⅝-inch rivets are used. For main members the thickness shall not be less than one-fortieth of the distance between end rivets for single lattice, and one-sixtieth for double lattice. For bracing members the corresponding figures shall be one-fiftieth and one seventy-fifth. Shapes of equivalent strength may be used.

Five-eighths-inch rivets shall be used for latticing flanges less than 2½ inches wide, and ¾ inch for flanges from 2½ to 3½ inches wide; ⅞-inch rivets shall be used in flanges 3½ inches and over, and lattice bars with at least two rivets shall be used for flanges over 5 inches wide.

The diameter of the rivets in lattice bars shall not exceed one-third of the width of the bar and there shall be at least two rivets in each end of lattice bars connected to flanges more than 5 inches wide.

The inclination of lattice bars with the axis of the member shall be approximately 60° for single latticing and 45° for double latticing and when the distance between rivet lines in the flanges is more than 15 inches, if single riveted bars are used, the lattice shall be double and riveted at the intersection.

End tie plates on main members shall have a length not less than one and one-fourth times the distance between the lines of rivets connecting them to the flanges, and intermediate plates not less than three-quarters this length. Their thickness should not be less than one-fiftieth this distance for main members and one-sixtieth for bracing members.

In a latticed column it is evidently essential that each rib between points of connection of the lattice bars shall be strong enough as a column to carry its share of the total load, hence the distance apart of the lattice bars when measured along the rib should be such that L/r for the rib, L being taken as the distance between latticing rivets, should be no larger than the corresponding term for the whole column; this, however, is seldom a limiting factor in the design of the latticing, the empirical rule as to maximum slope of the lattice bars being usually sufficient to cover this point.

The American Railway Engineering Association also specifies that the slenderness ratio of the portion of the flange included between the lacing-bar connections shall not be more than 40 or more than two-thirds of the slenderness ratio of the member.

In connection with this subject, it should be said that the columns of the famous Forth Bridge are of circular section, thus requiring no lattice bars or diaphragms and forming an ideal section so far as strength is concerned. These columns, however, were built in position, a method entirely different from American practice, in which the columns are built in the shops of the fabricating company and shipped intact to the bridge site, a method that limits the size of the column.

147. Rivet Pitch.—The rivet pitch in built-up columns should be small enough to ensure that wrinkling of the different parts between the rivets should not occur and to distribute properly the stress throughout the cross section at the ends and at intermediate points where concentrated loads may be applied. The common rule is to use no spacing along the column axis greater than 7 times the diameter of the rivet or 12 times the thickness of the thinnest connected piece except for web stitch rivets and to use at the ends and other points of application of the load a maximum pitch of 4 times the diameter of the rivet for a length equal to $1\frac{1}{2}$ times the maximum width of the member. If the bending moment carried by the column is large, as may be the case if loads of considerable eccentricity are applied, the rivet pitch should be investigated by the methods used for plate girders.

148. Distribution of Normal Stresses on Cross Sections of Straight Bars.—If the resultant stress on any cross section of a bar does not pass through its centroid, the force is said to be *eccentric.* The effect of eccentric application of the load is to subject the section to a combination of direct stress and bending moment and to cause a maximum fiber stress considerably greater than would otherwise be the case. Such a loading should be avoided if possible. A similar condition arises if the resultant force on the cross section is due to a direct force acting at the center of gravity and a bending moment due to transverse flexure instead of eccentricity; and the two cases may be treated in the same manner.

General equations for the fiber stress at any point of a cross section of any shape due to a combination of direct stress and

bending moment are complicated and will not be given here, the reader being referred for a complete treatment of the subject to Spofford's "Theory of Continuous Structures and Arches."

The usual problem, that of determining the extreme fiber stress on a symmetrical cross section of a straight bar, may be accomplished as follows: Consider first a straight bar subjected to a resultant thrust, acting parallel to its axis but not applied at the center of gravity of the cross section; and consider the bar to be so short that column action may be disregarded. Let the cross section and point of application of the load be as shown in Fig. 229.

Let V = the vertical component, lb., of a resultant thrust acting at point a.

A = area of cross section, sq. in.

I_h = moment of inertia of cross section about axis hh.

I_v = moment of inertia of cross section about axis vv.

f = compressive fiber stress at any corner (extreme fiber with respect to both axes).

Then,

$$f = \frac{V}{A} \pm \frac{Vxb}{2I_v} \pm \frac{Vyd}{2I_h}$$

The last two terms of this equation give the fiber stress due to the bending moment resulting from the eccentric application of the load. If the piece is subjected to a direct axial thrust and transverse loads, the same equation would apply, but Vx and Vy would have to be replaced by M_h and M_v, respectively, the bending moments due to transverse loads acting in planes hh and vv, respectively.

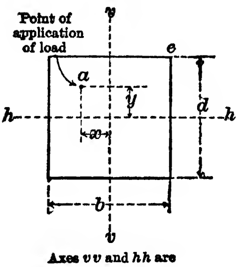


FIG. 229.

The proper signs to use for the last two terms may be determined from the character of the bending moment for the corner under consideration, with respect to the hh and vv axes; *e.g.*, for the compression at corner e the equation would be

$$f = \frac{V}{A} - \frac{Vxb}{2I_v} + \frac{Vyd}{2I_h}$$

Ordinarily, if an eccentric load is used, it is applied in one of the principal axes, in which case the expression for f would include but two terms. If the applied force is a pull instead

of a thrust, the same equation holds, but a positive result would give the tensile fiber stress.

The serious effect of an eccentric load may be readily determined by considering the cross section of one of the columns designed in Art. 143. Suppose, for example, that the resultant force on the cross section of the column shown in Fig. 224, instead of being applied at the centroid, is applied at a point 2 in. to the right of axis *vv* and 2 in. above axis *cl*. The compressive stress in the column will then evidently be a maximum at the corner marked x and will be given by the following equation:

$$f = \frac{430{,}000}{33.43} + \frac{430{,}000 \times 2 \times 9}{1{,}175} + \frac{430{,}000 \times (2 - 1.30) \times (9.0 - 1.30)}{1{,}532} = 20{,}900 \text{ lb.}$$

149. Effect of Combined Flexure and Thrust or Pull on a Column or Tension Member.—The effect upon a column of an eccentric application of an axial force in increasing the stress due to the bending of the column has been discussed in Art. 135. Columns and tension members are also frequently subject to flexure as well as to axial stress due to the lateral forces applied between their ends, such as the weight of a horizontal bridge chord or the eccentric application of loads by brackets such as in a building column carrying girders or brackets.

The author knows of no theoretical treatment of this problem the validity of which has yet been demonstrated by adequate experiments. It is common practice to compute the fiber stress due to flexure by the application of the beam formula, add it to the fiber stress due to the axial load, and design the member so that the maximum unit stress shall not exceed the allowable unit stress as determined by the column formula. This is the method adopted in the 1935 specifications of the American Railway Engineering Association.

150. Building Columns under Eccentric Loads.—The following discussion shows a method of computing the bending moment due to an eccentric load applied to a column of a one-story building by a traveling crane running on a track supported by brackets.

If the column is assumed as pin-ended, the curve of bending moments will be as shown in Fig. 230. The column is held at

the top by connection to the truss and at the bottom by friction at the base and by foundation bolts; hence, the bending moment Px of the eccentric load is resisted by horizontal forces at the ends of the column which form a couple the value of which is also Px. The maximum bending moment occurs at the load and depends upon the height at which the latter is placed. The maximum possible value is evidently Px, which would occur with the load at either end of the column. The curve of moments is represented as changing suddenly at the point of application of the load; this is not strictly correct, however, since such a condition could not actually occur if the load were applied to a bracket, as the latter would distribute its bending effect by means of the rivets connecting it to the column.

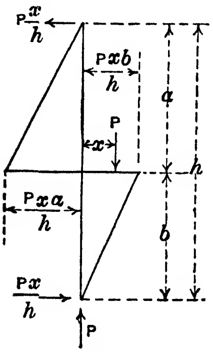


FIG. 230.—Curve of bending moments. Eccentrically loaded column.

It is seldom that such columns need to be treated as pin-ended, since the ends are usually partially if not completely fixed. The effect of fixing the ends is to reduce the bending moments considerably.

Columns in high buildings are often eccentrically loaded and must be carefully studied. As such columns are usually continuous over a number of stories and held more or less rigidly at each floor by the floor-beam connections or by wind bracing, this problem is a difficult one to treat mathematically and will not be considered at this point.

151. Design of Cast-iron Columns.—The design of a cast-iron column differs somewhat from that of a steel column; hence, the following treatment of hollow circular cast-iron columns under eccentric load is appended.

Mode of Procedure.

1. Design the column for its direct load, assuming a reasonable unit stress.

2. Make the metal sufficiently thick to ensure a good casting. A thickness of 1 in. should, in general, be used, although in exceptional cases ¾-in. or thinner metal may be permitted. The 1937 New York Building Laws specify that cast-iron columns shall have an outside diameter or side of at least 5 in. and thickness of metal of at least one-twelfth of the diameter or least dimension of cross section, with a minimum of ¾ in.

3. Compute the maximum fiber stress in the column as designed in accordance with the foregoing requirements. In this computation, any reasonable eccentricity of the load must be considered.

4. If the fiber stress thus obtained differs too much from the allowable stress, revise the computation.

The following method illustrates the design of such a column and shows a method of determining the eccentricity:

Assume beams A and B, Fig. 231, each to have a live reaction of 10,000 lb. and a dead reaction of 5,000 lb. and to be 12 in. wide. Then, the maximum load on the column = 80,000 lb. Assuming

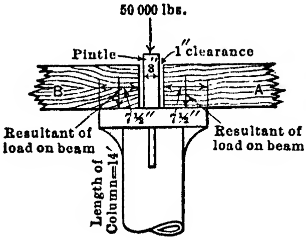


FIG. 231.

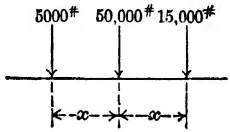


FIG. 232.

a unit stress of 4,000 lb. for a trial section gives 20.0 sq. in. A column 8 in. in external diameter and of 1-in. material has an area of 22.0 sq. in. and will do for a trial section. Since it is possible that one of the beams may be fully loaded and the other only partially loaded, it is evident that the resultant of the loads on the column may not act through its center of gravity and that in consequence the column will be eccentrically loaded. The maximum eccentricity will occur when one of the beams, say beam A, has its full live load and the other is unloaded. The loading will then be as shown in Fig. 232, and the resultant will act at a distance from the line of action of the center load equal to $x - \dfrac{60{,}000x}{70{,}000} = 0.14x$ and will have a value of 70,000 lb. The total load in this case is less than the maximum, but the effect of the eccentricity may be sufficient to make this the critical case.

In order to obtain the actual eccentricity, it is necessary to ascertain the value of x. This involves the design of the column cap and the location of the line of action of the resultant reaction on each beam. This latter cannot be ascertained with exactness,

since its distribution depends upon the relative elasticity of the beam, column, and column cap and upon the crushing strength of the wood. If the beam, cap, and column were to be perfectly rigid, then the reaction on each beam would be distributed uniformly over its bearing surface; on the other hand, if the column and cap were to be rigid and the beam elastic, the tendency would be to throw all the pressure to the edge of the cap and to make that the point of application of the resultant. This latter condition could, however, not really be reached, since the wood would be crushed at the point of bearing, which would relieve the pressure there and distribute it over a greater length of beam. The true position of the resultant is evidently somewhere between the center of bearing and the edge of the cap.

To design the cap and determine the position of the resultant reaction, let the following assumptions be made:

1. Pressure varies uniformly from a maximum at edge of cap to zero at end of beam.

2. Pressure at edge of square cap under maximum load equals allowable crushing strength of the wood across the grain, which may be assumed as 350 lb. per square inch for yellow-pine beams, or 4,200 lb. per lineal inch for a 12-in. beam. Figure 233 shows the distribution of pressure at the end of one of the beams, based upon the assumption just made. The distance d may be determined by dividing the maximum beam reaction by one-half the allowable crushing strength per lineal inch. For the case under consideration, this gives $7\frac{1}{2}$ in., approximately.

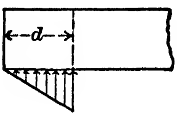


FIG. 233.

The value of x is

$$1\tfrac{1}{2}\text{ in.} + 1\text{ in.} + (\tfrac{2}{3})(7\tfrac{1}{2}\text{ in.}) = 7.5\text{ in.}$$

The eccentricity under the partial loading is then $(7.5)(0.14) = 1.05$ in. The eccentricity due to bending of the column will be neglected here, as being an unnecessary refinement for a material as variable as cast iron; hence, the fiber stress due to the eccentricity will be

$$\frac{4 \times 70{,}000 \times 1.05}{0.049(8^4 - 6^4)} = 2{,}140\text{ lb.}$$

To determine whether the column is safe, this eccentric stress should be added to the maximum stress due to the direct load as determined by the column formula, and the sum should not exceed the allowable unit stress in a short column. If formula (28) Art. 138 is applied, the maximum fiber stress due to direct load is given by the expression

$$f = \frac{P}{A} + \frac{32L}{d} = \frac{70{,}000}{22} + \frac{32 \times 14 \times 12}{8} = 3{,}180 + 670 = 3{,}850$$

The eccentric stress added to this gives a total of 5,990 lb., which is less than 6,100 lb., the allowable unit stress by the formula for short columns; hence, the column has an area that is only slightly in excess of the required amount and may be used.

152. Design of Iron and Steel Tension Members.—The design of tension members involves little more than the selection of

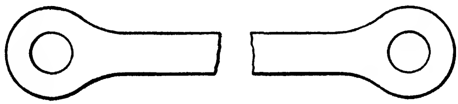


FIG. 234.

bars with sufficient net area to carry the total stress without exceeding the allowable unit stress. Steel or iron tension members may be divided into two general types, *viz.:* solid bars rectangular or circular in cross section, and built-up members composed of structural shapes riveted together. Solid bars are used generally in pin trusses for diagonals and bottom-chord members and in Howe trusses for verticals. Built-up members are generally employed for tension members in riveted trusses and for the end hangers in pin trusses.

Of the first type of member, the eyebar shown in Fig. 234 is used most commonly. Such bars are made by most of the large steel manufacturers and are fully described in their handbooks. The heads of these bars are designed so that the bar, if tested to destruction, will fail in the body rather than in the head, and the engineer should specify that full-sized tests should give this result and not attempt to proportion the heads. In determining clearance, the dimensions of the heads given by the makers may be used, noting that the diameter of the head depends upon the size of pin hole. Eyebars may be manufactured to any thickness above the minimum size quoted by the

makers, but a thickness above 2 in. should not generally be employed, since such thick bars are not likely to be of the best material. A good rule to observe in selecting bars is to keep the thickness between 1 and 2 in. Eyebars are generally used in

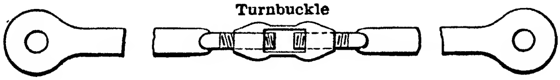


FIG. 235.

pairs, since an odd number of bars would give a poor arrangement on the pin. For counters, adjustable eyebars such as those shown in Fig. 235 may be used, the two bars being connected by a turnbuckle or sleeve nut; iron rods with loops formed by welding such as those shown in Fig. 236 may be used if the

FIG. 236.

counter stresses are small. For the verticals of Howe trusses, iron rods, with screw ends fastened by nuts bearing on washers supported by the top chord, are generally employed.

In proportioning adjustable members, allowance must be made for the decrease in section due to the screw threads. It is usually

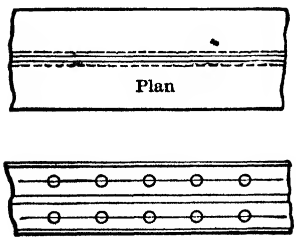


FIG. 237.

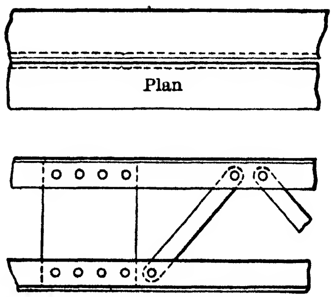


FIG. 238.

advisable to upset the screw end, *i.e.*, to make it of larger diameter than the body of the bar, so as to give sufficient area at the root of the thread to make the bar as strong there as elsewhere. For short rods, however, the labor cost involved in this process may be greater than the saving of material would warrant.

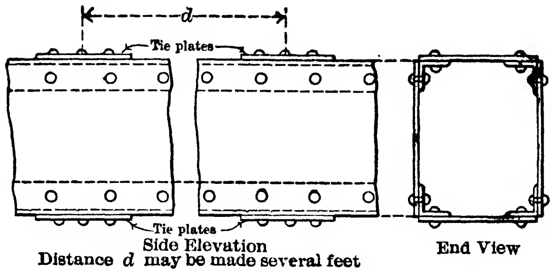


FIG. 239.

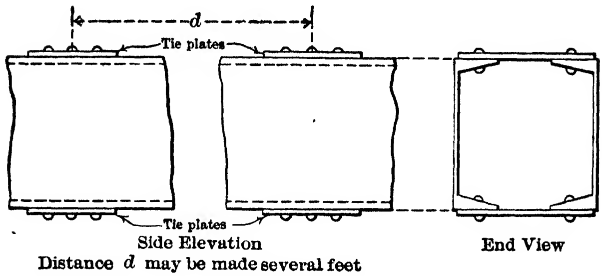


FIG. 240.

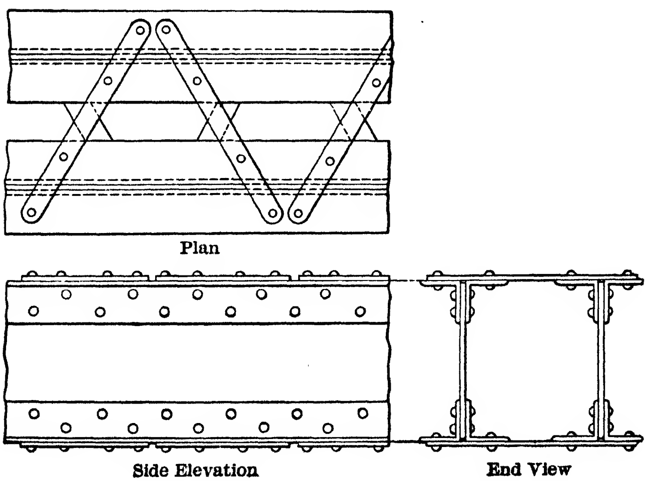


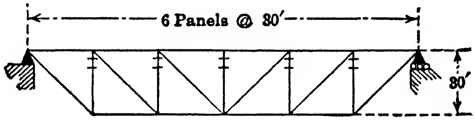
FIG. 241.

Riveted tension pieces may be made of various sections. Figures 237 to 241 show typical members and need no explanation. Although these members do not need latticing or tie plates to keep the separate parts from buckling, some connection between them should be used to make the different parts act together. The design of these details must be left to the judgment of the engineer.

Problems

64. Design the vertical truss members of this single-track bridge with no counters, using channels; make allowance for reversal of stress, if necessary, according to the following rule:

Members subject to alternate stresses of tension and compression shall be proportioned for the stresses giving the largest section, on the assumption that each stress is increased by 50 per cent of the smaller.



PROB. 64.

Dead load all on top chord, 1,000 lb. per foot.

Live load on top chord, Cooper's E_{60} locomotive.

Use for impact 43 per cent.

Use not less than 12-in. channels for verticals, and place channels 7½ in. back to back, flanges to be turned out.

CHAPTER XII

PIN AND RIVETED-TRUSS JOINTS

153. Bridge Pins Described.—A bridge pin may be considered as a large rivet that has to carry bending moment as well as shear and bearing. The difference between a bridge pin and a rivet is due to construction. A rivet is driven while red-hot and is then headed, usually under a high pressure, so that it completely fills its hole and binds together so tightly the different pieces through which it passes that there is little, if any, opportunity for it to become distorted through bending. A bridge pin, on the other hand, is always made somewhat smaller than the pinhole, and the attempt is not made to hold together tightly by the pin the members coming on it; hence, it can bend and must be designed to resist bending moment as well as shear. It must also have sufficient bearing area on each connected piece to make it safe against failure by crushing of either pin or member, this frequently being secured by increasing the thickness of the member by the addition of a plate or plates rather than by an increase in the diameter of the pin.

154. Arrangement of Members on Pin.—The actual design of a pin as carried out in practice is a very simple process after the arrangement of the different members upon the pin is once satisfactorily accomplished. To arrange properly the members is, however, a somewhat complicated problem, since the arrangement on one pin cannot be worked out independently but must be studied with due regard to its effect upon the other parts of the truss.

The following rules should be observed in arranging the different members:

1. Allow sufficient clearance. This is extremely important, since insufficient clearance gives trouble in erection. For trusses of ordinary spans, the heads of all eyebars coming on the pin should be assumed as $1/16$ in. thicker than their normal thickness for bars 8 in. or less in width, $1/8$ in. for bars more than 8 in. and not more than 12 in., and $3/16$ in. for bars more than 12 in. wide,

and the total clearance between riveted members should be at least ½ in. Figure 242 shows the method of determining clearance in a simple case.

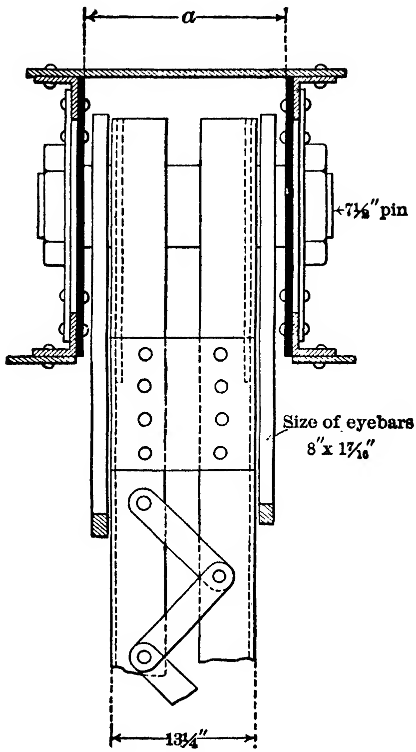


FIG. 242.

Distance a = distance between chord webs, and should, on the basis of ⅞-in. rivets with heads ⅝ in. deep, be made

$$13\tfrac{1}{4} + 2(1\tfrac{1}{2}\text{ in.}) + 2(\tfrac{5}{8}) + \tfrac{1}{2} = 18\text{ in.}$$

If rivets are countersunk and chipped, this distance may be reduced by 1¼ in. If rivet heads are flattened or are countersunk but not chipped, the distance between channels may be varied accordingly. (Note that rivet heads if countersunk but not chipped usually project ⅛ in. above the surface and that they are frequently flattened without being countersunk, so that

⅞-in. rivets project but ⅜ in. above the surface and smaller rivets still less.)

2. Arrange eyebars so that their center lines will be parallel or nearly so with the center line of the truss. It is seldom possible if compact joints and small pins are to be obtained to follow this rule very closely, but it is common to specify that no bar shall deviate from the center line of the truss by more than $\frac{1}{16}$ in. per foot in length of the bar. In cases where a greater allowance than this is necessary the bar should be bent to the proper slope before being annealed.

As it is sometimes difficult to arrange the different members so that all the conditions above will be observed, the student

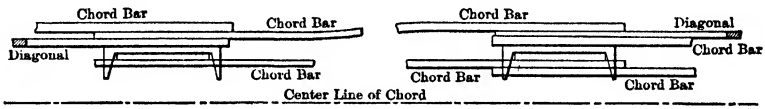


FIG. 243.

is advised to lay out to a large scale, say 1½ in. = 1 ft., the different joints of each chord. The distance apart of the various joints should also be plotted to scale, but this scale may be much smaller than the scale of details. The different members can then be drawn from joint to joint, and the deviation from the center line can be determined by scale. This method is indicated in Fig. 243 for two joints of a bottom chord. To carry out the method completely the chord should be drawn on a sheet of sufficient length to show all the joints or, if the truss is symmetrical, all the joints up to and including the joint on the far side of the truss center, and the top chord should be plotted above the bottom chord in a similar manner.

It will be noticed that in order to secure the above arrangement the channel flanges may have to be cut away. This is undesirable but is frequently necessary for members in which the flanges are turned out in order to avoid the necessity of using a pin of large diameter. If this is done, the channels should be reinforced by web plates extending throughout the distance over which the flanges are cut, unless the channel webs alone without the flanges are of sufficient strength to carry the compression. In investigating such a case in a compression member, the column formula should be applied, the distance from the

center of pinhole to the first row of rivets beyond the point where the flanges are cut being used for the unsupported length. The American Railway Engineering Association specifies that there shall be enough pin plates on forked ends to make the section of each jaw equal to that of the member.

155. Minimum Size of Pins.—Before it is possible to complete the arrangement of the various truss members, it is necessary to make some assumption as to the size of the pins, since it is usually necessary to add pin plates to the riveted members either for the purpose of strengthening the member against crushing on the pin or to make up for the area taken out by the pinhole, since the pin, unlike a rivet, does not completely fill the hole and hence cannot be counted upon to carry compression. To determine the size approximately requires some experience; the lowest limit is, however, usually fixed by the width of the widest eyebar connected to the pin as shown below.

Let f_b = allowable bearing stress per sq. in. on pin.
f_t = allowable tension stress per sq. in. in bar.
w = width, in., of widest bar coming on the pin.
t = thickness, in., of same bar.
d = diameter, in., of the pin.

Then,

f_bdt = bearing value of the bar on the pin

and

f_twt = tensile strength of the bar

Putting these equal gives

$$f_bdt = f_twt$$

Therefore,

$$d \geq w\frac{f_t}{f_b}$$

For example, if $f_t = 16{,}000$, $f_b = 24{,}000$, and the width of the widest bar coming on the pin is 6 in., the diameter of the pin should not be less than $\frac{16}{24} \times 6 = 4.0$ in.; hence, the pin in this case should be assumed as not less than 4.0 in. in diameter. Whether it should be assumed as larger is a matter that can be estimated only by experience, but it should be noted that it is wiser to assume the pin *too small* rather than too large, since, in the former case, pin plates, which are somewhat thicker than are needed, will be selected at first and these may be easily reduced

in thickness if it is found that the diameter of the pin should be larger than that assumed. The only exception to the statement that it is usually on the safe side to assume the pin too small is when a reinforcing plate is not needed to increase the bearing resistance on the pin but is required to make up for reduction in section by the pinhole. This sometimes happens near the center of the top chord of a simple truss, but in arranging the members on a pin it is wise always to allow for at least one pin plate upon the chord at every joint, a 3/8-in. plate if rivets do not require countersinking, and a 7/16-in. plate if the clearance is so small as to make countersinking necessary.

156. Stresses Causing Maximum Moment and Shear.—After the arrangement of the members is satisfactorily accomplished,

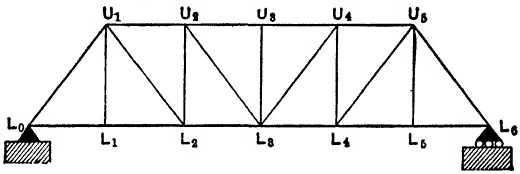


FIG. 244.

it is necessary to compute the maximum stresses that act simultaneously on each pin and that seem likely to produce critical bending moments and shears.

In order to obtain these simultaneous stresses, it is sometimes necessary to calculate anew the stresses in a number of bars under the loading that produces the maximum in one of them. For example, for the truss shown in Fig. 244, the maximum moment on pin L_2 may occur under the loading that produces maximum stress in either chord L_1L_2, diagonal U_1L_2, or chord L_2L_3; hence, it becomes necessary to compute the stresses in the bars connected by pin L_2 under all these conditions of loading.

In all cases, it is the horizontal and vertical components of the stresses that are desired, and the results should be checked by noting whether the pin is in equilibrium under the action of these components, *i.e.*, whether $\Sigma H = 0$ and $\Sigma V = 0$. There is one point here that may cause trouble. The floor beam in an ordinary bridge is frequently connected to the vertical above the pin; hence, the vertical stress that reaches the pin is not the stress in the vertical as a whole but is the stress in the vertical

below the floor beam. To avoid confusion, no attention should be paid to the actual stress in the vertical, whether the floor beam is above or below the pin, but the stress coming to the pin from the vertical should be placed equal to the vertical components of the diagonal stress. After the stresses are found, it is desirable to make sketches for each joint showing the stresses in the bars meeting at the joint.

157. Computation of Maximum Moment and Shear.—The next step is to determine the maximum bending moment and shear on the pin for each loading. For the moment, this can best be accomplished by plotting the curves of vertical and hori-

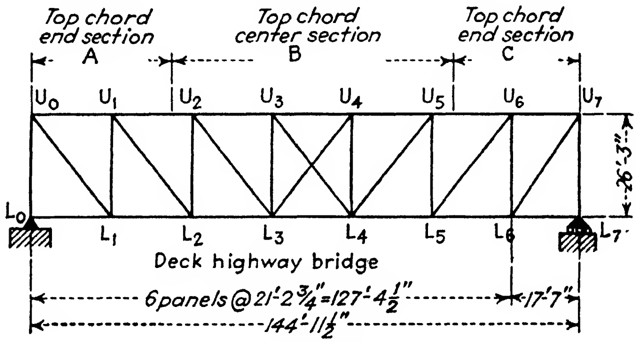


FIG. 245.

zontal moments and determining by inspection or trial the section where the maximum resultant moment occurs. This resultant can be determined with sufficient accuracy by graphical methods, since its value equals that of the hypothenuse of a right-angled triangle, the sides of which equal the vertical and horizontal moments, respectively.

It is seldom that the maximum shear needs to be carefully figured, since ordinarily the bending moment determines the size of the pin. The shearing stress should always be investigated, however, and in doubtful cases its maximum value determined by the method given above for bending moment.

If the size of the pin as computed differs materially from that assumed, the thickness of the pin plates should be investigated and revised if necessary. This should not be done too hastily, however, since it is customary to use but few different-sized pins, in a truss, and it may happen that the pin as com-

puted may not be the one that it is finally decided to use. Examples of pin computation will now be given.

158. Computation of a Top-chord Pin for Truss Shown in Fig. 245.

Problem: Determine the size of pin and thickness of bearing area of chord and vertical at joint U_2, using following allowable unit stresses:

Bearing on pin, 24,000 lb. per square inch.
Bending on pin, 27,000 lb. per square inch.
Shear on cross section, 13,500 lb. per square inch.

Solution: For this pin the only loading which needs to be considered is that which produces the maximum stress in diagonal U_2L_3. The reason for this is that the top chord is continuous at the joint and spliced elsewhere as shown. This is inconsistent with the theory upon which the computation of truss stresses is based but is the common practice and probably does not affect the stresses materially, though it simplifies greatly the construction. Under this condition the duty of this pin is to connect the diagonal to the top chord and vertical, the horizontal component of the diagonal stress being transmitted by the pin to the chord, and the vertical component to the vertical. The actual stress in the chord, therefore, is not an element in the pin design, and needs to be considered only in investigating the strength of the chord at the cross section through the pinhole. Figure 246 shows the maximum stress in the diagonal with its vertical and horizontal components.

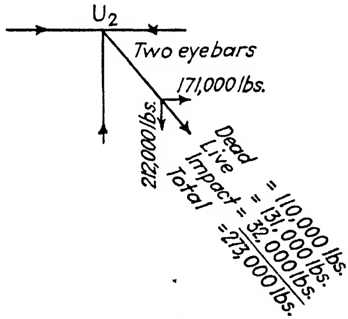


FIG. 246.

The allowable unit stress in the diagonal of 18,000 lb. per square inch tension, and 24,000 lb. per square inch bearing gives for the minimum-size pin required for bearing on the 6-in. diagonal bar $\frac{18}{24} \times 6$ in. = 4.50 in.

Since the stress in the diagonal is large, the size of pin that will be assumed in determining bearing areas will be taken as somewhat larger than the minimum size, say 5 in. The bearing area required by this assumption may then be computed, by assuming the stress to be distributed uniformly over a surface equal to the plane diametrical section of the pin.

Total thickness of bearing required on chord

$$= \frac{171,000}{24,000 \times 5 \text{ in.}} = 1.43 \text{ in.}$$

Total thickness of bearing required on vertical

$$= \frac{212,000}{24,000 \times 5 \text{ in.}} = 177 \text{ in.}$$

SCHEDULE OF SIZES FOR TRUSS SHOWN IN FIG. 245

Top chord, section A	1 cover plate	28″ × ½″
	2 top angles	4″ × 4″ × ½″
	2 bottom angles	5″ × 3½″ × ½″
	2 webs	22″ × ½″
Top chord, section B	1 cover plate	28″ × ½″
	2 top angles	4″ × 4″ × 7/16″
	2 bottom angles	5″ × 3½″ × 7/16″
	2 webs	22″ × 7/16″
	2 webs	14½″ × 7/16″
Top chord, section C	1 cover plate	28″ × ½″
	2 top angles	4″ × 4″ × ⅜″
	2 bottom angles	5″ × 3½″ × ⅜″
	2 webs	22″ × ½″
L_0L_1 and L_6L_7	2 channels	12″—25 lb.
L_1L_2	2 eyebars	7″ × 2″
L_2L_3	4 eyebars	7″ × 1 11/16″
L_3L_4	4 eyebars	7″ × 1 15/16″
L_4L_5	4 eyebars	7″ × 1 9/16″
L_5L_6	2 eyebars	7″ × 1 11/16″
U_0L_0	1 cover plate	21″ × 7/16″
	2 webs	20″ × ½″
	2 webs	13″ × 9/16″
	4 angles	5″ × 3½″ × 9/16″
U_1L_1	2 channels	15″—40 lb.
	2 plates	12″ × ½″
U_2L_2, U_3L_3, U_4L_4	2 channels	15″—33.9 lb.
U_5L_5	2 channels	15″—40 lb.
U_6L_6	2 channels	15″—40 lb.
	2 plates	12″ × 9/16″
U_7L_7	1 cover plate	21″ × 7/16″
	2 webs	20″ × ½″
	2 webs	13″ × 9/16″
	4 angles	5″ × 3½″ × 9/16″
U_0L_1	2 eyebars	7″ × 1 11/16″
	2 eyebars	7″ × 1 9/16″
U_1L_2	2 eyebars	7″ × 1″
	2 eyebars	7″ × 1¼″
U_2L_3	2 eyebars	6″ × 1 9/16″
U_3L_4	2 eyebars	5″ × ⅞″
U_4L_3	2 eyebars	5″ × 1″
U_5L_4	2 eyebars	6″ × 1¾″
U_6L_5	2 eyebars	7″ × 1″
	2 eyebars	7″ × 1 7/16″
U_7L_6	2 eyebars	7″ × 1 7/16″
	2 eyebars	7″ × 1 11/16″

In order to obtain these thicknesses, it is necessary to add a ½-in. pin plate to each half of the vertical. The chord thickness need not be increased for bearing, but a plate should be added to make up for the reduction in area due to the pinhole. This reduction =

$$5 \times 1\tfrac{3}{4} = 8.75 \text{ sq. in.}$$

A 22- by $\tfrac{7}{16}$-in. pin plate on each rib of the chord gives a net area of $16\tfrac{3}{4} \times \tfrac{7}{8} = 14.7$ sq. in., which is ample. A thinner plate should not be

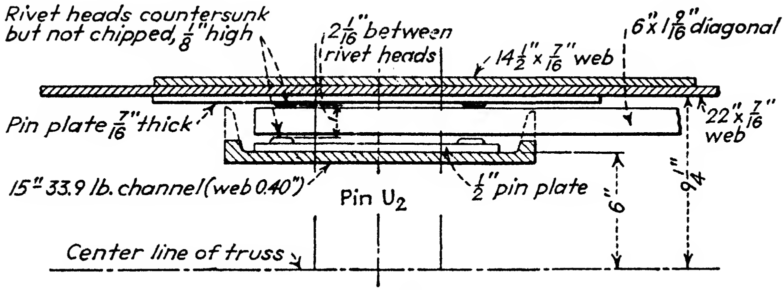


FIG. 247.

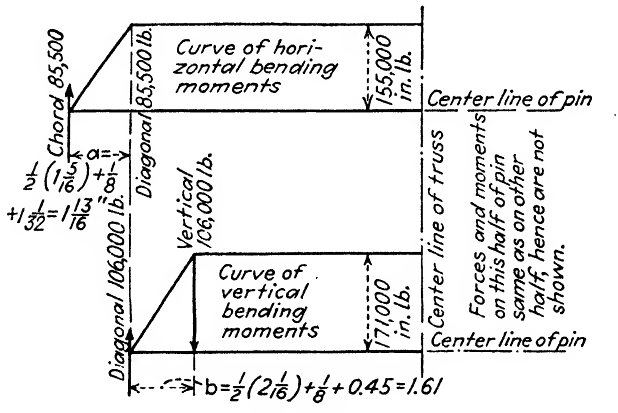


FIG. 248.

assumed, since the rivets may have to be countersunk, and it is inadvisable to countersink ⅞-in. rivets in a plate thinner than $\tfrac{7}{16}$ in.

The width adopted for the top chord is 18½ in. between the 22-in. webs, and for the vertical 12 in. out to out of webs. These widths are determined, principally, by the conditions at the end joint, which will not be considered here. It should be noted, however, that the width of the vertical is given for the distance out to out of webs, instead of between webs. This distance is made constant for all verticals, so that the lengths of the floor beams may be the same regardless of the thickness of the vertical

channel webs. The arrangement of members adopted at the joint is shown by Fig. 247.

The forces acting on the pin were assumed, in determining the required bearing area, to be distributed uniformly over a plane surface equal to the diametrical section of the pin. In computing moments, however, it is the usual custom to consider these forces as concentrated at the center of the bearing areas. The distances between the points of application of these stresses should be computed upon this basis, and the horizontal and vertical bending moments on the pin determined. The results of these computations are shown in Fig. 248.

It is evident that the maximum moment in this case is the resultant of the maximum horizontal and the maximum vertical moments, since these both occur at the same section. This is found graphically, as shown by Fig. 249, and equals 231,000 in.-lb.

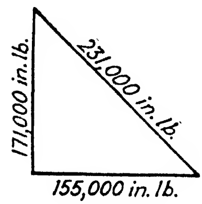


FIG. 249.

With an allowable fiber stress in bending in the pin equal to 27,000 lb., a 4½-in. pin is required to carry this moment (see table for Maximum Bending Moments on Pins in "Carnegie Pocket Companion"). The area of a 4½-in. pin is 15.9 sq. in. which, at a unit stress of 13,500 lb. per square inch assumed to be uniformly distributed over the cross section, is good for 215,000 lb. shear. The maximum shear is 106,000 lb., which is less than the allowable shear.

Since the vertical pin plates were computed for a 5-in. pin, they are insufficient for a 4½-in. pin; hence, a recomputation should be made on the assumption of a pin somewhat larger than 4½ in., say 4¾ in.

159. Computation of a Bottom-chord Pin for Truss Shown by Fig. 245.

Problem: Determine the size of pin and thickness of bearing area on vertical at joint L_2, using same unit stresses as for pin U_2.

Solution: For this pin, two loadings must be considered:

1. That which produces maximum stress in chords L_1L_2 and L_2L_3.
2. That producing maximum stress in diagonal U_1L_2.

For the first case, the chord stresses are identical with the maximum stresses, since the chords of this truss were computed for a uniform load per foot, and hence these stresses may be written down at once. The difference between the chord stresses equals the horizontal component of the diagonal stress, from which the vertical component is readily obtained. The stress transferred to the vertical from the pin equals the vertical component of the diagonal stress. The stresses for this case are shown by Fig. 250.

For the second case, it is necessary to compute the stresses in the chords produced by the loading that causes maximum stress in the diagonal. This computation requires but little additional work, even if a concentrated load system is used, since the position of the loads is known and the left reaction

would have been determined in making the shear computations. The stresses for this case are shown by Fig. 251.

Had the maximum chord stresses been computed for a concentrated load system, it might have been necessary to compute the pin for three instead of two cases, since the position of the loads for maximum stress in chord L_1L_2 might have been different from the position for maximum stress in the chord L_2L_3. It will be noticed that impact is not included with above-

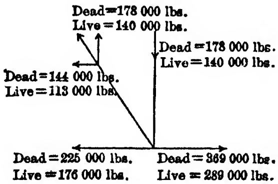


FIG. 250.

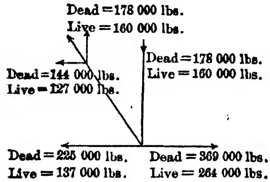


FIG. 251.

mentioned stresses. The reason for this is that the allowance for impact if figured by the method given in Art. 17 would in general give different percentages for the different bars, with the result that the forces on a pin would not always balance. It is necessary, therefore, in such cases to compute the dead and live moments separately and determine the impact as a factor of the moments and not of the bar stresses. For this particular case, inasmuch as the stresses are computed for a uniform load, the loaded lengths for maximum stress in all members meeting at the pin would be the same

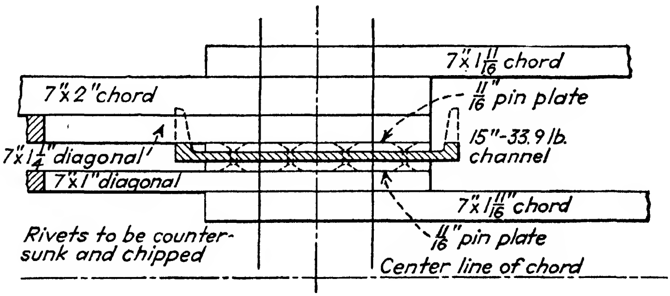


FIG. 252.

under any loading condition; hence, impact could be included in the bar stresses.

The minimum size of pin for this joint is $^{18}\!/_{24} \times 7$ in. = 5.25 in. For the loading of the second case, the vertical stress, with impact added, equals 393,000 lb.; hence, the total thickness required for bearing on the $5\frac{1}{4}$-in. pin is $\dfrac{393{,}000}{24{,}000 \times 5\frac{1}{4}} = 3.12$ in. The thickness of the channel web is 0.40 in.; hence to each web must be added 1.36-in. pin plates, or, say two $1\frac{1}{16}$-in. plates. The proposed arrangement of the different members coming on the

pin is shown in Fig. 252. This arrangement is one that gives a satisfactory location of the bars as regards the other joints of the truss.

Figures 253 and 254 show curves of moments for both loadings and should be understood without difficulty. It will be noted that in determining distances between loads each eyebar is assumed to be $\frac{1}{16}$ in. thicker than its nominal size.

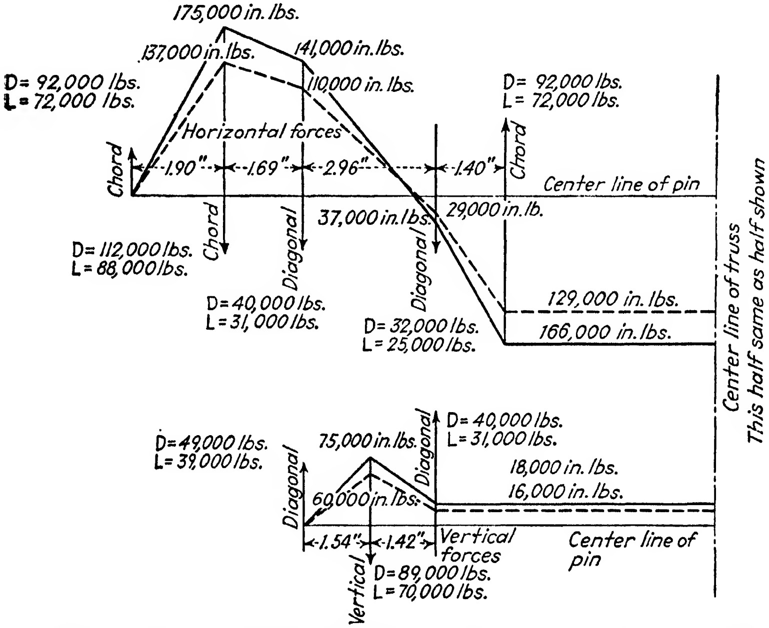


FIG. 253.—Curves of moments for Case 1. Full lines are dead moments.

It is evident that the maximum moment on the pin occurs for Case 1 and equals

$$
\begin{aligned}
D &= 175{,}000 \text{ in.-lb.} \\
L &= 137{,}000 \text{ in.-lb.} \\
I &= \underline{25{,}000} \text{ in.-lb. (by formula in Art. 17)} \\
& 337{,}000 \text{ in.-lb.}
\end{aligned}
$$

The size of pin required to carry this moment with unit stress of 27,000 lb. is $5\frac{1}{4}$-in. This is the size assumed in computing the thickness of the bearing plates on the vertical. It should be noted that for this case the thickness of these bearing plates has no influence upon the maximum moment on this pin which occurs at the next to the outermost chord bar.

Shear.—The allowable shear on the $5\frac{1}{4}$-in. pin at 13,500 lb. per square inch equals 292,000 lb.

A slight study of the pin and its applied loads shows that the maximum shear for Case 1 is

$$\begin{aligned} D &= 92{,}000 \text{ lb.} \\ L &= 72{,}000 \text{ lb.} \\ I &= \underline{13{,}500 \text{ lb.}} \\ & 177{,}500 \text{ lb.} \end{aligned}$$

For Case 2, the shear is still less; hence, the $5\frac{1}{4}$-in. pin is strong enough to carry the shear.

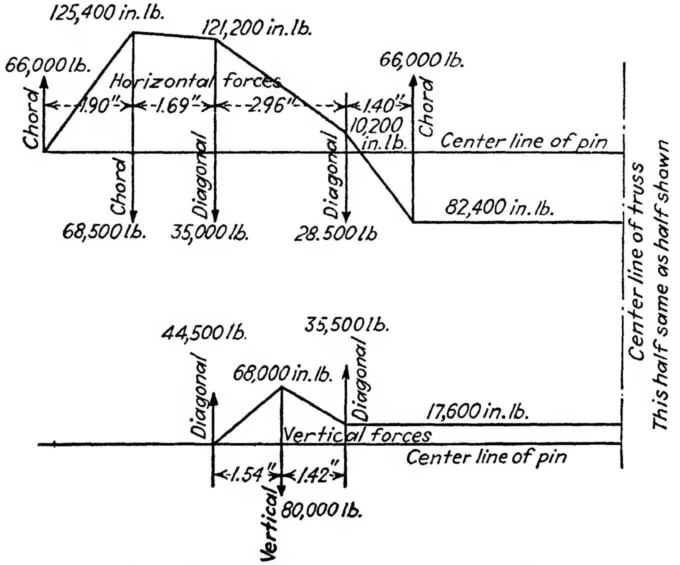


FIG. 254.—Curve of live moments for Case 2.

160. Effect upon Pin of Change in Arrangement of Members.— The student should consider carefully the comparatively great effect upon the moment of a slight change in the arrangement of the members on pin L_2. If the 2-in. chord bar were to be interchanged in position with the $1\frac{1}{4}$-in. diagonal, the maximum dead moment due to horizontal forces would be increased by 54,000 in.-lb. and the live moment proportionally. The effect of an interchange of the 2-in. chord bar with the adjoining $1\frac{11}{16}$-in. bar would be to increase the maximum horizontal dead moment by 291,000 in.-lb. and the live moment proportionally.

It is desirable to use as small pins as possible, so that the size of the eyebar heads may be kept within reasonable limits;

hence, the arrangement of the bars should be carefully studied, and the designer should bear in mind that an arrangement which will produce both positive and negative moments will usually give a satisfactory result. For example, if the arrangement of bars shown in Fig. 255 is changed to correspond to that shown in Fig. 256, the moment will be reduced, since in the first case the moment continually increases whereas in the second case the moment varies from positive to negative and then back to positive, its maximum value being far below that reached in the first arrangement.

The thickness of the bars also has an important effect upon the size of the pin, and a reduction can often be made by reducing one bar of a member in thickness and increasing another by

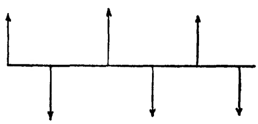


FIG. 255.

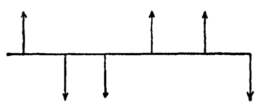


FIG. 256.

the same amount. This, of course, cannot be done if the member is composed of only two bars, since in such a case both bars must be equal in size to preserve the symmetry of the truss.

161. Pin-plate Rivets.—The determination of the number of rivets required in the pin plates sometimes requires careful study. The student should, however, have no difficulty in solving this problem if he is careful to use enough rivets to carry from each plate the stress which it receives from the pin, assuming that it receives that proportion of the total stress which its thickness bears to the total thickness of bearing. Due allowance should be made, in case several pin plates are needed, for the effect of intermediate plates upon the strength of the rivets, and it is often found desirable to make plates of different lengths so that something of the effect of a tight filler may be obtained.

162. Pin Nuts.—The nut commonly used on bridge pins is a special nut that is much thinner than the ordinary hexagonal or square nut, since its function is not to carry tension into the pin, but merely to hold the bars in place. It should be held in position by a cotter pin, since nuts not held have been known to be loosened by the impact of trains and to fall off. On very large trusses, nuts are sometimes replaced by washers which

are held in place by a rod passing through a hole bored along the longitudinal axis of the pin.

163. Packing Rings.—In order that the bending moment on the pin may not differ from the computed value, it is necessary that the eyebars should be held in the position assumed in the computations. To do this, it is sometimes necessary to use washers or collars between some of the bars. These are sometimes made of thin plates bent around the pin and sometimes of short pieces of iron pipe. When the bar is restrained by the other members so that the clearance is not more than $\frac{1}{4}$ to $\frac{1}{2}$ in., the use of such washers is unnecessary.

164. Riveted-truss Joints.—The design of the joints of riveted trusses is of equal importance with the design of the main members and should receive most careful study. The observance of the following rules is necessary in order to prevent eccentric application of the forces to the members meeting at a joint and consequent increase in fiber stress in the main members.

1. Center of gravity lines of members meeting at a joint should intersect at a point.

2. Connection rivets in a member should be arranged symmetrically about the axis passing through its center of gravity, with as few rivets as practicable in a line parallel to its longitudinal axis.

3. Members composed of a single angle, or of two angles back to back, should be connected to plates by means of lug angles in the manner illustrated by Fig. 257. The use of the lug angle is desirable, not only to diminish the eccentricity of application of the load, but also to decrease the size of the connection plate which would otherwise be necessary.

4. If stress at any joint is to be transferred from one member into a gusset plate and thence transferred to another member, the group of connection rivets in the second member should have its center of gravity coincident as nearly as possible with the point of intersection of the two members.

5. The arrangement of the connection rivets in a tension member should be such as to reduce the cross-sectional area of the member as little as possible, consistent with economy in the connection plate. In order that this result may be obtained, it is usually desirable to have not more than two rivets at right angles to the line of action of stress in the row farthest from the

point of intersection of the members meeting at the joint and to make the distance between this row and the row next to it as much as 5 in. in order that rivets in outstanding legs of angles or in channel flanges may stagger completely with the connection rivets. It should be observed that the connection rivets transfer the stress gradually from a member to a plate and that in consequence the required net area of the member decreases in passing from the edge of the plate toward the end of the member; hence, as the latter point is approached, the reduction in area of the member due to rivet holes may be very large without reducing the strength of the member.

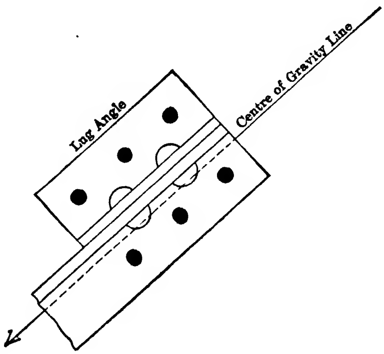


FIG. 257.

6. The size and thickness of connection plates should be determined by the following considerations:

a. The size of connection plates must be sufficient to enable the rivets necessary for connecting the different members to be properly located. In general, it is desirable to use a small rivet pitch, usually for $\frac{7}{8}$-in. rivets a $3\frac{1}{2}$-in. pitch, except where a larger pitch is required by the application of rule 5.

b. The net section across the plate at right angles to the line of action of a member must be sufficient to carry that proportionate part of the stress in the member that is transferred to the plate by the rivets between the given section and the end of the plate.

c. If the resultant stress upon *any section* of a connection plate is eccentrically applied, as determined by assuming each rivet on one side of the section to carry to the plate its proportion-

ate part of the total stress in the connecting member, the plate must be made of sufficient thickness to withstand the effect both of this eccentricity and the direct stress upon the section. This condition usually determines the thickness of the connection plates.

d. For joints where the main connection plates prove to be exceptionally thick owing to their action as eccentrically loaded

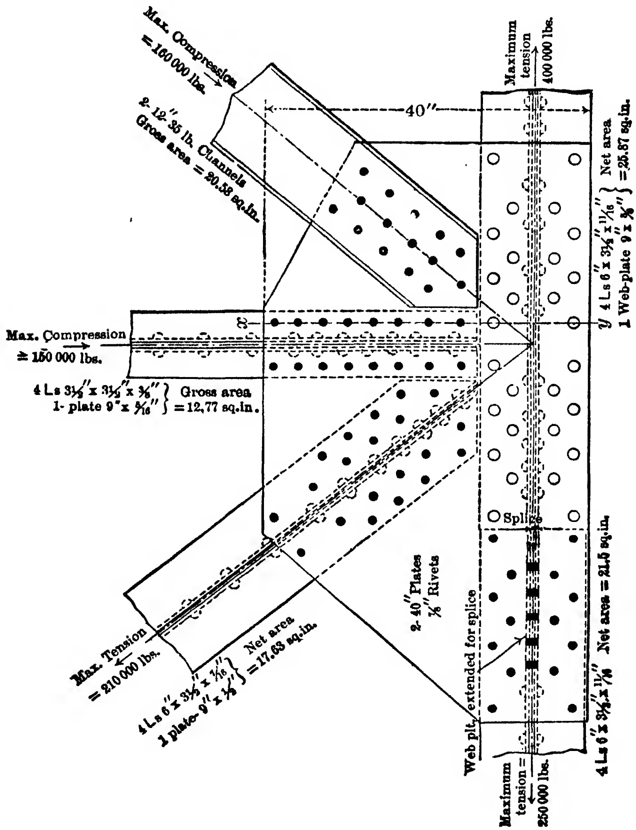


FIG. 258.

members, it is advisable to use additional plates equal in width to the chord and attached thereto with sufficient rivets to develop their full strength.

Figure 258 shows a typical joint in a riveted truss and illustrates the application of some of these principles. It also shows a splice in a tension member in which the connection plate is used

as a splice plate. This is a common practice; and, whether the splice is of a tension member or a compression member, sufficient rivets should be used in the splice plates to carry the entire stress, no dependence being placed upon the abutting of the ends of the members.

The following example illustrates the character of the computations necessary to determine the thickness of such a plate:

Problem: Determine the necessary thickness of the connection plates shown in Fig. 258, using an allowable unit stress in bending of 16,000 lb. per square inch.

Solution: Inspection of the plate indicates that section xy is probably the critical section, since it contains many rivet holes and the resultant stress on either side of it is large in magnitude and applied at some distance below the center of gravity of the cross section. The strength of the plate at this section will therefore be investigated.

The forces acting to the right of xy are the proportionate part of the chord stress, carried into the plate by the chord rivets, and the total stress in the diagonal. Evidently the stress due to the chord is the more important factor, since its line of action is further from the center of the cross section; hence, the condition of loading corresponding to maximum stress in this chord bar will be assumed. Computations show that for this condition the stress in the diagonal is 100,000 lb. The stress passing into the plate from the chord rivets will be taken as the product of the number of rivets to the right of xy and the allowable stress per rivet, it being assumed that the total number of rivets is little if any in excess of the number actually needed. The assumption that the thickness of the plate will be such as to cause the rivets to be limited by shear rather than bearing and that the allowable unit stress in shear is 12,000 lb. per square inch gives a total force of $14 \times 7,200 =$ 100,800 lb., the rivets cut by the section being included, since they bear upon the portion of the plate to the right of xy rather than to the left.

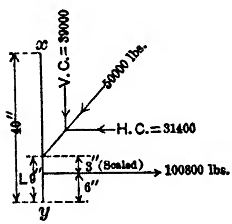


FIG. 259.

The forces acting upon section xy of one of the two connection plates will, therefore, be as shown in Fig. 259.

It has already been shown in considering plate-girder web splices that the effect of a row of rivet holes such as exists at section xy in reducing the strength in bending will probably be amply allowed for if the moment of inertia is considered as three-quarters of the value for the gross cross section. If this allowance is made, the maximum stress in the plate, on assuming its thickness as t, will be given by the following expression:

$$f = \frac{100{,}800 - 31{,}400}{(40 - 10)t} + \frac{4}{3} \times \frac{6(100{,}800 \times 14 - 31{,}400 \times 11)}{t(40)^2}$$

$$= \frac{69{,}400}{30t} + \frac{106{,}580}{20t} = \frac{7{,}642}{t}$$

Using an allowable value of f as 16,000 lb., $t = 7{,}642/16{,}000 = \frac{1}{2}$ in. = required thickness. This thickness would develop more than the shearing value of the rivets and is consequently sufficient, at least for the section investigated.

If the computed thickness should be much greater than ½ in., say ⅞ in., it probably would be economical to add plates 12½ in. wide centered on the chord axis and of sufficient length to develop their full stress through connecting rivets.

A more accurate determination could be made if thought desirable by actually determining the net moment of inertia, and other sections may be tested in a similar manner if doubt exists as to the critical section.

Consideration should also be given to the effect of a long unstayed edge of plate upon the maximum allowable unit stress in compression in the gusset plates, and, if necessary, such edge should be supported. It is the writer's custom to specify that compression edges of main gusset plates shall be stayed against buckling by angles if unsupported lengths exceed twelve times the plate thickness where maximum allowable stresses occur at the edge and for stresses less than maximum to permit unstayed lengths to be increased proportionately.

CHAPTER XIII

GRAPHICAL STATICS

165. Graphical and Analytical Methods Compared.—It is generally possible to solve by graphical methods all statical problems that can be solved analytically, and for certain classes of problems such methods are somewhat simpler and more rapid than analytical methods, such, for example, being the case in the problems of Arts. 89 and 90. As a general rule, however, analytical methods are more satisfactory both in accuracy and speed. The engineer should nevertheless be thoroughly familiar with the principles of graphical statics so that he may be prepared to apply them, particularly in checking analytical computations. A knowledge of them is also necessary in order that engineering literature may be read intelligently. For a comprehensive treatment of the subject the student is referred to Graphische Statik, by Muller-Breslau.

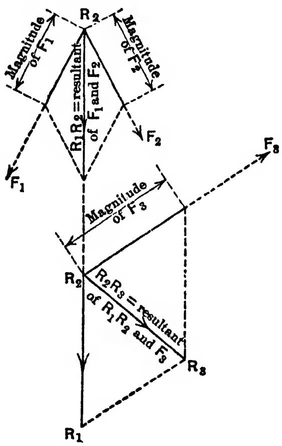


Fig. 260.

166. Force and Funicular Polygons. The most obvious method of determining graphically the magnitude, direction, and point of application of the resultant of a set of coplanar forces may be briefly stated as follows:

Plot the correct position and direction of the forces as indicated in Fig. 260 by F_1, F_2, and F_3. Prolong any two forces, such as F_1 and F_2, until they meet, thus obtaining the point of application of the resultant of these two forces. Determine the magnitude and direction of this resultant force by the parallelogram of forces. In a similar manner, combine this resultant with one of the other forces, and continue the process until

the resultant of all the forces has thus been determined in direction, point of application, and magnitude. This process may be continued indefinitely if the forces are not parallel but fails for parallel forces, since for such forces it gives only the magnitude and direction of the resultant, the point of application being indeterminate. This method is simple in its application, but the fact that it is not applicable to the case of parallel forces and that it does not give compact diagrams makes the following general method more desirable:

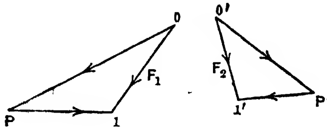


FIG. 261.

Let the force F_1 be resolved into any two components, such as $0P$ and $P1$ of Fig. 261, and let the force F_2 be resolved into the two components $0'P'$ and $P'1'$. Since the effect of any force is equal to that of its components, it is evident that $0P$ and $P1$ may be substituted for F_1 and $0'P'$ and $P'1'$ for F_2 without changing the result; hence, the resultant of F_1 and F_2 equals the resultant of the four components $0P$, $P1$, $P'1'$, and $0'P'$. Since F_1 and F_2 may be resolved into components at any point and in any direction and since $P1$ and $P'1'$ may be made parallel it is evident that $P1$ and $P'1'$ may be made to coincide in direction. If they can also be made equal, then the resultant of F_1

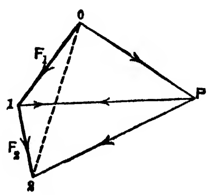


FIG. 262.

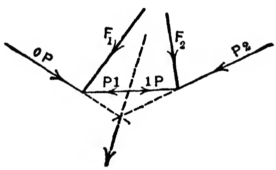


FIG. 263.

and F_2 equals the resultant of $0P$ and $0'P'$ and acts in the same direction. The components corresponding to $P1$ and $P'1'$ will be equal, parallel, and opposite in direction if the forces F_1 and F_2 are resolved, as shown in Fig. 262, in which F_1 and F_2 are given in direction and magnitude but not in position, P being taken at any convenient point.

In Fig. 263, the forces are shown in their correct positions and the components $0P$, $P1$, $1P$, and $P2$ are plotted so that $P1$ and $1P$ coincide in position and are equal in magnitude and

opposite in direction; therefore, the resultant of F_1 and F_2 acts at the intersection of $0P$ and $P2$, its magnitude and direction being given by the side 02 of the triangle of forces, 012, in Fig. 262.

Had the forces F_1 and F_2 been parallel, the same method could have been used, as is illustrated by Fig. 264.

In this method the point P is called the *pole,* 012 the *force* polygon, the figure *abcd* the funicular or equilibrium *polygon,* the lines $P0$, $P1$, and $P2$ connecting the pole and the apices of the force polygon the *rays,* and the corresponding lines in the funicular polygon the *strings.* The force polygon serves to determine the *magnitude* and *direction* of the resultant, and the funicular

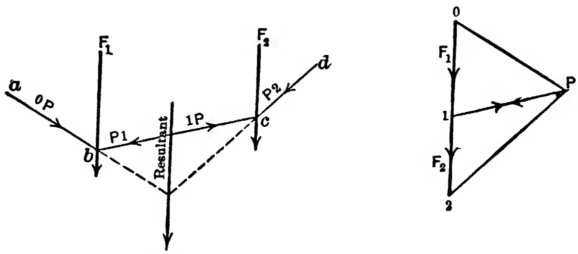


FIG. 264.

polygon fixes its *position* by determining a point on its line of application. It is evident that the method is simple, compact, and perfectly general. The following brief statement of the method may now be made.

To find the resultant of a series of coplanar forces, lay off the forces $F_1, F_2, \ldots, F_n$ to any desirable scale, thus forming the force polygon, locate the pole P at any desirable point, draw the rays $P0, P1, \ldots, Pn$ and the strings $P0, P1, \ldots, Pn$ parallel to these rays. In constructing these strings, draw $P0$ till it meets F_1, $P1$ till it meets F_2, $P2$ till it meets F_3, etc., each string being drawn from the point of intersection of the previous string and the appropriate force. The resultant will act through the point of intersection of the first string $P0$ and the last string Pn and will be given in magnitude and direction by $0n$ of the force polygon. If the forces are in equilibrium, the force polygon must be a closed figure, *i.e.,* point 0 and point n must coincide, since only under this condition can $\Sigma H = 0$ and $\Sigma V = 0$. The funicular polygon must also close, *i.e.,* the string $P0$ and the string Pn must coincide, since otherwise the resultant

force that equals the resultant of the two components represented by these strings would be a couple. For concurrent forces, *i.e.*, forces all of which meet at a point, closure of the force polygon is sufficient to show that equilibrium exists, since such forces are in equilibrium if $\Sigma H = 0$ and $\Sigma V = 0$, a condition that obviously exists if the force polygon closes.

It is clear that, unless the pole is located on the line $0n$, the coincidence of the first and last strings of the funicular polygon will involve the closure of the force polygon. This is illustrated by Fig. 265, in which the funicular polygon, shown by full lines, closes, since $P0$ and $P4$ coincide, a result that obviously would not occur if 0 and 4 in the force polygon were not to coincide.

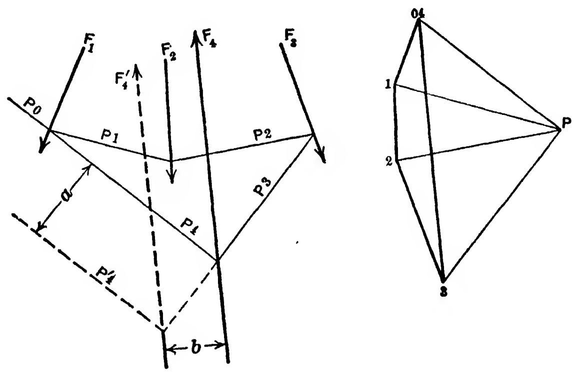


FIG. 265.

Were the forces in this case to consist of F_1, F_2, and F_3 only, the first and last strings of the funicular polygon could not coincide unless the pole were to be located on the line 03 or, in the more general case, on the line $0n$. If F_4 were to act in the position indicated by the dotted line marked F_4', the force polygon would close as before; but the last string of the funicular polygon would not coincide with the first string but would instead have the dotted position $P'4$, and the resultant of the forces $0P$ and $P'4$ would be a couple with a value of $(P0)a = (F_4)b$.

167. Characteristics of the Funicular Polygon.—The strings of the funicular polygon represent the bars of a framework which would be held in equilibrium by the applied forces and in all of which the stresses would be either direct tension or direct compression. An infinite number of such frameworks can be

selected, their position and shape being determined by the location of the pole.

Since each intermediate string represents two forces that are equal in magnitude and opposite in direction, the resultant *of all the forces will be held in equilibrium by the forces represented by the extreme strings; hence, this resultant acts at the point of intersection of the extreme strings, as has already been* stated. The resultant of any set of consecutive forces is also held in equilibrium by the extreme strings corresponding to that particular set of forces; hence, it acts at the point of intersection of these extreme strings. This may be illustrated by Fig. 265, in which the resultant of F_1 and F_2 acts at the intersection of $P0$ and $P2$; of F_1, F_2, and F_3, at the intersection of $P0$ and $P3$; of F_1, F_2, F_3, and F_4, at the intersection of $P0$ and $P4$, *i.e.*, at any point along $P0$ or $P4$, an obviously correct conclusion, since the resultant of F_1, F_2, F_3, and F_4 equals zero, these forces being in equilibrium.

The following general rule may be applied to the determination of the point of application of the resultant: The resultant of any set of consecutively numbered forces acts through the point of intersection of the two strings, one of which is designated by a number equal to that of the highest numbered force and the other by a number 1 lower than the lowest numbered force. For example, the resultant of a series of forces, F_4 to F_7, acts through the point of intersection of $P3$ and $P7$. In applying this rule the forces and strings should be numbered in the exact manner used in the illustrations.

168. Reactions.—Since in order that a set of forces may be in equilibrium the force and funicular polygons must close, it is evident that the reactions of a given structure may be determined graphically if their values are such as to make these two polygons close. The method of doing this is clearly shown by the following examples:

Problem: Determine by use of the funicular polygon the reactions for the beam shown by Fig. 266.

Solution: In order that the funicular polygon may close, the first and last strings must lie on the same line. To ensure that such will be the case, draw the string $P0$ through the point of application of the left reaction, since this is the only known point on the line of action of this reaction.

Draw the remainder of the funicular polygon in the usual manner, and draw also the line marked *Closing Line*, which should connect the point of intersection of the string $P4$ and the reaction R_R, with the point of intersection of $P0$ and R_L. The first and last strings $P0$ and Pn of the funicular polygon may now be made to coincide by drawing the line PK in the force polygon parallel to the closing line and fixing the position of K by drawing from 4 a line parallel to the reaction the direction of which is known, *i.e.*, to R_R. $4K$ will equal the right-hand reaction and $K0$ the left-hand one,

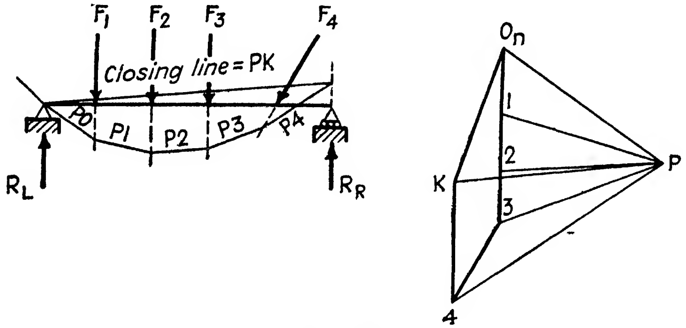


FIG. 266.

since, by using these as applied forces and drawing the funicular polygon for the six forces F_1, F_2, F_3, F_4, R_R, and R_L, the first and last strings will coincide.

Problem: Determine by the funicular polygon the reactions for the truss shown by Fig. 267.

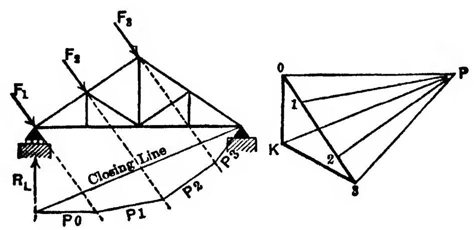


FIG. 267.

Solution: For this case the left reaction is the one that is known in direction. The funicular polygon has therefore been constructed by drawing $P3$ first, thus reversing the usual method of construction. The closing line is drawn from the right point of support to the intersection of $P0$ with the left reaction. The value of the right reaction is given by $3K$ and of the left reaction by $K0$.

Further examples need not be given, as no new methods are required. The essential thing for the student to grasp is that the closing line should connect the points of intersection of the reactions and the extreme strings (*i.e.*, the strings holding the resultant of the applied loads in equilibrium)

and that the point K in the force polygon should be so located as to enable the reactions and forces in the force polygon to be read consecutively beginning with the left-hand force (or reaction). Each reaction may be identified by observing that the rays in the force polygon between which it lies correspond to the strings of the funicular polygon intersecting on its line of action.

169. Graphical Method of Moments.—It is evident that the moment of any set of forces about a given axis may be obtained by scaling the distance from the axis to the line of action of the resultant of the given forces and multiplying this value by the resultant. To illustrate: Let point a (Fig. 268) represent the trace of the axis, and let the problem be to find the moment about a of the forces F_1, F_2, and F_3. The resultant of these

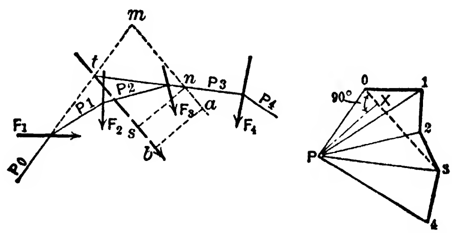


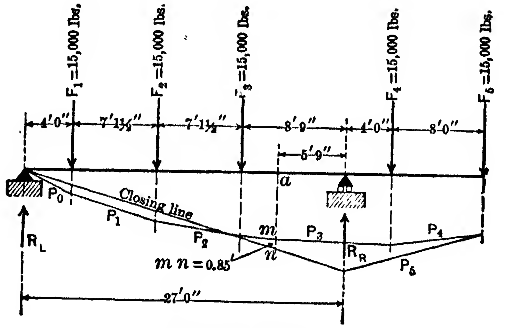
FIG. 268.

forces = 03, and it acts through t, the point of intersection of $P3$ and $P0$ produced; hence, its moment about a = 03 to the scale of force multiplied by ab to the scale of distance.

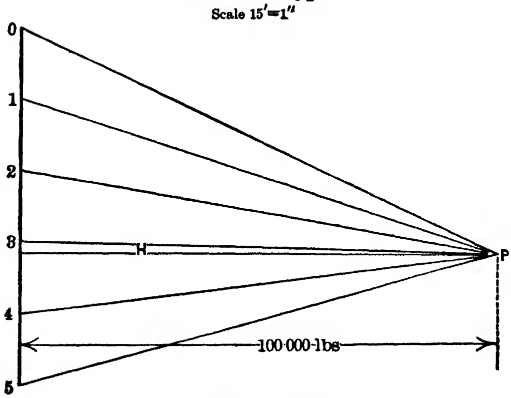
The method above is very simple, but the following modification of it is more useful. Draw through a a line ma parallel to 03. Then the moment about a of the given forces equals 03 multiplied by ns, the distance from ma to the line of action of the resultant. Draw from P in the force polygon a line PX perpendicular to 03. Then the Δntm is similar to $\Delta P30$; therefore, $PX/ns = 03/mn$ (altitudes of two Δs are to each as their bases). Therefore, $(PX)(mn) = (03)(ns)$ = moment desired. The theorem thus deduced may be stated as follows:

To find the moment about any point of any number of coplanar forces, construct a funicular polygon corresponding to a force polygon having forces laid off consecutively, draw through the point a line *paralled to the resultant of the forces*, and find its intercept between the *strings holding the resultant in equilibrium*. This intercept, measured to the scale of distance, multiplied by

the perpendicular distance, hereafter called H, from the pole of the force polygon to the resultant of the given forces, measured to the scale of force, equals the desired moment. For a horizontal beam carrying vertical loads, this equals the product of the intercept of the vertical ordinate through a and the horizontal pole distance.



Funicular Polygon
Scale 15'=1"



Force Polygon
Scale 40,000#=1"

FIG. 269.

It follows that for such a beam the funicular polygon corresponds to a curve of moments for the given loads, referred to the closing line as an axis, the moments being measured by vertical ordinates.

The character of the moment can usually be determined by inspection. If doubt exists, the point of application of the resultant of the forces on one side of the section should be located;

with this and the direction of the resultant as given in the force polygon known, the character of the moment can be easily seen. For a horizontal beam with vertical loads, the fact that the moment is zero wherever the closing line intersects the funicular polygon and hence changes sign at such a point may often be used to advantage in determining the character of the moment, as is illustrated in the problem that follows.

Problem: Determine by the graphical method of moments the bending moment at section a of the beam shown in Fig. 269.

Solution: This problem involves the determination of the moment of the forces R_L, F_1, F_2, and F_3 about a horizontal axis passing through a. Since the forces are all vertical, draw through a a vertical line, and measure to the scale of distance its intercept between the strings holding the given forces in equilibrium. These are the closing line and $P3$; hence, the moment equals the product of mn to scale of distance and H to scale of force. The result thus obtained equals $0.85 \times 100{,}000 = -85{,}000$ ft.-lb. This is negative, since it is of the same character as the moment in the overhanging end, the point of zero moment occurring to the left of point a.

170. Graphical Method of Moments with a Concentrated Load System.—The application of the graphical method of moments for a system of moving wheel loads may be easily made as follows:

Lay off the forces to any convenient scale, and locate the pole of the force polygon so that its normal distance from the force line measured to scale of force equals some even number, say 100,000 lb. Plot the loads to any convenient scale, and draw the funicular polygon, considering the uniform load as equivalent to a series of equal concentrated loads equally spaced. If the given load system is likely to be used for a number of spans, the funicular polygon should be made comprehensive enough to permit its use for any span likely to be investigated. Such a polygon is shown by Fig. 270, the force polygon being omitted.

With the funicular polygon constructed, the operation of finding the moment with any load at a given point of an end-supported span is very simple. Suppose it be desired to determine the moment at the center of a 60-ft. span with load F_{13} at the center.

Lay off on the funicular polygon the distance 30 ft. to the left of F_{13}, and an equal distance to the right of the same load, and draw verticals to intersect the funicular polygon at s and t.

The ordinate *mn*, to the scale of distance, multiplied by the distance *H* in the force polygon equals the desired moment. The moment thus obtained = 19.2 × 100,000 = 1,920,000 ft.-lb. In this manner, several loads may be tried, and that giving the largest ordinate will give the maximum moment at the center of this span. This method may be conveniently used to verify the results of analytical computations, and a diagram once prepared for a standard loading, like Cooper's E_{60}, and a long span should be of material value to the designing engineer.

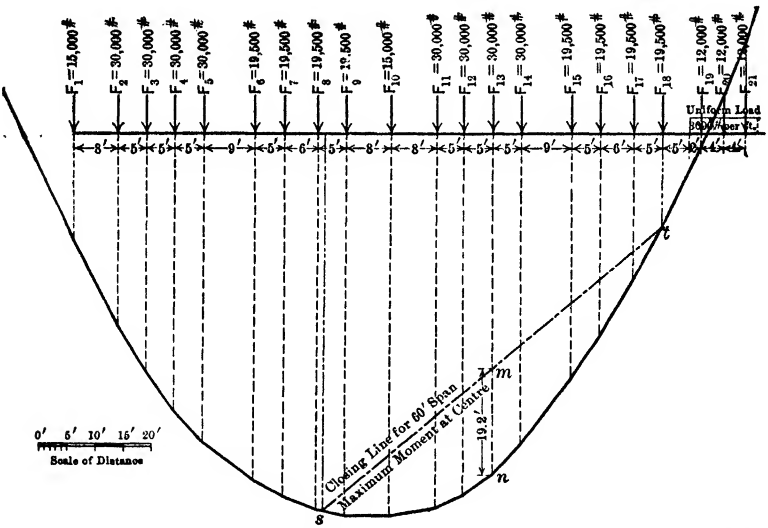


FIG. 270.—Funicular polygon for Cooper's E_{60} loading. Force polygon omitted. *H* of force polygon = 100,000 lb. Loads are wheel loads.

171. Graphical Method of Shear.—Since the shear at any section of a beam equals the resultant, parallel to the given section, of the forces on either side of the section, its value may be determined graphically by the force polygon, the reaction having previously been determined by the method of Art. 168. The following method, however, is somewhat better adapted to the treatment of concentrated load systems and should be thoroughly understood.

Consider the beam shown in Fig. 271, and let the problem be to determine the shear at a distance *x* from the left end with the first load of a concentrated load system at *x*. Draw the

force and funicular polygons in the usual manner, making $P0$ horizontal for convenience, prolong the string $P0$, and draw the vertical bc. Then the Δabc of the funicular polygon is similar to the $\Delta P0K$ of the force polygon; hence,

$$\frac{bc}{ab} = \frac{K0}{P0}$$

Therefore,

$$bc = \frac{(ab)(K0)}{(P0)}$$

But

$$\frac{ab}{P0} = \frac{L}{H}$$

Hence, if H is made equal to L, bc will equal $K0$. But $K0 = R_L$

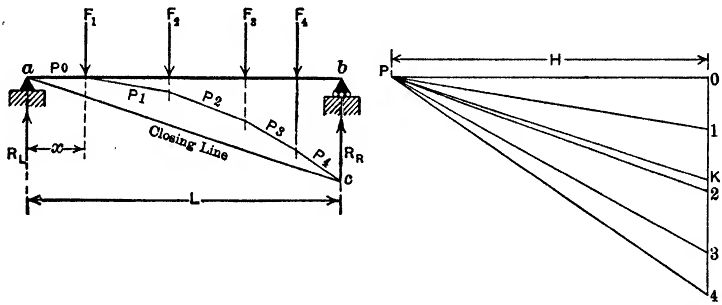


FIG. 271.

equals the shear at x with F_1 at x; hence, the following theorem may be stated:

For a simple beam supported at the ends and loaded with vertical loads the shear at a distance x from the left end, with the first load which is on the span at x, equals the vertical ordinate measured to the scale of *force* between the funicular polygon and the first string produced at a distance $L - x$ from the first load *provided that the pole is so chosen as to make H and L equal.* This latter condition may be readily secured by constructing the force polygon with the loads at one point of support and the pole at the other. The vertical ordinate between the first string $P0$ and the funicular polygon at a distance $L - x$ from the first load has the same value whether the loads, force, and funicular polygons are laid off as in Fig. 271 or as in Fig. 272, since one of these funicular polygons may be superposed upon the other, if drawn

to same scale, by inverting it and turning it end for end. If, therefore, the given loads are laid off, as in Fig. 272, the shear at a distance x from the left end of a simple end-supported beam may be found when the *first load is at* x by laying off the distance $L - x$ to the *left* of F_1 and scaling to the scale of force the ordinate between $P0$ and the funicular polygon. This is equivalent to scaling the ordinate between $P0$ and the funicular polygon at the section of the beam where it is desired to determine the shear. In order to determine the maximum shear at a given section due to a concentrated load system, it may be necessary to try several loads at the section. If load 2 is moved up to the section, the left reaction should be determined in the manner previously stated, *i.e.*, by measuring the ordinate at the first load which will now be at the distance $x - d$ from the left end of the beam. The shear may now be obtained by subtracting the first load from this reaction. This shear may be found directly by scaling the ordinate between the funicular polygon and a horizontal drawn through (1) of the force polygon. This method is applicable only if the first load remains on the span. If F_1 goes off the span during the process of moving up, the ordinate should be measured at the distance $L - x$ from F_2, but the intercept should be the vertical distance between string $P1$ and the string above it, since the values of vertical ordinates are in nowise affected by the fact that the first string is inclined rather than horizontal, this corresponding merely to a change in the vertical position of the pole and not a change in its horizontal position. If floor beams are used, the shear must, of course, be determined by subtracting from the reaction the proper percentage of the loads between the support and the panel point at which the load under consideration is located. If the load system is to be used for a number of spans, the diagram should be drawn for the longest span and the scale for any given span determined by proportion. The application of the graphical method of shear is clearly shown by Fig. 273.

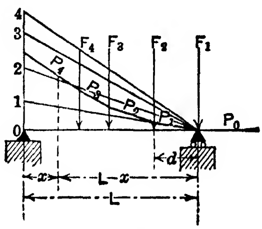


FIG. 272.

It should be noticed that the funicular polygons will be exactly alike, whatever the span chosen, *provided that* the ratios of the

scales of forces are inversely proportional to the spans, *e.g.*, the funicular polygon of Fig. 273 is constructed for a 200-ft. span to a scale of forces of 60,000 lb. = 1 in.; but if constructed for a 100-ft. span with a scale of forces of 120,000 lb. = 1 in. as also indicated, the same funicular polygon would be obtained. It follows that this polygon may be used for any span by multiplying the scaled ordinate by the ratio between the span for which the polygon is constructed and the given span, regardless of whether the latter is longer or shorter than the original span.

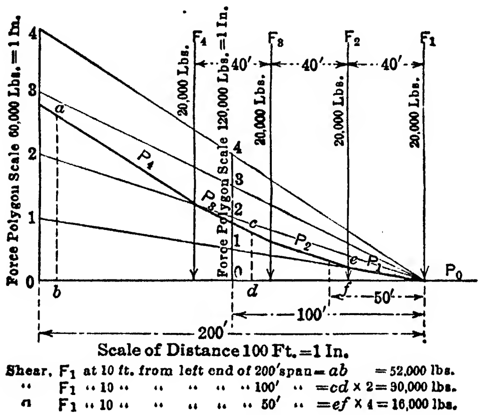


FIG. 273.

172. Funicular Polygon through Several Points.—It is clear that if a hinged framework is constructed identical in shape to a given funicular polygon and with its bars designed to carry the string stresses of the polygon, it will be in equilibrium under the loads for which the funicular polygon is drawn. This fact may be used in determining the proper outline for a voussoir arch intended to carry a set of fixed loads but cannot be used for moving loads. On the other hand, if a funicular polygon cannot be drawn within the limits of an arch the shape of which has already been decided upon, it is reasonable to suppose that such an arch will not be stable, and it is upon this hypothesis that a commonly accepted theory of stone arches is based. The theory of such arches will not be taken up at this point; but since in studying them it is often useful to be able to draw a funicular polygon through certain fixed points, methods of doing this will be derived.

Funicular Polygon through One Point.—Since the pole may be chosen anywhere and any string of the funicular polygon drawn through a given point, it is evident that an infinite number of funicular polygons may be drawn through one point.

Funicular Polygon through Two Points.—Since the first and last strings of a funicular polygon must always meet on the line of action of the resultant of the set of forces for which the polygon is drawn, it is evident that a funicular polygon may be drawn through two points by first drawing any funicular polygon through the first point and plotting the resultant of the set of forces in direction and position; the appropriate string of the desired polygon may then be constructed through the second

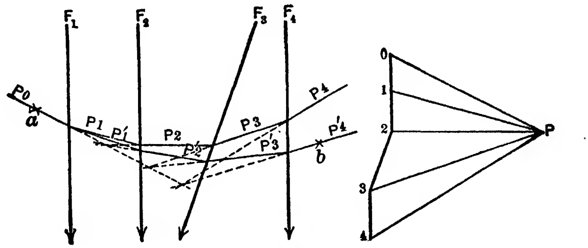


FIG. 274.

point in question and the intersection of the resultant and the string drawn through the first point. This method is illustrated by Fig. 274, in which the original funicular polygon is constructed with a pole located at random, the string $P0$ being drawn through point a. The final funicular polygon may then be constructed so that $P'4$ will go through point b, $P0$ remaining unchanged in position, by drawing $P'4$ through point b and the intersection of $P0$ and $P4$. $P'3$ may next be constructed by connecting the point of intersection of $P'4$ and F_4 with the point of intersection of $P0$ and $P3$, and in a similar manner $P'2$ and $P'1$ may be located. This construction is evidently consistent with corresponding strings of the two funicular polygons intersecting on the resultant of the forces held in equilibrium by these strings. If the intersections as obtained by the method just given are not on the sheet, the second funicular polygon may be constructed by locating a new pole P' and constructing an entirely distinct polygon. The method of doing this is illustrated by Fig. 275, in which the line of action of the resultant of all the

forces is plotted and the new strings $P'0$ and $P'4$ drawn at random through points a and b, respectively, to meet at any point upon the line of action of this resultant. The new pole may then be located by drawing from 0 and 4 in the force polygon rays parallel to $P'0$ and $P'4$, respectively, their intersection locating the new pole P'. If it is desired to have other than the first and last

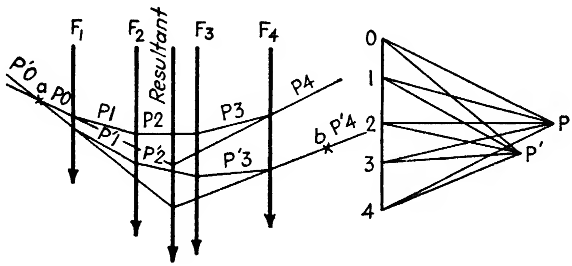


FIG. 275.

strings pass through the two points, the resultant of the forces held in equilibrium by the desired strings should be used.

The following important theorem is also of use at times, *viz.*: For any set of loads, the intersection of corresponding strings of two funicular polygons drawn with different poles will lie on a line parallel to the line joining the poles. To prove this, consider the two funicular polygons, shown in Fig. 276, with the poles P and P'.

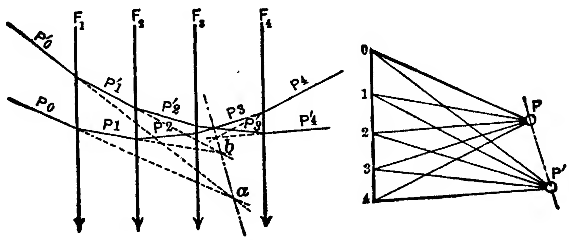


FIG. 276.

The force F_1 may be *resolved* into the two components $0P$ and $P1$; consequently, that force may be replaced by these components without changing existing conditions. The force F_1 will be *held in equilibrium* by the two forces $1P'$ and $P'0$. Consequently, the resultant of these two forces is equal and opposite to the resultant of the two forces $0P$ and $P1$; hence, the forces $0P$, $P1$,

$1P'$, and $P'0$ are in equilibrium, and therefore the resultant of $0P$ and $P'0$ is equal and opposite to the resultant of $P1$ and $1P'$. The resultant of $0P$ and $P'0 = P'P$ and acts at the intersection a of the strings $P0$ and $P'0$; the resultant of $P1$ and $1P' = PP'$ and acts at the intersection b of the strings $P1$ and $1P'$. Since these resultants are in equilibrium, they must not only be equal but act along the same line; *i.e.*, both must act along the line ab, and hence ab must be parallel to the actual direction of these forces, *i.e.*, to PP'. In the same way, it may be shown that the resultant of $P1$ and $1P'$ acts in the same line but in the opposite direction to the resultant of $P'2$ and $2P$, this direction being parallel to PP'; hence, the point of intersection of $P'2$ and $2P$ also lies on the pro-

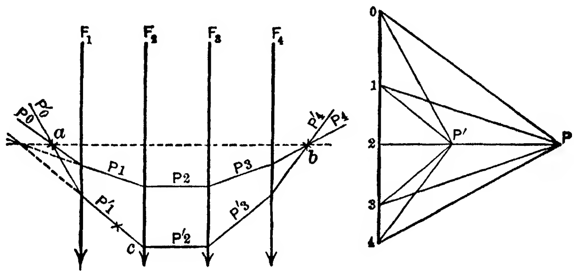


FIG. 277.

longation of the line ab. In the same manner the intersection of all the corresponding strings may be proved to be on the line ab; hence, the theorem is proved.

Funicular Polygon through Three Points.—Application of the theorem just stated enables a funicular polygon to be passed through three points. The following is the mode of procedure: Construct a polygon through either two of the given points, say the points a and b, Fig. 277, and connect these points by a straight line. If a new polygon is drawn such that the intersections of its strings with the corresponding strings of the new polygon lie on line ab, it will also pass through the points a and b; hence it is merely necessary to draw a new string through the third point c and the intersection of the corresponding string of the first polygon with the line ab, and finish the polygon by the method given for a polygon through two points. The figure shows the application of this method, it being assumed that the polygon $P0$, $P1$, etc., has already been drawn through two points.

It will be noted that in the construction it was necessary to locate the pole P' in order that $P'2$ might be drawn, since the intersection of $P2$ and the line ab does not come on the sheet. The remainder of the polygon was drawn by using the intersections of the strings of the original polygon and the line ab.

Alternative Method for Funicular Polygon through Two and Three Points.—Another simple and useful method of drawing a funicular polygon through two points a and b is as follows: Assume the forces held in equilibrium by the strings that are to pass through the given points to act upon a beam supported at a and b, Fig. 278. Assume the reaction at b to be fixed in direction, vertical in this case, and determine graphically both reactions for these loads, drawing one of the strings through a. The funicular polygon thus drawn will pass through point a by construction but the value of the reactions will be *independent* of the *position* of the funicular polygon. If a new funicular polygon is now drawn with its pole at *any* point on a line drawn from the closing point K of the force polygon parallel to the line ab and with the same string passing through a, as in the original polygon, this polygon will pass through both points, since its closing line passes through a, and is parallel to ab. This construction is illustrated by the figure, in which the pole P' of a funicular polygon with the string $P'0$ passing through a and $P'4$ through b might lie at any point on a line drawn from K parallel to ab.

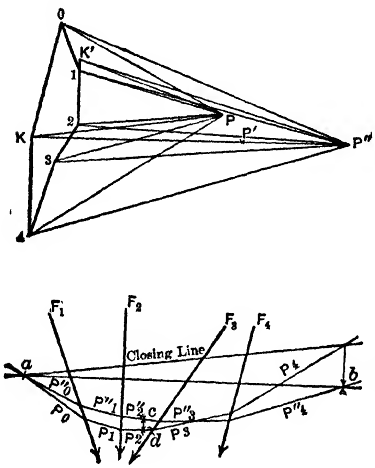


FIG. 278.

To extend this method to a third point c, proceed as follows: Draw a vertical through c till it intersects the original funicular polygon at d. From the pole P, draw a line parallel to ad to its intersection K' with a vertical through 2. Draw from K' a line parallel to ac. The intersection of this line with the line

KP' locates P'' which is the pole of a funicular polygon passing through a, b, and c.

This method consists essentially of the determination of the reactions upon a beam ac, due to the forces F_1 and F_2, and the location of the closing point K' of the force polygon corresponding to these reactions. A funicular polygon with its pole at any point on a line from K' parallel to ac will pass through points a and c. Since, as has already been shown, a funicular polygon with its pole at any point on KP' will pass through a and b, it is evident that a polygon with its pole at the intersection of these two lines will pass through the three points a, b, and c.

Problems

65. Determine graphically by the funicular-polygon method the bending moment at a section located 6 ft. to right of left support for each beam shown in Probs. 6 to 10, Art. 44.

66. Construct a funicular polygon passing through the three hinges of the arch shown in Prob. 61, Art. 132, assuming uniform load of 2,000 lb. per horizontal linear foot applied to top chord.

Scale of distance, 1 in. = 40 ft.

Scale of force, 1 in. = 100,000 lb.

67. Construct a funicular polygon for wheel loads of Cooper's E_{40} loading.

Scale of distance, 1 in. = 5 ft.

Scale of force, 1 in. = 20,000 lb.

Locate P opposite and to right of point 10 of force polygon, *i.e.*, opposite point in force polygon at the bottom of the line representing the first truck wheels of the second locomotive. Value of H = 100,000 lb. Polygon should be drawn on a sheet with outside dimensions 25 in. × 37 in.

CHAPTER XIV

DEFLECTION SLOPE AND CAMBER

173. Elastic and Nonelastic Deflection of Trusses.—The deflection of a truss is due to changes in length of the members and may be divided into two parts, elastic and nonelastic. The former may be caused by stresses or by differences in temperature of the various members and disappears upon the removal of the loads or the return to uniform temperature; the latter is due to play at the joints and occurs when the falsework used in erection is removed, being particularly important for pin-connected trusses in which considerable play usually exists in the pinholes.

A knowledge of the deflection is often desirable and is necessary in proportioning the lifting devices at the ends of a swing bridge and in planning the erection of cantilever and continuous structures. A method based upon deflections also furnishes a convenient mode of determining stresses in a statically indeterminate structure.

174. Truss Deflection. Trigonometrical Method.—A rigid truss is generally composed of triangles, all the properties of which may be determined if the lengths of the three sides are known. The vertices of the triangles coincide with the joints of the truss; hence, the various positions of a joint with respect to a pair of rectangular axes may be determined for any length of the sides of the triangles, *i.e.*, of the members of the truss. It is evident, therefore, that to find the vertical or horizontal elastic deflection of a joint, say the center joint of the bottom chord, it is only necessary to compute by trigonometry its position with respect to a fixed horizontal or vertical axis both before and after the application of the load causing the deflection. If the axis passes through the original position of the joint, the movement under the load may be found without the former computation.

The length of each member after the application of the load may be found by adding to its original length the change of length due to tension or by subtracting the change of length due to compression.

If the nonelastic deflection is desired, the same method may be used; but for this case the change in length of a member will equal the average play in the pinholes at the two ends instead of the change due to stress.

Although this method is simple in theory and application, it is very laborious and is not used in practice.

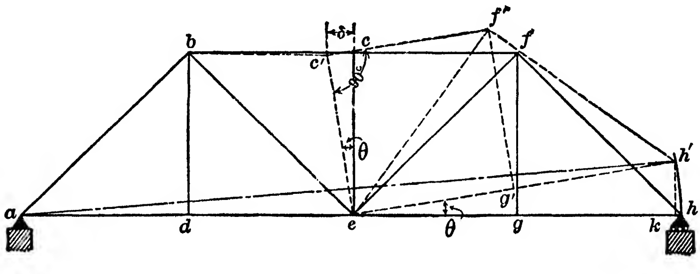


FIG. 279.

175. Truss Deflection. Method of Rotation.—The deflection of any joint of a simple truss due to a change in length of one bar only may be readily determined by investigating the resulting rotation of one portion of the truss with respect to the other, the latter being assumed as fixed. By considering the effect of each bar separately and summing up the results, the final deflection may be determined. To illustrate this, consider the truss shown by full lines in Fig. 279, and let it be desired to determine the vertical deflection δ_e, of panel point e, due to a decrease in length δ of bar bc. Evidently bce is the only triangle that will be changed in shape by the change in length of this bar. If the portion abe of the truss is assumed for the time being to remain fixed in position, the figure $abc'f'h'ea$ will represent the new position of the truss. The value of δ for any ordinary change of length is very small compared with the original length of the bar; *e.g.*, for a steel bar subjected to a stress of 15,000 lb. per square inch, δ is approximately 1/2,000 of the length of the bar. It follows that the angular rotation of bar ec is extremely small,

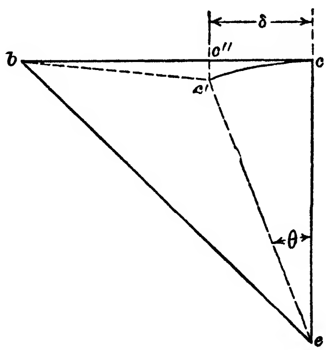


FIG. 280.

and in consequence the distance cc'' in Fig. 280 may be assumed as equal to δ as shown, the error being negligible.

The angular movement θ of bar ce evidently equals the angular movement of eh. The sine of this angle equals $\delta/c'e = \delta/ce$; hence, $h'k = eh'\frac{\delta}{ce}$. But $eh' = eh$; hence, $h'k = \frac{eh}{ce}\delta$ = the vertical deflection of point h with respect to the axis ae, which is assumed to remain unchanged in position. Actually, however, point h remains on the abutment and ae changes its position. The correct position of the distorted truss may be found by rotating it about a horizontal axis passing through a until ah' becomes horizontal. This rotation will cause c to drop below its original position by the amount which it is now below ah', *i.e.*, by one-half of $h'k$ (the effect of the slight difference in length

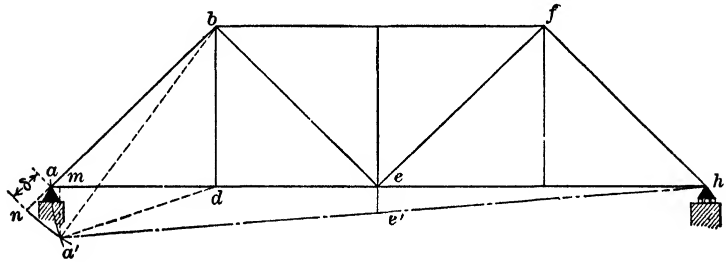


FIG. 281.

between ek and eh' being neglected); hence the vertical deflection δ_e of point e due to the change δ in bar bc will be given by the following equation:

$$\delta_e = \frac{1}{2}\delta\frac{eh}{ce}$$

In a similar manner the deflection of other points due to the change in length of this bar or others may be obtained.

In order to illustrate this method more completely, its application to the problem of determining the vertical deflection of point e resulting from an increase δ in the length of bar ab will also be given. For this case the portion $dbfhed$ of the truss will be considered stationary. The condition of the truss after the change in length of the bar will be as shown, greatly distorted, in Fig. 281, and $e'e = \frac{1}{2}a'm$ will be the actual vertical upward deflection of point e.

The value of $a'm$ may be determined as follows: Let the distortion of the triangle *abd* be as shown, greatly exaggerated, in Fig. 282. Then *bn* equals the new length of bar *ab*, and the new position of *a* will be at the intersection of arcs swung from *b* and *d* as centers, with radii *bn* and *da*, respectively. The fact that δ is very small compared with the sides of the triangle makes negligible the error in assuming the tangents na' and aa' to coincide with the corresponding arcs and gives the condition shown in Fig. 282, from which it at once follows that aa' equals $\delta/\sin\alpha$. But $\sin\alpha = bd/ab$; hence $aa' = \delta\frac{ab}{bd}$ and therefore $ee' = \frac{1}{2}\delta\frac{ab}{bd}$. This method is evidently much simpler in application than the trigonometrical method previously mentioned but is nevertheless awkward for general use, since it involves an entirely separate solution for each bar. It is given here to illustrate graphically the distortion due to change in length of a single bar.

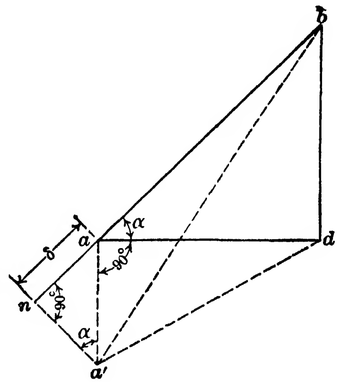


FIG. 282.

176. Truss Deflection. Method of Work.—The method that follows is a modification of one usually attributed to Prof. Otto Mohr and provides a very simple and ingenious solution of the problem of determining the deflection of a truss. The method is based upon the fact that, if a truss is loaded by a *single load* at any point and is then deflected by the application of *other forces*, the external work done by the load equals the internal work done by the bar stresses caused by the load, since a system of concurrent forces in equilibrium may be moved a small distance by an external force without the performance of work by the system. Such a condition occurs at each joint in a truss, the forces being the bar stresses due to the load, and the movement of the joint being that due to the external forces producing the deflection of the truss. A slight approximation actually occurs in the application of the method, since it is assumed that the bar stresses due to the load remain constant during the distortion of the structure; actually, these change

slightly owing to the change in the angles between the various members meeting at the joint, but the error is extremely small for trusses formed of material with a high modulus of elasticity, since the change in these angles is inappreciable for working stresses. The method is inapplicable for a truss that would distort greatly under load, such, for example, as a truss formed of spiral steel springs or rubber bands.

In order to apply this method, consider the truss shown in Fig. 283, and let it be desired to determine the deflection δ_2 of panel point L_2, due to the shortening of bar U_1U_2 by the amount δ, this shortening being due to any cause whatever, such, for example, as a stress in the bar, a difference of temperature in the

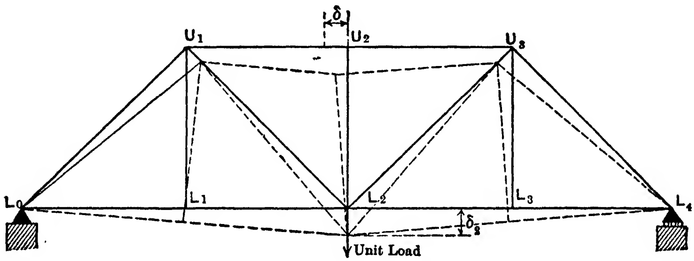


FIG. 283.

bar as compared with other bars in the truss, or an adjustment of its length by a turnbuckle. Figure 283 shows by full lines the truss before the length of bar U_1U_2 is changed and by dotted lines the distorted position of the truss. Evidently the external work that would be performed by a vertical load of unity[1] hanging at L_2 during the change of length of the bar would be unity $\times\ \delta_2$. This load would cause a compression s in bar U_1U_2, and this internal force would have to move the distance δ during the change in length of the bar and hence would perform an internal work of $s\delta$. Equating the internal and external work gives $1 \times \delta_2 = s\delta$; therefore the vertical deflection δ_2 of point L_2, due to a change in length δ of bar U_1U_2, equals the product of the numerical coefficient of the stress s in U_1U_2 and the change in length of the bar. Were the load of unity inclined instead of vertical, s would have a different value, and δ_2 would be the

[1] This unit load may be considered as an infinitesimal load which of itself has no influence upon the actual deflection of the truss.

deflection along the inclined line of action of the load of unity. A comparison of the results obtained by this method and the method of rotation shows them to be equal.

The signs must be carefully considered. If tension and increase in length are both denoted by positive signs, the deflection will be in the *direction* of action of the load of unity if the resulting value of $s\delta$ is *positive* and in the *opposite* direction if this product is negative. For the case considered, s is compression, and δ is a shortening. Hence, each has a negative sign and the product will be positive; therefore, the deflection will be in the direction of action of the load of unity, *i.e.*, downward.

The correctness of this method of dealing with the signs may be readily seen by examining the case of a single vertical bar with a force of unity acting downward at its lower end. The stress s in this case is tension and hence has a positive sign. If the length of the bar is increased by the amount δ, the product $s\delta$ will also have a positive sign, showing, according to our convention, that the lower end of the bar deflects in the direction of action of the unit force, *i.e.*, downward. If the bar is shortened, $s\delta$ will have a negative sign indicating a deflection of the lower end of the bar in a direction contrary to the direction of action of the unit force, *i.e.*, upward. Both these conclusions are obviously correct, and the *direction* of the deflection with respect to the direction of action of the unit force would evidently be unchanged if the unit force were to be applied to the bar through a series of other bars instead of directly, and if it were to be inclined or horizontal instead of vertical.

To apply the method to all bars of a truss, it is only necessary to obtain the summation of the various products. The final formula for deflection may then be written

$$\delta_n = \Sigma s\delta$$

in which δ_n = the component of the deflection, of any joint n, of the truss in any desired direction

s* = stress in any bar of the truss due to a load of unity acting at joint under consideration and in the direction of the desired deflection

* s as used in this chapter is to be considered as the numerical coefficient of the actual stress due to a load of one unit, *i.e.*, as an abstract number. It will be spoken of, however, as the stress. For any given bar, it equals

δ = change in length of the bar of the truss in which the stress s occurs

$\Sigma s\delta$ = summation of the products $s\delta$ for all bars of the truss

If $\Sigma s\delta$ is found to have a positive value, the deflection will be in the direction of action of the force of unity; if a negative value, it will be in the opposite direction. If the actual deflection of a given joint is desired, the deflection in both a horizontal and a vertical direction must be obtained and the resultant found.

The usual problem is to determine the deflection in a given direction of a given joint due to applied loads such as the weight of the structure itself, or to live loads in a given position. For this case, δ will be the change in length due to the stress caused by the applied loads; hence, the formula may be written

$$\delta_n = \sum \frac{sSL}{AE}$$

in which δ_n = deflection, ft. of any joint in any desired direction

L = length of any bar, ft.

A = area of same bar, sq. in.

S = stress, lb., in same bar due to applied loads

E = modulus of elasticity, lb. per sq. in.

If E is constant for all bars of the truss, as is usually the case, it is simpler to express the change in length of each bar in terms of E and substitute the final numerical value of E after the summation is complete.

177. Truss Deflection Illustrated.—The following example illustrates clearly the application of this method.

Problem: Let the problem be the determination of the vertical deflection of point L_2 of truss shown in Fig. 284 for a uniform live load of 2000 lb. per foot over the entire truss applied at the bottom chord.

Solution: The simplest method of solution of such a problem is to prepare a table in which separate columns are assigned for the terms s, S, L, and A; for the change in length of the bar; and for the product of the change in length of the bar and the stress s. The table on page 362 is prepared in this manner and is self-explanatory.

The summation of the numbers in the last column of the table gives $+\frac{2,170,000}{E}$ and equals the vertical deflection *downward* of L_2. Were this

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the ratio between the deflection of the point under consideration and the change of length of the bar.

summation to have a negative sign it would equal an upward deflection of this point. The numerical value of the deflection may be obtained by substituting the value of E. If this is taken as 29,000,000,
δ = 2,170,000/29,000,000 = 0.0749 ft. = 0.90 in. = ⅞ in., approximately.

An inspection of the table shows that the stresses due to load unity should be computed before the change in length of the bars, since if the stress in any bar caused by this load is zero the deflection due to this bar is zero and its change in length need not be figured.

Had the problem been that of computing the horizontal movement of the roller end, the load of unity should have been applied horizontally at that end. The only truss bars that would be stressed by this load would be those in the bottom

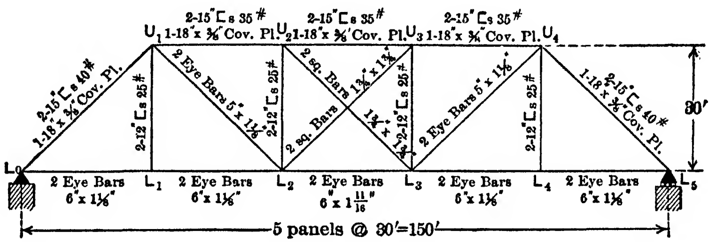


FIG. 284.

chord in which the stress would be 1.00 (+) were the load taken as acting to the right. The deflection would then be found by the summation of the changes in length of the bottom-chord bars, which equals (+) 0.00920 × 5 = (+)0.046; hence the horizontal movement of the roller end of the truss under the load of 2,000 lb. per foot = 0.046 ft. or 0.55 in. to the right. Had it been desired to find the elastic deflection due to the dead load, the dead stresses should have been used instead of the stresses due to the load of 2,000 lb. per foot.

For the nonelastic deflection due to play in the pinholes, the change in length of each bar can be written directly and the third, fourth, and fifth columns in the table on page 362 omitted. For example, if the allowable play in the pinholes at L_1 and L_2 is $\frac{1}{32}$ in., the change in length of bar L_1L_2, *i.e.*, the change in distance center to center of pins, would be $\frac{1}{32}$ in. (+), and this value should be written in the sixth column.

TABULAR VALUES FOR DEFLECTION OF POINT L_2 OF TRUSS SHOWN IN FIG. 284

(Slide rule used throughout)

Bar	Stress due to load of unity at $L_2 = s$	Stress due to applied load $= S$	Length of bar, ft., $= L$	Gross area of cross section, sq. in., $= A$	$E \times$ change in length of bar, ft., $= E\delta = SL/A$	$E \times$ deflection due to each bar, ft., $= Es\delta$
L_0U_1	0.848(−)	169,600(−)	42.4	30.15	238,000(−)	202,000(+)
U_1U_2	1.200(−)	180,000(−)	30.0	27.21	198,000(−)	238,000(+)
U_2U_3	1.200(−)	180,000(−)	30.0	27.21	198,000(−)	238,000(+)
U_3U_4	0.800(−)	180,000(−)	30.0	27.21	198,000(−)	158,000(+)
U_4L_5	0.565(−)	169,600(−)	42.4	30.15	238,000(−)	134,000(+)
L_0L_1	0.600(+)	120,000(+)	30.0	13.50	267,000(+)	160,000(+)
L_1L_2	0.600(+)	120,000(+)	30.0	13.50	267,000(+)	160,000(+)
L_2L_3	0.800(+)	180,000(+)	30.0	20.25	267,000(+)	214,000(+)
L_3L_4	0.400(+)	120,000(+)	30.0	13.50	267,000(+)	107,000(+)
L_4L_5	0.400(+)	120,000(+)	30.0	13.50	267,000(+)	107,000(+)
U_1L_1	0.000	60,000(+)	30.0	unnecessary		0
U_2L_2	0.000	0	30.0	unnecessary		0
U_3L_3	0.400(−)	0	30.0	unnecessary		0
U_4L_4	0.000	60,000(+)	30.0	unnecessary		0
U_1L_2	0.848(+)	84,800(+)	42.4	11.25	320,000(+)	271,000(+)
U_2L_3	0.000	0	42.4	unnecessary		0
U_3L_2	0.565(+)	0	42.4	unnecessary		0
U_4L_3	0.565(+)	84,800(+)	42.4	11.25	320,000(+)	181,000(+)

NOTE.—The computations for the deflection of any panel point of a symmetrical truss symmetrically loaded and having an *even* number of panels, may often be reduced materially by substituting, for a single load of unity at the panel point under consideration, two loads of unit, one at the panel point and the other at the corresponding panel point on the other half of the truss, the bars in only one-half the truss being considered and in each bar the value of s due to both loads of unity being used. The correctness of this method depends upon the fact that the stress in any main bar in the left half of the truss due to a load of unity at any panel point on the right half of the truss is the same as the stress in the corresponding bar in the right half of the truss due to a load of unity at the corresponding panel point on the left half of the truss. It is obviously correct for the center panel point.

178. Deflection of Beams and Girders.—Simple formulas for the deflection of beams and girders of constant cross section supported at both ends or fixed at one end are derived in all standard works on mechanics. These derivations are usually based upon the general equation for the elastic curve, *viz.:*

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

The more common cases are represented by Fig. 285, in which the deflections given are the maximum deflections in inches, provided that linear dimensions are in inches and forces in

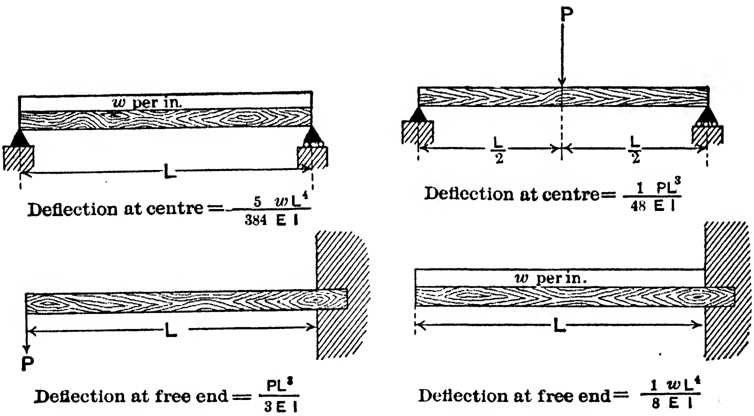


FIG. 285.

pounds. For more complicated cases of loading or for girders with variable cross sections, the method of work applied in the same manner as for trusses, the fibers of the beam being substituted for the bars of the truss, is generally simpler.

A general formula for the deflection may be developed by this method in the following manner:

Let M = moment at section *de* of the given beam (*see* Fig. 286) due to the external forces causing the deflection of point *a*.

m = moment at same section due to load unity acting vertically at *a*.

δ = longitudinal distortion, due to the external forces, of the prism *fdeg* at a distance y from the neutral axis.

δ_a = deflection of point a due to the external forces.
f_1 = longitudinal fiber stress due to moment m at a distance y from the neutral axis.
f_2 = fiber stress at same point, due to moment M.
w = internal work done in prism of length dx, depth dy, and width b, with its center at a distance y from the neutral axis of the beam, by the load unity, during the distortion of the beam by the application of the external forces.

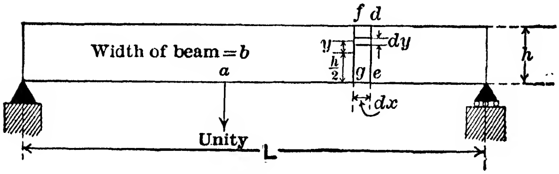


FIG. 286.

W = total internal work done in the beam by the stresses due to the load of unity during the beam's distortion by the external forces.
I = moment of inertia of beam, in.^{4}

Then,

$$\delta = \frac{f_2 dx}{E} = \frac{My}{EI}dx$$

and

$$w = (f_1 b dy)\left(\frac{My}{EI}dx\right)$$
$$= \left(\frac{my}{I}bdy\right)\left(\frac{My}{EI}dx\right)$$
$$= \left(\frac{Mm}{EI}\right)\left(\frac{y^2bdy}{I}\right)(dx)$$

Therefore,

$$W = \int_0^L \int_{-h/2}^{h/2} Mm\frac{y^2dby}{EI^2}dx$$

but

$$\int_{-h/2}^{h/2} \frac{y^2bdy}{I} = \frac{I}{I} = 1$$

Hence,

$$W = \int_0^L \frac{Mm}{EI}dx$$

The external work due to the load of unity $= 1 \times \delta_a$; hence,

$$\delta_a = \int_0^L \frac{M m_0 dx}{EI} \tag{29*}$$

The application of Eq. (29) to a beam of constant cross section is illustrated by the following problem:

Problem: Let it be required to find the deflection at the load for the case shown in Fig. 287.

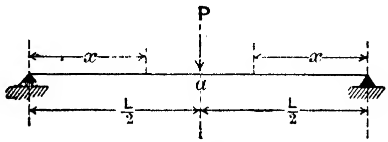


FIG. 287.

Solution: Consider a section at distance x from left end. Then

$$M = \frac{Px}{2} \qquad \text{and} \qquad m = \frac{x}{2}$$

Since the beam is symmetrical, the integral for the entire length of beam may be taken as double that for the left-hand half; therefore, the value of δ_a is given by the following equation:

$$\delta_a = 2\int_0^{L/2} \frac{Px}{2} \cdot \frac{x}{2} \cdot \frac{dx}{EI} = \frac{1}{48}\frac{PL^3}{EI}$$

Were the beam to be loaded for its entire length with a load of p per foot, M in the above equation would be $\frac{pL}{2}x - \frac{px^2}{2}$; hence, the equation for deflection would be

$$\delta_a = \frac{p}{2}\int_0^{L/2} \frac{(Lx - x^2)(x)dx}{EI} = \frac{5}{384}\frac{pL^4}{EI}$$

For beams of variable section, formula (29) may be applied by integrating for different portions of the beam and then adding the results. Suppose, for example, that in the first case the middle half of the beam had the value I_1 for the moment of inertia and the two end quarters each had the value I_2. The total deflection would then be expressed by the following equation:

$$\delta_a = 2\int_0^{L/4} \frac{Px^2}{4}\frac{dx}{EI_2} + 2\int_0^{L/4} \frac{P\left(\frac{L}{4} + x\right)^2}{4}\frac{dx}{EI_1}$$

* Note that in this equation m_0 is in units of distance and hence the equation is homogeneous and also that the equation may be applied to a curved bar of moderate curvature by substituting ds and s both measured along arch axis for dx and L, respectively (see also Art. 131).

The case just given illustrates the application of the method to an end-supported girder with a single cover plate on each flange extending over the central half of the girder. If more cover plates are used, it is necessary to write more terms, but the same general method is applicable. If the girder varies in depth as well as in flange, it may be divided into as many sections as seems desirable and the average moment of inertia of each section used, the equation for the deflection including as many terms as there are moments of inertia.

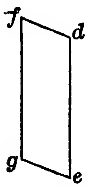


FIG. 288.

Before leaving this method, it should be observed that both by it and the elastic-curve method, by which the results shown in Fig. 285 are usually obtained, the deflection due to shear is neglected. The effect of positive shear at the section under consideration would be to distort the prism *fdeg* in the manner shown by Fig. 288 and hence to cause some deflection. The value of the deflection due to shear is, however, relatively small and may be neglected.

179. Slope by Method of Work.[1]—The slope of the neutral axis of any member subjected to flexure only may be found by the method of work in a manner similar to that used for deflections by substituting a unit moment for the unit load. This may be proved as follows:

Consider the beam shown in Fig. 286, and let the problem be the determination of the change in slope, *i.e.*, the angular deflection of the neutral axis of the beam at the normal section through a due to the application at any point of an external load P acting normal to beam axis.

Let α = change of direction, radians, of neutral axis at section under consideration = change in direction of the normal to this axis.

Let m = moment, in.-lb., at any section due to moment of 1 in.-lb. applied at section under consideration.

Let other nomenclature be the same as in the previous article.

By mechanics, the external work done by the unit moment as the direction of the section changes, equals $m\alpha$; also, from previous article, the internal work $= W = \int^{L} \frac{Mmdx}{EI}$.

[1] First presented in detail by Prof. George F. Swain (see *Proc. Am. Soc. C. E.*, March, 1919).

Therefore,

$$\alpha \text{ (1 in.-lb.)} = \int_0^L \frac{Mmdx}{EI}$$

Hence,

$$\alpha = \int_0^L \frac{Mm_1dx}{EI} \tag{30}$$

in which m_1 is the numberical coefficient of the moment m. This equation gives α as a ratio, which should be the case.

The sign of the result indicates the direction of the angular movement. A positive result shows the angular movement to be in the same direction as that of the moment of unity; *e.g.*, if the unit moment is clockwise and the sign of the final result is positive, the change of slope is also clockwise. In the computations, care must be used to express M and m_1 with their proper signs.

This method can be equally well applied to a truss or to a structure composed both of members in bending and members in direct stress. In either case the work due to direct stresses should be found in the same manner as in the method of truss deflections, *viz.*, by determining the summation of the products of the change in length of each member due to the applied load and the direct stress in it due to the unit moment.

The following problems illustrate the method.

Problem: Determine change of slope α of neutral axis at point of application of load for the beam shown in Fig. 288*a*.

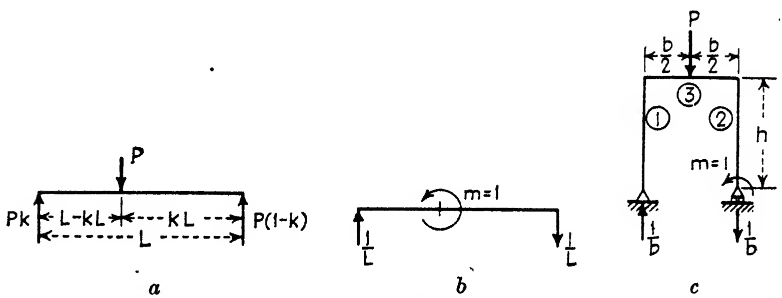


FIG. 288.

Solution: Assume unit counterclockwise moment applied to beam at point of application of load. This moment and the reactions due to it are shown in Fig. 288*b*.

The expression for the slope at the point under consideration will then be as follows:

$$\begin{aligned}\alpha &= \int_0^{L(1-k)} \frac{(Pkx)}{EI}\left(\frac{x}{L}\right)dx + \int_0^{kL} - \left[P\frac{(1-k)x}{EI}\left(\frac{x}{L}\right)\right]dx \\ &= \frac{P}{L}\left(\Big|_0^{L(1-k)}\left[\frac{kx^3}{3EI}\right] - \Big|_0^{kL}\left[\frac{(1-k)x^3}{3EI}\right]\right) \\ &= \frac{P}{L}\left(\frac{k}{3}\cdot\frac{L^3(1-k)^3}{EI} - \frac{(1-k)k^3L^3}{3EI}\right) \\ &= \frac{PL^2k}{3EI}(1-k)(1-2k)\end{aligned}$$

To verify this equation, note that slope $= 0$ when $k(1-k)(1-2k) = 0$ and that this equation when solved gives the following values, which are manifestly correct:

$$k = 0, \qquad k = \tfrac{1}{2} \qquad \text{and} \qquad k = 1$$

Problem: Determine change of slope α at bottom of column 2 of Fig. 288c, due to application of load P.

Solution: Assume a unit counterclockwise moment applied to structure at bottom of column 2. Both of the reactions due to this moment are vertical and equal $1/b$. Their directions are as shown in the figure. The expression for the change in slope is as follows, using same nomenclature as heretofore and, in addition, letting

S = direct stress in any member due to applied load

s = direct stress in any member due to unit moment m

A_1, A_2, E_1, E_2, I_1, I_2, etc., represent the area, the moment of inertia, and the modulus of elasticity, respectively, of the various members.

Then

$$\alpha = \sum\frac{sSL}{AE} + \int\frac{Mm\,dx}{EI}$$

The direct stresses as in the various members are as follows:

$$\begin{aligned}&\text{Bar 1, } S = -\frac{P}{2}, \qquad s = -\frac{1}{b} \\ &\text{Bar 2, } S = -\frac{P}{2}, \qquad s = +\frac{1}{b} \\ &\text{Bar 3, } S = 0 \qquad\quad s = 0\end{aligned}$$

Therefore,

$$\alpha = \frac{P}{2}\frac{h}{A_1E_1}\frac{1}{b} - \frac{P}{2}\frac{h}{A_2E_2}\frac{1}{b}$$

$$+ \int_0^{b/2}\frac{\frac{P}{2}\frac{1}{b}x^2dx}{E_3I_3} + \int_0^{b/2}\frac{\frac{P}{2}x\left(1 - \frac{1}{b}x\right)dx}{E_3I_3}$$

$$= \frac{Ph}{2bA_1E_1} - \frac{Ph}{2bA_2E_2} + \frac{P}{2}\cdot\frac{(b/2)^2}{2E_3I_3} = \frac{Ph}{2b}\left(\frac{1}{A_1E_1} - \frac{1}{A_2E_2}\right) + \frac{P}{16}\frac{b^2}{E_3I_3}$$

If $A_1 = A_2$ and $E_1 = E_2$,
then

$$\alpha = \frac{P}{16}\frac{b^2}{E_3 I_3}$$

180. Graphical Method of Truss Deflection. *Williot Diagram.*—The method of work given in Art. 177 furnishes a simple and accurate method of determining the deflection in a given direction of a particular joint in a truss. It is, however, occasionally desirable to determine the actual distortion at several or all joints, a problem that can be solved somewhat more readily by graphical than by analytical methods.

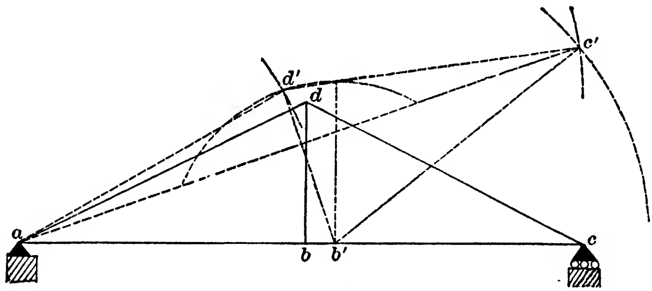


FIG. 289.

It is obvious that the actual movement of any truss joint, due to changes in the bar lengths, may be determined graphically in the following simple manner:

Let *abcd*, Fig. 289, be the truss before the bar lengths are changed, point *a* being fixed in position. Let the new lengths of the bars be *ab'*, *b'c'*, *c'd'*, *d'a*, and *b'd'*. Assume for the present that bar *ab* remains unchanged in direction. If its length is now changed to *ab'*, *b* will move to *b'*; and if, from *a* and *b'* as centers, arcs with radii *ad'* and *b'd'* are swung, these arcs will intersect at *d'*, which will be the new position of *d* on the assumption that *ab* remains unchanged in direction. In a similar manner, arcs swung from *b'* and *d'* with the new lengths of the other bars as radii will give the new position of *c* at *c'*; hence, *ad'c'b'* will be the actual shape of the truss after distortion takes place. Its position is, of course, incorrect, since point *c* should remain on the abutment. Hence, the line *ac'* should actually be horizontal, and by so considering it the deflections of all points may be obtained; *e.g.*, the vertical deflection of point *b* equals the normal distance

from b' to ac', the horizontal deflection of point $c = ac' - ac$, etc.

This method, which may be extended to cover any form of truss, is impracticable in practice, since accurate results cannot be obtained without laying out the diagram to a very large scale, owing to the minute changes in bar lengths for materials having the high moduli of elasticity of structural materials. To overcome this difficulty a modification of this method by which

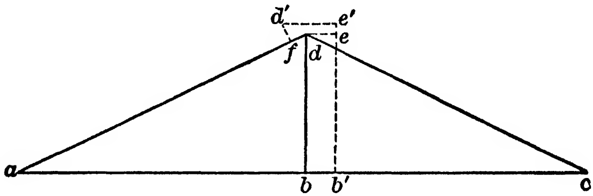


FIG. 290.

changes of length only are dealt with was developed by the French engineer Williot and will now be given.

Let the truss shown in Fig. 290 be identical with that given in Fig. 289, and assume the same changes in bar lengths to occur. Assume also as before that bar ab is fixed in direction and that b moves to b' when distortion occurs.

Let $b'e$ be parallel to bd and equal to it in length.

Let ee' be the change in length of bd (an increase).

Let df be the change in length of ad (a decrease).

If normals to ad and $b'e'$ are erected at points f and e', they will meet at d', which will be the new position of d for any rigid truss of ordinary structural materials, since the distortions are so small compared with the bar lengths that the normal $e'd'$ may be considered as coinciding with the arc swung from b' as a center with radius $b'e'$ and normal fd' may also be considered as coinciding with the arc swung from a as a center with radius af. Since the figure $dee'd'f$ may be drawn to *any scale* without reference to the truss itself, as shown in Fig. 291, it is evident that the actual displacement of point d with reference to axis ab may be found with great accuracy. In a similar manner, each triangle of which the truss is composed may be dealt

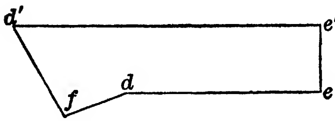


FIG. 291.

with and the displacement of each vertex found with reference to any one of the sides of the triangle as an axis.

If this process is carried out for the entire truss, each displacement may be determined with respect to bar *ab* as an axis by using for each new axis the new position of a bar which is common to any triangle previously considered and that under consideration; *e.g.*, to find the new position of point *c*, with reference to axis *ab*, use the new position of the bar *bd*, *i.e.*, *b'd'*, as an axis.

This is illustrated in Fig. 292, where *ab'd'* is the new position of the triangle *abd*, resulting from the changes in length of bars *ab*, *bd*, and *ad*. To find the position of *c'* due to the combined distortion of the two triangles *abd* and *bcd*, it is evidently neces-

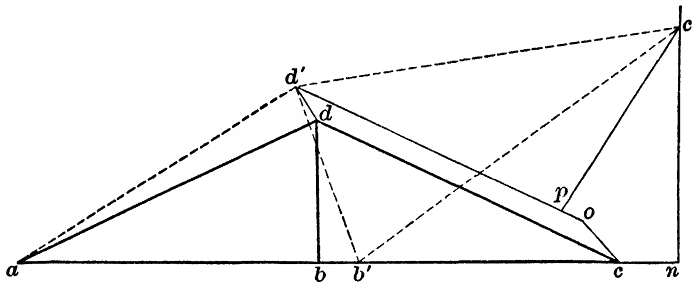


FIG. 292.

sary to determine the intersection of two arcs, one swung from *b'* as a center with the new length of bar *bc* as a radius, and the other from *d'* with the new length of bar *dc* as a radius. Here, as before, it is essentially correct to replace the arcs by their tangents. The process may be accomplished by laying off *cn* equal to the combined increase in length of *ab* and *bc*, *co* equal and parallel to *dd'*, *op* equal to the decrease in length of *cd*, and *nc'* and *pc'* perpendicular, respectively, to *bc* and *dc*. The point of intersection *c'* of these normals is the new position of point *c*.

It should be observed that the new position of point *c* is a function of the change in shape of triangle *abd* as well as of triangle *bcd*, even if bar *ab* actually remains horizontal, since bar *bd* cannot be changed in length without influencing the shape of triangle *bcd*.

Since *dd'* of Fig. 291 equals, numerically, *co* of Fig. 292 and *de* of Fig. 291 equals the change in length of bar *ab*, it is evident that

Fig. 291 may be used as a basis for determining the movement of point c. The resulting diagram is called a *Williot diagram* and is shown in Fig. 293.

In this diagram,

df = decrease in length of bar ad
de = increase in length of bar ab
en = increase in length of bar bc
ee' = increase in length of bar bd
$d'p$ = decrease in length of bar dc

If all of the changes above are plotted parallel to the original directions of the bars, then

dd' = displacement of the point d
dc' = displacement of point c

In order to avoid confusion, it is desirable to letter the initial point the same as the panel point used for an origin, in this case, point a; hence, if in the diagram letter d is replaced by a, the displacement will in every case be the distance from a to the intersections of the proper normals. A Williot diagram for a more complicated truss is shown by the full lines of Fig. 294.

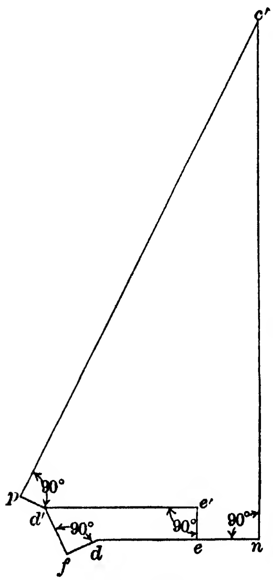


FIG. 293.

The following rules for the construction of the Williot diagram may now be given:

1. Draw a diagram of the truss, and select one member as an axis, or fixed reference bar, assuming this bar as fixed in direction with one end fixed in position. Choose, if convenient, a bar that does not change its direction under the given loading.

2. From some convenient point, plot the axial deformation of this reference bar parallel to and in the direction of the distortion of its free end. Letter the origin and the point just plotted, hereafter called the *second* point, to correspond with the lettering of the fixed and free ends, respectively, of the chosen reference bar.

3. Select two bars which with the fixed reference bar form a triangle. Lay off from the origin the axial deformation in

magnitude and direction of the far end of that bar which is connected to the fixed end of the reference bar. From the second point, similarly lay off the deformation of the other bar. At the

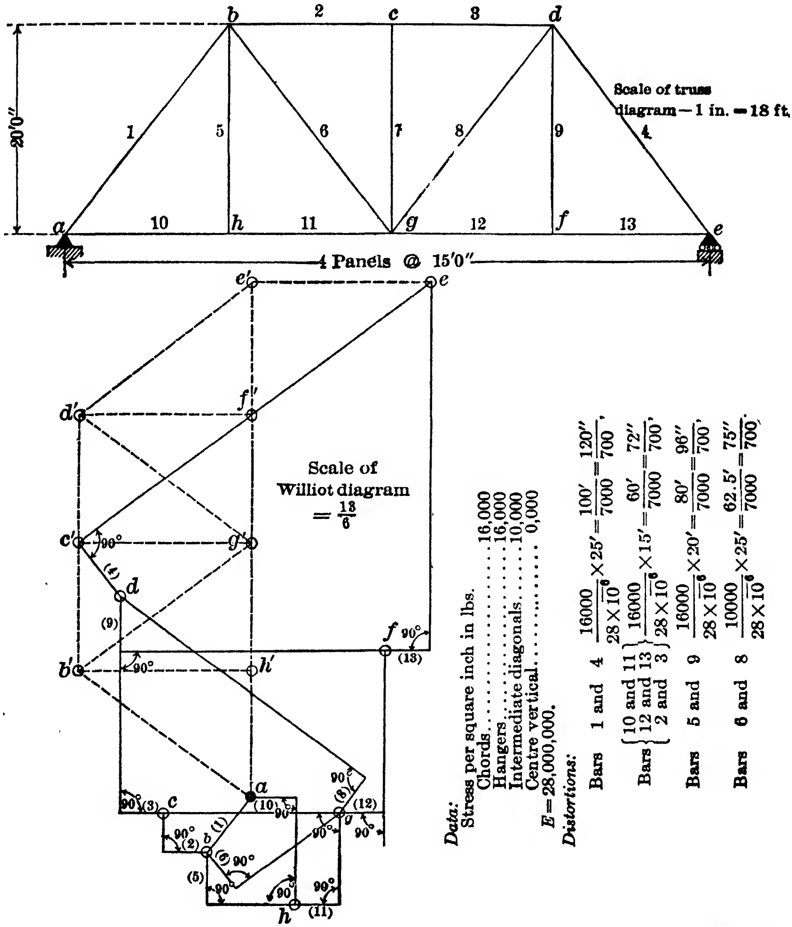


Diagram is drawn assuming a to be fixed in position and ab in direction. Distances from a to b, h, c, g, d, f, and e equal actual movements of these points referred to axis ab.
Figures in brackets indicate bar numbers and are placed on the lines which show the changes in length of the corresponding bars.

FIG. 294.

extremities of these plotted deformations, erect perpendiculars thereto. The distance from the origin to the point of intersection of these perpendiculars equals the distortion of the point of

intersection of the two bars selected under (3) *with reference to the origin*, and the point of intersection should be lettered the same as the corresponding point in the truss diagram.

4. Consider the bars forming a triangle, of which the displaced positions of the ends of one leg are given by two of the three points now plotted. From each of these two points, on a line parallel to the bar of the triangle that is connected thereto, lay off the axial deformation of this bar in the direction of the motion of its far end. At the extremities of these plotted deformations, erect perpendiculars. The distance from this point of intersection to the origin equals the distortion of the apex of the triangle under consideration with reference to the *origin* and should be lettered to correspond to this apex in the truss diagram.

5. Continue thus till all points are located.

6. The distance from the origin to any point of intersection of a pair of perpendiculars equals the actual deflection of the panel point to which these perpendiculars apply, provided that the reference bar remains fixed in direction.

181. Correction of the Williot Diagram.—The Williot diagram shows the actual movement of the joints only when the bar that is assumed to be fixed in direction actually remains fixed during the change in shape of the truss and when the origin also remains fixed. The latter condition is readily obtained by selecting for the origin a fixed point of support, and the former may sometimes but not always be secured; *e.g.*, if the truss shown by Fig. 294 is loaded with a uniform load per foot, bar *cg* will remain vertical while the truss deflects. As in many cases no bar remains fixed in direction, this method would be incomplete unless some means can be found for correcting the displacements thus obtained to allow for the rotation about the assumed axis. If the displacements found by the Williot diagrams for the truss of Fig. 294 are plotted, the truss will appear as shown by dotted lines in Fig. 295 with the distortions exaggerated owing to the plotting of the displacements on a different scale from the truss diagram. Since point *e* should remain on the abutment, its true movement being horizontal, the correction necessary to apply to the diagram must be such as would be produced by rotating the whole truss about *a* until e' drops to the horizontal line through *a*, *i.e.*, until *e* drops through the distance $e'k$

(since here again the arc swung with a as a center and ae' as a radius differs in position from the tangent by only an infinitesimal amount, the actual distance $e'k$ being very small).

The amount of movement of the other points of the truss due to this rotation will bear the same relation to the amount of

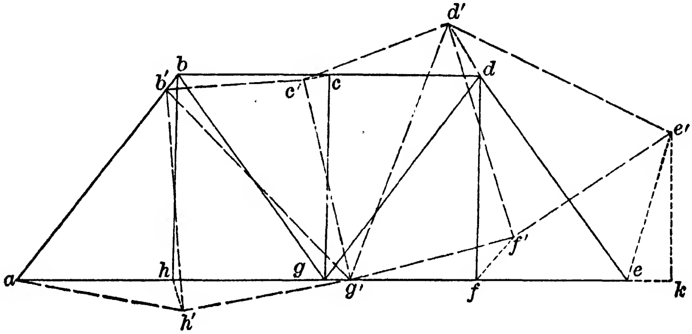


Fig. 295.

movement of e as the distance from a to these points bears to the distance ae.

In Fig. 296, all the full lines are perpendicular to the corresponding bars of the actual truss shown in Fig. 297. In consequence any triangle such as $a'd'e'$ is similar to the corresponding triangle ade; hence,

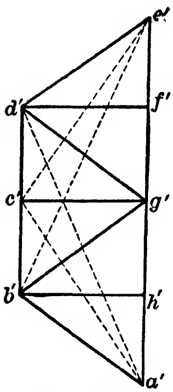


Fig. 296.

$$\frac{a'd'}{ad} = \frac{a'e'}{ae}$$

Therefore,

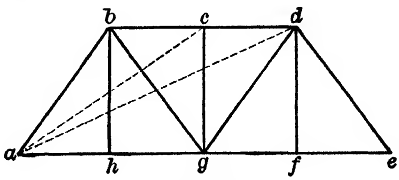


Fig. 297.

$$a'd' = a'e' \times \frac{ad}{ae}$$

In a similar manner, it may be shown that

$$a'c' = a'e' \times \frac{ac}{ae}, \qquad a'b' = a'e' \times \frac{ab}{ae}, \qquad \text{etc.}$$

Hence, if $e'a'$ equals the movement of point e due to rotation, $d'a'$ will equal the movement of point d, $c'a'$ the movement of point c, etc. To obtain the true displacement of the various points the displacements determined in Fig. 294 may be corrected by the foregoing relations. A simple method of accomplishing this has been devised by Prof. Mohr and is illustrated by Fig. 294. It consists of the insertion in the Williot diagram of a figure corresponding to Fig. 296 with a' at a and e' on a horizontal through e. The correct displacement of a point will then be given in *direction* and *magnitude* by the distance from the corresponding point of the inserted truss, shown dotted, to the same point as located by the

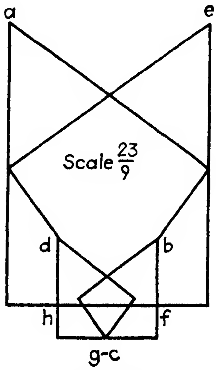


FIG. 298.

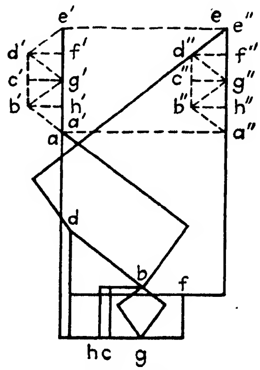


FIG. 299.

Williot diagram; *e.g.*, the correct displacement of point $c = c'c$. The truth of this is easily seen; ac = movement shown by the diagram, $c'a$ = movement due to rotation, and hence $c'c$ equals actual displacement of point c. In other words, rotation causes point c to move from c' to a, and distortion from a to c, the resultant movement equaling $c'c$.

This method of correction for rotation is simple, and no confusion need arise if the following rules are observed:

1. Draw a line through the displaced position, as given on the Williot diagram, of the truss joint at the expansion point of support, parallel to the known direction of movement of this point, *i.e.*, in general, parallel to the surface upon which the expansion rollers move.

2. Draw through the point on the Williot diagram, corresponding to the joint at the fixed point of support of the structure,

a line perpendicular to the line in the truss connecting the two points of support of the truss, and determine its point of intersection with the line previously drawn.

3. Insert in the Williot diagram a truss diagram drawn with all its bars *perpendicular* to corresponding bars in the actual truss, *i.e.*, drawn in a position perpendicular to the original position of the truss. The location and scale of this new truss diagram is fixed by locating the joint corresponding to the expansion point of support at the point of intersection previously determined [see (2)] and the joint corresponding to the fixed point of support at the corresponding point on the Williot diagram.

4. The correct displacement of each joint of the truss may now be determined in magnitude and direction by scaling the distance from the joint as given on the correction diagram drawn as described under (3) and the position of the joint as shown on the Williot diagram.

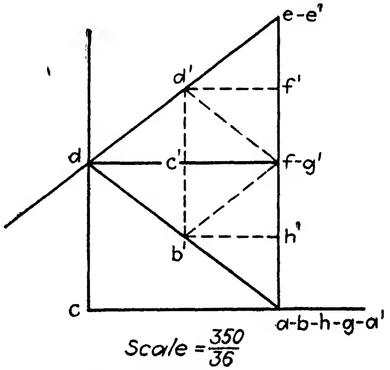


FIG. 300.—Displacement diagram for truss shown in Fig. 294. Point *a* and bar *ah* assumed fixed in position and direction respectively. Change in length assumed to occur in bar 2 only. (Decrease of $^{60}/_{7000}$ ft.)

Figures 298, 299, and 300 illustrate more fully the graphical method as applied to the truss shown in Fig. 294. In Fig. 298, bar *cg* has been assumed as fixed in direction. This agrees with the actual condition if the truss is loaded uniformly, and the displacement diagram needs no correction for rotation. Figure 299 is drawn with *hg* fixed in direction and point *h* in position. This needs to be corrected for rotation, and the correction diagram is given both for the truss shown and for the same truss with point *a* free to move horizontally. Figure 300 is drawn to show the effect of a change in length of one bar only. The correction diagram must also be drawn for this case and is shown in the figure.

182. Elastic-load Method of Truss Deflections. End-supported Trusses.—This method of determining truss deflections is

useful at times and for some trusses simpler to apply than the methods previously given if it is desired to determine the vertical deflection only of all panel points of a given truss.

Deflection Due to Changes in Lengths of Chords and Inclined End Posts.—Consider the deflection of any point b, of the end-supported truss, shown in Fig. 301, due to decrease in length Δ_3 of bar 3 due to any cause whatsoever.

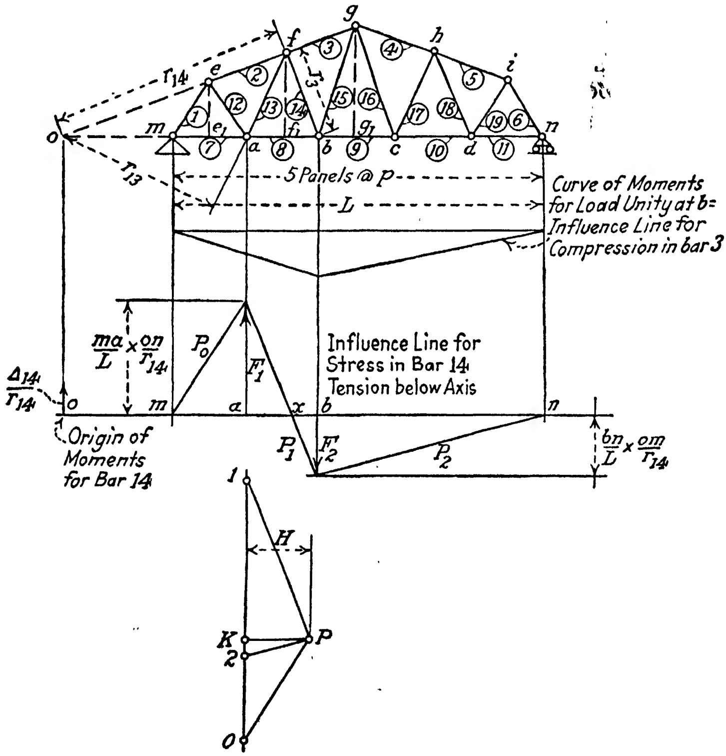


FIG. 301.

Let the positive moment at b due to downward load unity at b equal M_b; then the compressive stress in bar 3 due to load unity at b equals M_b/r_3, and the downward deflection at b equals $M_b\Delta_3/r_3$. Now, if a downward force Δ_3/r_3* is placed at b, the positive

* Δ_3/r_3 is really a ratio and not a force, but for the purposes of this discussion, it will be considered as a force.

moment at b due to this force $= M_b\Delta_3/r_3 =$ the downward deflection at b due to decrease in length Δ_3 of bar 3. The name *elastic load* is given hereafter to terms such as Δ_3/r_3.

If a curve of moments is constructed for the elastic load Δ_3/r_3 at b, the deflection of any other panel point on either top or bottom chord due to change in length of bar 3 will be given by the ordinate at the given panel point to this curve of moments since the ordinate at any other panel point such as h is to the ordinate at b as the stress in bar 3 due to load unity at h is to the stress in bar 3 due to load unity at b. It follows that the curve of moments drawn for the elastic load Δ_3/r_3 at b is a curve of panel-point deflections caused by change in length of bar 3. It should be observed that this curve of moments corresponds in shape to the influence line for deflection of point b.

In a similar manner, the elastic load may be determined for any chord bar either in top or bottom chord, or for an end diagonal, and the moment curve drawn for it. The elastic load for a bottom-chord bar should be placed at a top-chord panel point. If the change in length of a top-chord member or of an end diagonal is an increase, then the elastic load for that particular member should act up instead of down, and similarly if the change in length of the bottom chord is a decrease.

Evidently the deflection at any panel point due to changes in length of several or all chord and end-diagonal bars may be determined by plotting moment curves for each bar upon the same axis and taking the sum of the ordinates at the given point to these curves of moments. These curves may, however, be combined in one by placing the elastic loads for the various bars on the structure simultaneously and drawing one curve of moments for all these loads, in which case the ordinate to this curve at any point gives the deflection at the given point due to the changes in length of all the members considered.

The deflection at any point due to changes in length of any or all such bars may also be determined by computing the moment at the point due to the combined elastic loads, instead of measuring the ordinates to a moment curve. It may also be readily determined graphically by the graphical method of moments.

The following rule may now be stated for determining deflections due to chords and inclined end posts by this method for either chord of any end-supported truss:

a. Determine the elastic load for each member by dividing its change in length due to any cause by the normal distance from the member to the origin of moments used in finding the stress in the given member by the method of moments. The elastic load for any bar should be taken as acting downward if the change in length of that bar corresponds in character to that due to the stress caused by a downward force acting at its origin of moments; otherwise, it should be taken as acting upward.

b. Place the various elastic loads computed under (*a*) at their appropriate origins of moments.

c. Determine the deflections of any panel point due to these bars by determining the moment at that panel point due to all the elastic loads. A positive beam moment corresponds to downward deflection. This moment may be computed analytically or determined graphically by means of a funicular polygon.

Deflection Due to Changes in Length of Web Members.—The deflection due to changes in length of intermediate web members of end-supported trusses cannot, in general, be determined by the method developed for chord members, since the character of the stress in such a member under a moving load changes as the load crosses the span. For such members, the following method may be adopted:

Considering the truss in Fig. 301, the influence line for stress in an intermediate web member, such as bar 14, is as shown in the figure tension being plotted below the axis. The deflection at either *a* or *b* due to increase in length Δ14 of this bar equals the ordinate at that point multiplied by Δ_{14}.* The term Δ_{14}/r_{14} may, therefore, be made a factor in any expression for deflection. This influence line corresponds *in shape* to a funicular polygon for two loads: F_1 at *a* acting up, and F_2 at *b* acting down, the polygon being drawn with the first string passing through left-hand point of support. The force polygon for this case is also shown in the figure. Since the closing line is horizontal, this funicular polygon corresponds *in shape* to the moment curve for this loading. (The product of the ordinate at any point between the polygon and the horizontal line joining the supports and of H

* Deflection is downward at points where the stress in the bar as shown by the influence line is of the same character as the stress required to produce the deformation of the bar; otherwise, it is upward.

in the force polygon equals the moment at that point.) Positive moment is plotted below the axis.

If we now place an *upward-acting* elastic load equal to Δ_{14}/r_{14} at point o, the truss reactions at n and m due to this load would have the following values:

$$\text{At } m, \frac{\Delta_{14}}{r_{14}} \cdot \frac{on}{L} \text{ downward}$$

$$\text{At } n, \frac{\Delta_{14}}{r_{14}} \cdot \frac{om}{L} \text{ upward}$$

The moments due to these reactions at a and b would have values as follows:

$$\text{At } a, \frac{\Delta_{14}}{r_{14}} \cdot \frac{on}{L} \cdot ma \text{ negative}$$

$$\text{At } b, \frac{\Delta_{14}}{r_{14}} \cdot \frac{om}{L} \cdot bn \text{ positive}$$

Each of these moment values equals the product of the ordinate to the influence line at point a or b, as the case may be, and the change in length Δ_{14} of bar 14 and hence equals the deflection at the given point both in magnitude and direction. If the two forces F_1 and F_2 are, therefore, of such values as to give a resultant upward force at o equal to Δ_{14}/r_{14}, a moment curve constructed for these two loads with $H = \Delta 14$ will be the curve of deflections for the lower chord panel points due to deformation of bar 14.

The values of F_1 and F_2 corresponding to the condition above will evidently be as follows:

$$F_1 = \frac{\Delta_{14}}{r_{14}} \cdot \frac{ob}{ab}, \qquad F_2 = \frac{\Delta_{14}}{r_{14}} \cdot \frac{oa}{ab}$$

In a similar manner, the elastic loads for bar 13 will have the following values:

$$\text{At } a, F_1 = \frac{\Delta_{13}}{r_{13}} \cdot \frac{ob}{ab}$$

$$\text{At } b, F_2 = \frac{\Delta_{13}}{r_{13}} \cdot \frac{oa}{ab}$$

If the deflections of the upper chord panel points are required, the elastic loads should be placed at the upper chord panel

points. The elastic loads for bar 14 should in this case be placed at f and g and will have the following values:

$$\text{At } f,\ F_1 = \frac{\Delta_{14}}{r_{14}} \cdot \frac{og_1}{f_1g_1} = \frac{\Delta_{14}}{r_{14}} \cdot \frac{og}{fg}$$
$$\text{At } g,\ F_2 = \frac{\Delta_{14}}{r_{14}} \cdot \frac{of_1}{f_1g_1} = \frac{\Delta_{14}}{r_{14}} \cdot \frac{of}{fg}$$

Similarly, for bar 13 the elastic loads should be placed at e and f will have the following values:

$$\text{At } e,\ F_1 = \frac{\Delta_{13}}{r_{13}} \cdot \frac{of_1}{f_1e_1} = \frac{\Delta_{13}}{r_{13}} \cdot \frac{of}{fe}$$
$$\text{At } f,\ F_2 = \frac{\Delta_{13}}{r_{13}} \cdot \frac{oe_1}{f_1e_1} = \frac{\Delta_{13}}{r_{13}} \cdot \frac{oe}{fe}$$

The direction of the elastic load at either panel point may be found by determining by inspection the character of the stress

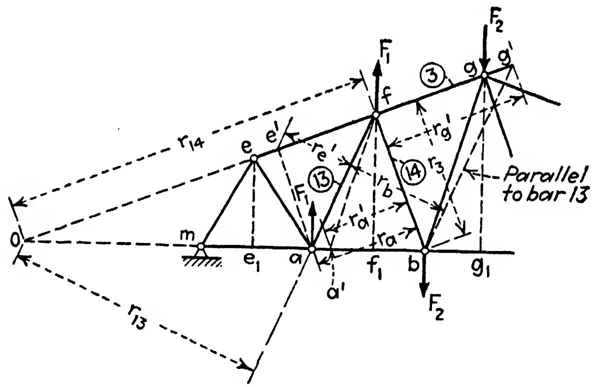


FIG. 302.

in the given bar due to the load applied at the panel point under consideration; if this stress is such that it would of itself cause a deformation in the bar of the character of the actual deformation, the elastic load at this point should act down and that at the other point up. If this convention is adopted, positive beam moment will be consistent with downward deflection as would be the case in a beam and as is the case for deflection due to chords.

Since the value of each elastic load, as given by the foregoing expressions, is either indeterminate or infinite in the case of a

parallel-chord truss in which o is at infinity, another method must be developed for this case. Such a method follows for the case of bottom-chord deflection.

Draw from b, Fig. 302, a line parallel to bar 13, and let its intersection with bar 3 prolonged be at g'. Let the normal distance from g' to bar 14 equal $r_{g'}$, and the normal distance from a to bar 14 equal r_a. The following equations may now be written for the elastic loads for bar 14:

$$\frac{oa}{ab} = \frac{of}{fg'} = \frac{r_{14}}{r_{g'}}$$

Also,

$$\frac{ob}{ab} = \frac{r_{14}}{r_a}.$$

But

$$F_1 = \frac{\Delta_{14}}{r_{14}} \cdot \frac{ob}{ab}, \qquad \text{and} \qquad F_2 = \frac{\Delta_{14}}{r_{14}} \cdot \frac{oa}{ab}$$

Hence,

$$F_1 = \frac{\Delta_{14}}{r_a}, \qquad \text{and} \qquad F_2 = \frac{\Delta_{14}}{r_{g'}}$$

For inclined-chord trusses, the above values are, in general, less simple to use than the values previously determined, but for

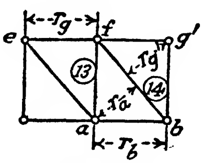


FIG. 303.

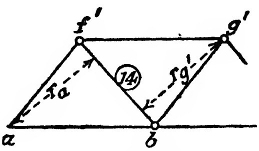


FIG. 304.

parallel-chord trusses they are simpler to apply. For example, for Pratt and Warren trusses with parallel chords, the conditions become those shown in Figs. 303 and 304 in which the values r_a and $r_{g'}$ are equal; hence, the elastic loads for bar 14 should be applied at a and b and be equal and opposite. The elastic loads for bar 13 should also be applied at points a and b, the lever arm in this latter case for the Pratt truss being the panel length, as would be true for other verticals except submembers. It follows that, in the case of Pratt and Warren trusses which are not subdivided, the elastic loads can be com-

puted with great ease. The direction of the elastic loads may be determined in the manner previously given.

183. Deflection of a Cantilever Arm. *Elastic Load Method.*—The deflection of any panel point of a cantilever arm of a truss supported at two points, such as that shown in Fig. 305 due to change in length of any or all chord bars located in the portion of the truss between supports, can be found by extending the end string of the elastic-load moment polygon beyond the point of support adjoining the cantilever arm to the panel point in question and determining the value of the ordinate at the panel point. This method may be proved as follows:

Let it be desired to determine the deflection of panel point L_1 (Fig. 305), due to decrease Δa in length of bar a.

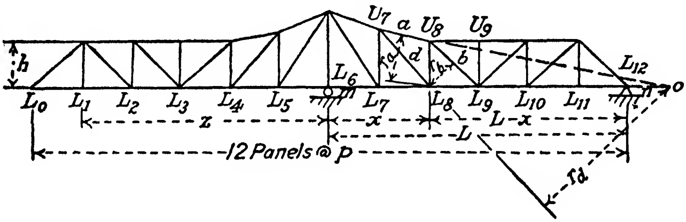


FIG. 305.

The elastic load for bar $a = \Delta a/r_a$ and should be placed at L_8. The moment at this point due to this load $= +\dfrac{\Delta a}{r_a} \cdot \dfrac{L-x}{L} \cdot x.$

If the last string in the moment polygon is extended to L_1, the upward ordinate at L_1

$$= \frac{\Delta a}{r_a} \cdot \frac{(L-x)x}{L}\left(\frac{z}{x}\right) = \frac{\Delta a}{r_a} \cdot \frac{L-x}{L} \cdot z$$

If this deflection is computed by the method of work, the result will be as follows:

Stress in a due to load unity at $L_1 = +\dfrac{z}{L}\left(\dfrac{L-x}{r_a}\right)$

The deflection will then equal $-\dfrac{\Delta a}{r_a} \cdot \dfrac{z}{L}(L-x)$, which equals value previously found and is upward as shown by the sign.

The deflection of points on the cantilever arm due to changes in length of web members located in the portion of the truss between supports may also be found in a similar manner. This

may be shown by taking any web member such as bar b, Fig. 305, and considering the effect of an increase Δ_b in length.

Each of the elastic loads for this bar for bottom-chord deflection equals $\Delta b/r_b$ and should be placed at L_8 and L_9, that at L_9 acting down.

The moment due to them at point $L_8 = -\frac{\Delta b}{r_b} \cdot \frac{p}{L} \cdot x$. The downward ordinate at L_1 to the end string of the moment curve for these loads equals

$$\frac{\Delta_b}{r_b} \cdot \frac{p}{L} \cdot x \cdot \frac{z}{x} = \frac{\Delta_b}{r_b} \cdot \frac{z}{L} p$$

The stress in bar b due to load unity at L_1 equals

$$\frac{z}{L} \cdot \frac{\sqrt{p^2 + h^2}}{h} = \frac{z}{L} \cdot \frac{p}{r_b}$$

Therefore, downward deflection at $L_1 = \frac{\Delta_b}{r_b} \cdot \frac{pz}{L}$ = value previously determined.

If the top chord is inclined, the truth of the method may be readily shown as follows: The deflection of L_7, which may for the present be considered at a distance x from the left support, due to a change in length of bar d, equals $\frac{\Delta d}{r_d} \cdot \frac{on}{L} \cdot x$; therefore, the deflection at a point such as L_1, obtained by prolonging the end string of the moment polygon equals $\frac{\Delta d}{r_d} \cdot \frac{on}{L} \cdot x \cdot \frac{z}{x}$. But load unity at L_1 would give stress in bar d equal to $\frac{z}{L} \cdot \frac{on}{r_d}$; hence, deflection of L_1 due to this bar equals $\frac{\Delta d}{r_d} \cdot \frac{on}{L} \cdot z$, which equals the value previously found.

The foregoing methods apply only to bars located in the portion of the truss between points of support. Changes of length of bars in the projecting end affect the deflection of points in that end only and may be readily determined by the usual analytical method.

184. Effect of Submembers. *Elastic-load Method.*—The elastic-load method as hitherto developed is not applicable to

trusses with submembers, since the curves of deflections for trusses with submembers are different in character from either of those shown in Fig. 301. For a truss such as that shown in Fig. 306, proceed as follows:

First, determine the deflection of all panel points by assuming change in length of each of the submembers (U_1L_1, U_3L_3, and U_5L_5) to be zero; then, correct the deflections of L_1, L_5, and U_3 by

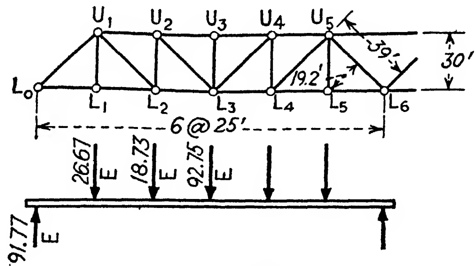


FIG. 306.

the changes in lengths of bars U_1L_1, U_3L_3, and L_5U_5, adding the change in length for a bottom-chord panel point if the submember acting at the point is in tension and for a top-chord panel point if the submember acting at the point is in compression, provided that in both cases the normal deflection of the truss is downward and therefore positive.

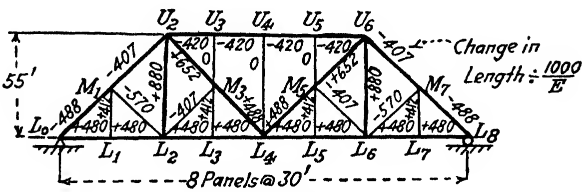


FIG. 307.

For trusses with secondary bars other than simple hangers, such as the truss shown in Fig. 307, proceed as follows:

a. Determine elastic loads for a truss identical with that shown by the heavy lines in Fig. 307, and determine deflection of panel points of this truss by the method previously given for similar trusses. Evidently, under the assumed conditions, the deflection of a point like L_3 will be the average of the deflection of L_2 and L_4.

b. Consider the subsystem trusses such as $L_0M_1L_2L_0$, $L_2M_3L_4L_2$, etc., to act as separate trusses, and determine the deflection of points L_1, L_3, etc., as panel points of such separate trusses.

c. Determine actual deflections of panel points L_1, L_3, L_5, and L_7 by correcting the deflection of these panel points as found under *a* by the deflections of the corresponding panel points of the subsystem trusses.

The foregoing methods are clearly illustrated by the following numerical examples.

Problem: Determine the deflection of the lower chord of the Pratt truss shown in Fig. 306, assuming that the stress per square inch of gross area in the various bars is as follows:

	Pounds
Top chord and inclined end posts	−14,000
Diagonals U_1L_2 and L_4U_5	+14,000
Bottom chord	+16,000
Diagonals U_2L_3 and L_3U_4	+10,000
Verticals U_1L_1 and U_5L_5	+16,000
Verticals U_2L_2 and U_4L_4	−12,000
Vertical U_3L_3	No stress

Solution: As previously stated, the effect of secondary members should be considered separately; hence, the distortion of bars U_1L_1, U_3L_3, and U_5L_5 will be ignored in constructing the deflection polygon.

The deformations of the various bars in feet are as follows, E being expressed in 1,000-lb. units.

$$\text{Bottom-chord bars} = \frac{16}{E} \times 25 = +\frac{400}{E} \qquad L_0U_1 \text{ and } U_5L_6 = \frac{14}{E} \times 39 = -\frac{546}{E}$$

$$\text{Top-chord bars} = \frac{14}{E} \times 25 = -\frac{350}{E} \qquad U_1L_2 \text{ and } U_5L_4 = \frac{14}{E} \times 39 = +\frac{546}{E}$$

$$U_2L_3 \text{ and } L_3U_4 = \frac{10}{E} \times 39 = +\frac{390}{E}$$

$$U_1L_1 \text{ and } U_5L_5 = \frac{16}{E} \times 30 = +\frac{480}{E} \qquad U_2L_2 \text{ and } U_4L_4 = \frac{12}{E} \times 30 = -\frac{360}{E}$$

The elastic loads for one-half of the truss including load at center are given in the table on page 388. Values for the other half are identical.

The elastic loads for this structure and their reactions are as shown in Fig. 306, and the deflections at various panel points of the bottom chord, the effect of bars U_1L_1 and U_5L_5 being neglected, equal the moments due to these loads and are as follows:

Deflection in feet, E in 1,000-lb. units,

$$\text{Panel point } L_1 = \frac{91.77 \times 25}{E} = \frac{2,294}{E}$$

$$\text{Panel point } L_2 = \frac{91.77 \times 50 - 26.67 \times 25}{E} = \frac{3{,}921}{E}$$

$$\text{Panel point } L_3 = \frac{3{,}921}{E} + \frac{(91.77 - 45.40) \times 25}{E} = \frac{5{,}080}{E}$$

The true deflection of panel point L_1 may now be determined by adding to its deflection as previously found the increase in length of bar L_1U_1, *viz.*, $480/E$ which gives $2{,}774/E$.

ELASTIC LOADS FOR TRUSS SHOWN IN FIG. 306
(+ sign indicates that elastic load acts downward.)

Bar	Panel points			
	U_1 and L_1	L_2	U_2	L_3
L_0L_1 and L_1L_2	$\frac{800}{30E} = +\frac{26.67}{E}$			
L_2L_3			$\frac{400}{30E} = +\frac{13.33}{E}$	
L_0U_1	$\frac{546}{E \times 19.2} = +\frac{28.44}{E}$			
U_1U_2		$\frac{350}{E \times 30} = +\frac{11.67}{E}$		
U_2U_3 and U_3U_4				$\frac{700}{30E} = +\frac{23.33}{E}$
U_1L_2	$\frac{546}{E \times 19.2} = -\frac{28.44}{E}$	$\frac{546}{E \times 19.2} = +\frac{28.44}{E}$		
U_2L_2		$\frac{360}{25E} = -\frac{14.40}{E}$		$\frac{360}{25E} = +\frac{14.40}{E}$
U_2L_3		$\frac{390}{19.2E} = -\frac{20.31}{E}$		$\frac{390}{19.2E} = +\frac{20.31}{E}$
L_3U_4				$\frac{390}{19.2E} = +\frac{20.31}{E}$
U_4L_4				$\frac{360}{25E} = +\frac{14.40}{E}$
Totals	$W_a = +\frac{26.67}{E}$	$W_b = +\frac{5.40}{E}$	$W_g = +\frac{13.33}{E}$	$W_c = +\frac{92.75}{E}$

This deflection may readily be checked analytically by placing unit loads simultaneously at L_1 and L_5, computing the combined deflection at L_1 and L_5, and dividing by 2. This method eliminates from the computation all intermediate web members between L_1 and L_5.

For the deflection of the upper-chord panel points the elastic loads for the verticals would have to be placed at top-chord panel points since the signs of the elastic loads for a vertical depend upon whether the loads are applied at top or bottom of the vertical.

The deflection of top-chord panel points for this case may however be obtained by correcting the deflections of bottom-chord panel points by the changes in lengths of the various verticals.

Problem: Determine the deflections of all bottom-chord panel points of truss shown in Fig. 307 due to changes in length of various bars as shown in the figure.

Solution: The elastic loads for the truss are first computed in accordance with the rule given under (*a*). Their values in kips are shown in the following table:

ELASTIC LOADS FOR TRUSS SHOWN BY HEAVY LINES, FIG. 307

Bar	At L_2 and U_2	At L_4 and U_4
L_0U_2	$+\frac{895}{40.55} = +22.07$	
U_2L_4	$-\frac{1,140}{40.55} = -28.11$	$+28.11$
L_4U_6		$+28.11$
U_2U_6		$+\frac{1,680}{55} = +30.55$
L_0L_4	$+\frac{1,920}{55} = +34.91$	
L_4L_8		
Total elastic loads	$+28.87$	$+86.77$

From the elastic loads, the deflections of the main panel points (L_2, L_4 and L_6) are determined by computing the elastic load moments at these panel points and correcting these values by the changes in length of U_2L_2 and U_6L_6. The resulting deflections are as follows, the numerical values in each case to be multiplied by $1{,}000/E$.

L_2, moment due to elastic loads $= 72.25 \times 60 = 4{,}335$

Change in length of U_2L_2 $\doteq \underline{880}$

Total deflection $= 5{,}215$

L_4, moment due to elastic loads $= 6{,}938 = 72.25 \times 120 - 28.87 \times 60 =$ total deflection

The deflections of L_1 and L_3 of the truss shown by heavy lines are determined as previously stated and are as follows:

$$\delta_{L1}' = \frac{5{,}215}{2} = 2{,}607$$

$$\delta_{L3}' = \frac{5{,}215 + 6{,}938}{2} = 6{,}076$$

The deflections of L_1 and L_3 for the subtrusses $L_0M_1L_2L_0$ and $L_2M_3L_4L_2$ follow.

SUBTRUSS $L_0M_1L_2L_0$

Bar	Elastic load at L_1 and M_1 combined
L_0M_1	488/20.27 = +24.07
M_1L_2	570/20.27 = +28.12
L_0L_2	960/27.5 = +34.91
Total elastic load at L_1 and M_1	= +87.10

Moment at L_1 due to elastic loads = +1,307
Change in length of bar M_1L_1 = + 412
Deflection of L_1, subtruss alone being considered, = +1,719

SUBTRUSS $L_2M_3L_4L_2$

Bar	Elastic load at L_3 and M_3 combined
L_2M_3	407/20.27 = +20.08
L_4M_3	−488/20.27 = −24.07
L_2L_4	+960/27.5 = +34.91
	Total = +30.92

Moment at L_3 due to elastic loads = +464
Change in length of bar M_3L_3 = +412
Deflection of L_3, subtruss alone being considered, = +876

In accordance with rule *c*, the true deflections of L_1 and L_3 are now found by combining the partial deflections already found. The computations follow.

Total deflection L_1 = 2,607 + 1,719 = +4,326
Total deflection L_3 = 6,076 + 876 = +6,952
The foregoing values should in each case be multiplied by $1,000/E$ to give actual deflections.

185. Maxwell's Theorem of Reciprocal Deflections.—Consider two points *a* and *b* of any truss such as that shown in Fig. 308.

Let δ_a = deflection of point a in direction XX due to the application of a force P at point b in direction YY.

δ_b = deflection of point b in direction YY due to the application of an equal force P at a in the direction XX.

Maxwell's theorem is that

$$\delta_a = \delta_b$$

The truth of this theorem is readily shown as follows:

Let S = actual stress in any bar of truss.

s_a = stress in any bar due to load unity acting at a in direction XX.

s_b = stress in any bar due to load unity acting at b in direction YY.

Then,

$$\delta_a = \sum s_a \frac{SL}{AE}, \quad \text{and} \quad \delta_b = \sum s_b \frac{SL}{AE}$$

But stress S in any bar due to load P acting at a in direction $XX = Ps_a$; and stress S in any bar due to load P acting at b in direction $YY = Ps_b$.

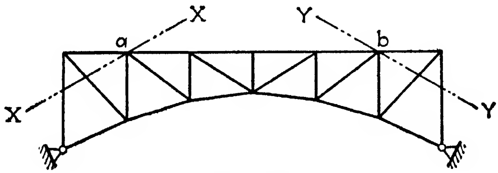


FIG. 308.

Therefore,

$$\delta_a = \sum \frac{s_a P s_b L}{AE} \quad \text{and} \quad \delta_b = \sum \frac{s_b P s_a L}{AE}$$

Hence,

$$\delta_a = \delta_b \qquad \text{Q.E.D.}$$

A similar relation between the deflection of any two points in a girder may be derived by a similar process by inserting proper values in the equation for deflection of girders, *viz.:*

$$\delta = \int \frac{Mmds}{EI}$$

The relation between the forces causing known deflections of any two points in any structure may be readily found by the application of Maxwell's theorem as follows:

Let Δ_a = known deflection of a given point a in direction XX due to the application of a force P_b at b in direction YY.

Δ_b = known deflection of a given point b in direction YY due to the application of a force P_a at a in direction XX.

Now,

$$\Delta_a = \sum \frac{s_a P_b s_b L}{AE}$$

Therefore,

$$P_b = \Delta_a \div \sum \frac{s_a s_b L}{AE}$$

And

$$\Delta_b = \sum \frac{s_b P_a s_a L}{AE}$$

Therefore,

$$P_a = \Delta_b \div \sum \frac{s_a s_b L}{AE}$$

Therefore,

$$\frac{P_b}{P_a} = \frac{\Delta_a}{\Delta_b} \qquad (31a)$$

The relation between the deflection of a and the change of slope at b can be determined in a similar manner.

Let δ_a' = deflection at a in direction XX due to application of moment M at a.

α_b = change in slope at b due to application of force P at a.

s_b' = stress in any bar due to application of moment of unit at b.

Then,

$$\delta_a' = \sum \frac{s_a M s_b' L}{AE}$$

and

$$\alpha_b = \sum \frac{P s_a s_b' L}{AE}$$

Therefore,

$$\frac{M}{P} = \frac{\delta a'}{\alpha b} \tag{31b}$$

The application of Eqs. 31*a* and 31*b* is the basis of a mechanical method known as *Beggs's method* of determining the reactions in statically indeterminate structures. By this method a predetermined deflection or change of slope at any given point *a* in a model of the truss under consideration is produced by a device known as a *deformeter gage* and the corresponding movement of any other point *b* is measured by a microscope.

186. Slope and Deflection. Moment-area Method. *Slope.*—Consider any beam supported at two points as shown in Fig. 309 and loaded in any desired manner, the curve of moments being as shown. It follows from the application of Eq. (30), the unit

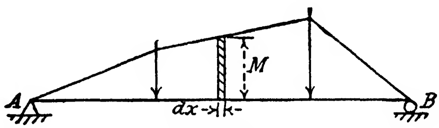


FIG. 309.

moment and the reactions due thereto being as assumed in Fig. 310, that slope at A

$$= \alpha_A = \int_0^L \frac{M}{EI}\left(1 - \frac{x}{L}\right)dx = \int_0^L \frac{Mdx}{EI} - \int_0^L \frac{Mxdx}{LEI}$$

Let F = area between moment curve and beam axis considered positive when above axis, and let x_a and x_b, respectively, be distance of its centroid from A and B, respectively.

Then,

$$\int_0^L Mdx = F$$

Also, $\int_0^L Mxdx = Fx_a$ which may be considered positive when clockwise; therefore, for a beam of constant values of E and I,

$$\alpha_A = \frac{F}{EI} - \frac{Fx_a}{LEI} = \frac{F}{EI}\cdot\frac{x_b}{L} \tag{32}$$

The foregoing value of α_A equals the reaction at A due to the application to the beam of an imaginary force F/EI located at

distance x_b from right end of beam. Its value is positive, *i.e.*, clockwise if F is positive. The value of Fx_b may be determined graphically, if desired, by the equilibrium-polygon method, the areas being divided into a series of small areas.

For cantilever beam with fixed end at B,

$$\alpha_A = \int_0^L \frac{Mdx}{EI} = \frac{F}{EI} \tag{33}$$

Since, in such a cantilever beam, F is negative for downward forces, α_A is negative, *i.e.*, counterclockwise which is evidently correct.

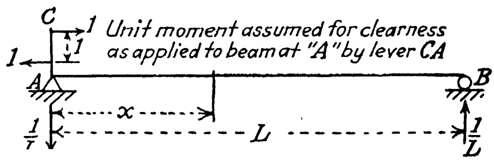


FIG. 310.

In a similar manner for an end-supported beam,

$$\alpha_B = -\frac{Fx_a}{EIL} \tag{34}$$

and for a cantilever beam with fixed end at A

$$\alpha_B = \frac{F}{EI} \tag{35}$$

For any end-supported or cantilever beam,

$$\alpha_A - \alpha_B = \frac{F}{EI} \tag{36}$$

For a beam in which E or I, or both, are variable, the equations of this article may readily be applied by dividing the moment area into sections corresponding to the portion of the beam over which E or I is constant, determining the slope corresponding to these sections, and combining the results.

The following examples illustrate the application of this method.

Problem: Determine expression for slope at B of an end-supported beam with E and I constant throughout and carrying a uniform load w per foot applied over its entire length.

Solution: The moment curve and value of F for this case are shown in Fig. 311.
Hence,

$$F = \frac{1}{12}wL^3 \qquad \text{and} \qquad x_a = \frac{L}{2}$$

Therefore,

$$\alpha_B = -\frac{Fx_a}{LEI} = -\frac{1}{24}\frac{wL^3}{EI}$$

If $L = 30$ ft., $w = 1{,}000$ lb. per ft., $I = 2{,}400$ in.-sq., and $E = 30{,}000{,}000$ lb. per square inch,

$$\alpha_B = -\frac{1}{24}\frac{1{,}000}{12}\frac{(30 \times 12)^3}{30{,}000{,}000 \times 2{,}400} = \frac{9}{4{,}000},$$

radians, and is counter-clockwise.

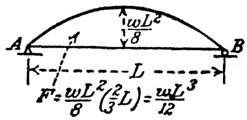


Fig. 311.

Problem: Determine slope at A of an end-supported beam with E and I constant throughout, carrying a single concentrated load P at center.

Solution:

$$F \text{ for this case} = \frac{P}{2} \cdot \frac{L}{2} \cdot \frac{L}{2} = \frac{PL^2}{8}$$

$$x_b = \frac{L}{2}$$

Therefore,

$$\alpha_A = \frac{PL^2}{16EI} \text{ corresponding to a clockwise slope at } A$$

Deflection.—Let it be desired to determine by the moment-area method the deflection of any point n of an end-supported beam loaded in any manner such as that shown in Fig. 309.

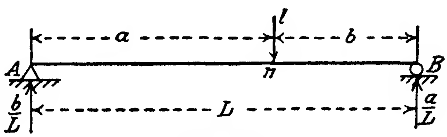


Fig. 312.

Applying load of unity at n gives the reactions shown in Fig. 312.

Applying Eq. (29), letting δ_n = vertical deflection downward at n, gives the following expression:

$$\delta_n = \int_0^a \frac{Mbxdx}{LEI} + \int_0^b \frac{Maxdx}{LEI}$$

Now, if we let F_A and F_B, respectively, equal value of moment area to left and to right, respectively, of a normal to beam axis through n, and x_a and x_b the respective distances of the centroids of these areas from A and B, respectively, both x_a and x_b being considered positive, the following expression may be written for a beam having E and I constant throughout:

$$\delta_n = \frac{b}{L} \cdot \frac{F_A x_a}{EI} + \frac{a}{L} \frac{F_B x_b}{EI} \tag{37}$$

= reaction at A due to a force corresponding to moment area to right of n plus reaction at B corresponding to moment area to left of n

For the center of a symmetrically loaded end-supported beam,

$$F_A = F_B = \frac{F}{2}, \qquad x_a = x_b, \qquad a + b = L$$

Therefore, for this case we have

$$\delta_n = \frac{Fx_a}{2EI} = \frac{Fx_b}{2EI} \tag{38}$$

If the beam is a cantilever with fixed end at B,

$$\delta_n = -\int_0^b \frac{Mxdx}{EI}$$

which for a beam with constant values of E and I becomes

$$\delta_n = -\frac{Fx_a}{EI} \tag{39}$$

In applying Eqs. (37) to (39), F should be taken as positive above the axis and the signs of x_a and x_b should be neglected. A positive result represents downward deflection.

Applications of the foregoing equations are illustrated by the following problems:

Problem: Determine deflection δ_c at center of an end-supported beam of span L, with E and I constant, due to a concentrated load P applied at center.

Solution:

$$F_a = \frac{P}{2} \cdot \frac{L}{2} \cdot \frac{L}{4} \cdot \frac{PL^2}{16} = F_b = \frac{F}{2}$$

$$x_a = \frac{2}{3}\frac{L}{2} = \frac{L}{3} = x_b$$

$$a = b = \frac{L}{2}$$

Therefore,

$$\delta_c = \frac{PL^2}{16} \cdot \frac{L}{3EI} = \frac{1}{48}\frac{PL^3}{EI}$$

Problem: Determine deflection δ_c at center of beam in previous problem due to a uniform load w per foot over entire length of beam.

Solution: The moment curve for this case and the position of the centroids are shown in Fig. 313.

$$F_a = \frac{2}{3}\frac{wL^2}{8} \cdot \frac{L}{2} = F_b = \frac{F}{2}, \qquad x_a = \frac{5}{8}\frac{L}{2} = x_b$$

$$a = b = \frac{L}{2}$$

Therefore,

$$\delta_c = \frac{wL^3}{24EI} \cdot \frac{5}{16}L = \frac{5}{384}\frac{wL^4}{EI}$$

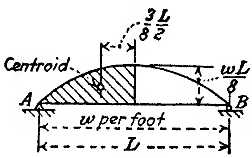


FIG. 313.

Problem: Determine deflection δ_B at free end B of a cantilever beam having E and I constant due to a concentrated load at B.

Solution:
For this case,

$$F = -\frac{PL^2}{2}, \qquad x_b = \frac{2}{3}L$$

Therefore,

$$\delta_B = \frac{PL^2}{2EI} \cdot \frac{2}{3}L = \frac{PL^3}{3EI}$$

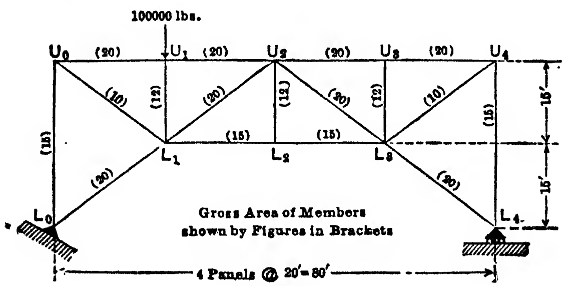
187. Camber Defined.—A structure is said to be *cambered* when so constructed that it will not assume its theoretical form until fully loaded. For beams and short-span girders, no provision for camber need be made. Girders and I beams of long span are often cambered by being slightly arched. Trusses, except in the case of comparatively short riveted spans, should always be cambered. This is particularly important in the case of pin bridges as ordinarily constructed with the splices of the top chord dependent for their strength upon the intimate contact of the planed ends of the chord sections. Camber is also artistically important in girders and in trusses with straight bottom chords in order to prevent the structure from having the appearance of sagging at the center when unloaded. The avoidance of long horizontal lines in important architectural buildings is also desirable and is well exemplified in the Parthenon at Athens.

188. Rules for Computing Cambers.—Short-span parallel-chord trusses are ordinarily cambered by the following more or less empirical rule: Make the top-chord panel lengths longer than those of the bottom chord by $\frac{3}{16}$ in. for every 10 ft. in horizontal projection. This process necessitates a corresponding change in the diagonals, but the verticals and bottom chords are unaffected.

For long spans or for trusses with curved top chords the cambering should be accomplished by decreasing the length of the tension members and increasing that of the compression members by an amount equal to their changes in length under the dead stresses and the live stresses due to full live load with due allowance for pinhole play, the geometrical lengths of the bars, as given in the truss diagram, being used as a basis. The lengths of the bars thus obtained correspond to the outline of the structure when assembled on falsework and not carrying its own weight. When the falsework is removed, the structure will deflect by an amount equal to the nonelastic deflection plus the deflection due to its own weight. If cambered in this manner the application of the live load, under which the changes in length of the bars were computed, should cause the truss to take the shape of the theoretical diagram. Continuous and cantilever trusses require special consideration.

Problems

68. *a.* Compute horizontal deflection of point L_4 of this steel structure due to the applied load shown, and state its direction. $E = 29{,}000{,}000$.

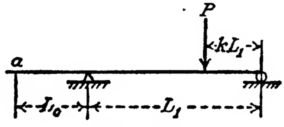


PROB. 68.

b. Determine the magnitude and direction of the horizontal force that must be applied at L_4 to deflect point L_4 horizontally an amount equal to the deflection determined in the first portion of this problem.

69. Determine the magnitude and direction of the horizontal movement of point L_4 of the structure of Prob. 68 due to heating the top chord 50°F. above the remainder of the structure.

70. Derive the expression for change of slope at point a of this beam due to application of load P; E and I constant.

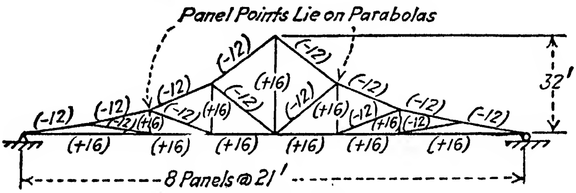


PROB. 70.

71. *a.* Derive expression for change of slope at right end of an end-supported beam of span L due to clockwise moment M_0 applied at left end of beam; E and I constant.

b. Derive expression for change of slope of right end of same beam due to application of clockwise moment M_0 at right end of beam.

72. Compute the elastic loads for the truss shown, assuming unit stresses in bars to be as shown in brackets on the diagram. State point of application of each elastic load, and determine deflections of upper chord panel points. Stresses in 1,000-lb. units.



PROB. 72.

73. Determine the vertical deflections of panel points, L_1, L_2, L_3, and L_4 of the truss shown in Fig. 307 by the analytical method, and by the Williot diagram, and compare results with those obtained by the elastic-load method. In constructing the Williot diagram, consider L_0 as fixed point and L_0M_1 as fixed bar, using scale 1 in. = 2,000,000/E and tabulating results in accordance with panel-point numbers. Use paper 15 by 24 in.

CHAPTER XV

STATICALLY INDETERMINATE GIRDERS AND TRUSSES

189. Continuous Girders. Definitions.—Continuous girders[1] are structures supported at more than two points and capable of carrying bending moments and shear at all sections throughout their entire length. Such structures are commonly used in reinforced-concrete buildings and for swing bridges, and their employment for other bridges is becoming more common. Continuous girders are subject to both positive and negative live moments over portions of their lengths and in consequence may require additional material to provide for the resulting reversal of stress; they may also be stressed by changes of temperature if the ends are fixed or are at different elevations. Partially continuous girders are girders supported at more than two points, but so built that the continuity is interrupted at one or more sections; such girders are generally trusses in which the continuity is broken by the omission of diagonals in one panel, as was noted in connection with cantilever trusses and swing bridges.

190. Reactions on Continuous Girders. Method of Computation.—The reactions on continuous girders can be accurately determined by the *theorem of three moments*, if the moment of inertia and modulus of elasticity of the material are constant throughout, a condition that sometimes exists for beams. If the moment of inertia and the modulus of elasticity of the material are not constant throughout, the reactions cannot be accurately computed until the cross-sectional areas are known, hence, an accurate determination of the stresses for such structures can be accomplished only by a series of approximations, the reactions first being approximately determined, the stresses and areas computed, and the computations revised to correspond to the new areas, the process being repeated as often as is necessary to obtain sufficiently precise results.

[1] As used here the word *girder* is intended as a general word to cover beams, plate girders, and trusses.

191. Derivation of the Three-moment Equation.—Let the girder shown in Fig. 314 have n spans. There will then be $(n + 2)$ unknown reactions, all the supports but one being on rollers, and hence $(n - 1)$ equations must be obtained from other conditions than those of statics. These equations may be deduced from the differential equation of the elastic curve, $d^2y/dx^2 = M/EI$, by the method which follows, each of the $n - 1$ resulting equations connecting the moments at three adjoining supports.

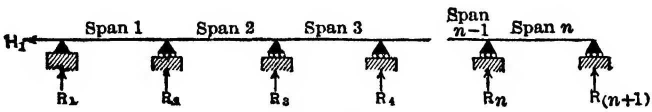


FIG. 314.

Let Fig. 315 represent a portion of a continuous girder with a constant moment of inertia and modulus of elasticity, the entire structure having n spans and the section under consideration including any portion of the girder supported upon three adjoining supports. The axis of the unloaded beam is assumed to be straight and the supports level. The assumption that each load acts at a distance kL from the adjoining support, simplifies greatly the resulting equations.

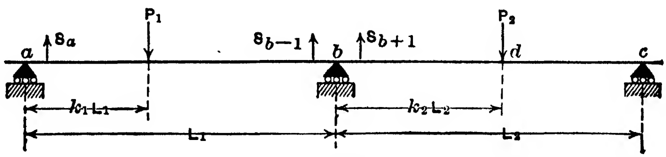


FIG. 315.

Let M_a, M_b, and M_c be the moments upon the beam at the supports, and assume these as positive when causing compression in the upper fiber of the portion of the beam under consideration.

Let S_a, S_{b-1}, and S_{b+1} be the shear at infinitesimal distances from the supports a and b, and let these be assumed as positive when acting as shown.

Let M equal the moment at any section of the girder.

Let t_c, t_{b+1}, and t_{b-1} equal the tangents at c and at infinitesimal distances on either side of b of the angle between the neutral axis of the deflected girder and its original position.

Let h = normal distances of points a, b, and c above some assumed axis parallel to abc and below it.

For the portion of the girder between b and c, the moment at a distance x from b is given by the following equations:

$$M = EI\frac{d^2y}{dx^2} = M_b + S_{b+1}x \text{ (for portion of girder between } b \text{ and } d) \tag{40}$$

$$M = EI\frac{d^2y}{dx^2} = M_b + S_{b+1}x - P_2(x - k_2L_2) \text{ (for portion of girder between } d \text{ and } c) \tag{41}$$

From (40) by integration we obtain

$$EI\frac{dy}{dx} = M_bx + \frac{S_{b+1}x^2}{2} + C_1EI \tag{42}$$

and

$$EIy = \frac{M_bx^2}{2} + \frac{S_{b+1}x^3}{6} + C_1EIx + C_2EI \tag{43}$$

When $x = 0$,

$$\frac{dy}{dx} = t_{b+1} \qquad \text{and} \qquad y = h$$

Therefore,

$$C_1 = t_{b+1} \qquad \text{and} \qquad C_2 = h$$

Hence,

$$EIy = \frac{M_bx^2}{2} + \frac{S_{b+1}x^3}{6} + t_{b+1}EIx + EIh \tag{44}$$

From (41) by integration we obtain

$$EI\frac{dy}{dx} = M_bx + \frac{S_{b+1}x^2}{2} - \frac{P_2x^2}{2} + P_2k_2L_2x + C_3 \tag{45}$$

and

$$EIy = \frac{M_bx^2}{2} + \frac{S_{b+1}x^3}{6} - \frac{P_2x^3}{6} + \frac{P_2k_2L_2x^2}{2} + C_3x + C_4 \tag{46}$$

When $x = k_2L_2$, $EI\frac{dy}{dx}$ in (45) = corresponding value in (42).

Therefore,

$$C_3 = t_{b+1}EI - \frac{P_2}{2}(k_2L_2)^2 \tag{47}$$

When $x = L_2$, $y = h$.
Therefore,

$$C_4 = EIh - \frac{M_b}{2}L_2^2 - \frac{S_{b+1}L_2^3}{6} + \frac{P_2}{6}L_2^3 - \frac{P_2}{2}k_2L_2^3 - t_{b+1}EIL_2 + \frac{P_2}{2}k_2^2L_2^3$$

Hence,

$$EIy = \frac{M_b}{2}(x^2 - L_2^2) + \frac{S_{b+1} - P_2}{6}(x^3 - L_2^3) + \frac{P_2}{2}(k_2L_2x)(x - k_2L_2) + t_{b+1}EI(x - L_2) + EIh + \frac{P_2}{2}k_2L_2^3(k_2 - 1) \quad (48)$$

The value of y given by (44) equals its value as given by (48) when $x = k_2L_2$; equating these values gives

$$0 = -\frac{M_b}{2}L_2^2 - \frac{S_{b+1}}{6}L_2^3 - \frac{P_2}{6}L_2^3(k_2^3 - 1) - t_{b+1}EIL_2 + \frac{P_2}{2}k_2L_2^3(k_2 - 1)$$

Hence,

$$t_{b+1}EI = \frac{P_2L_2^2}{6}(1 - k_2^3 + 3k_2^2 - 3k_2) - \frac{M_b}{2}L_2 - \frac{S_{b+1}}{6}L_2^2$$

But

$$M_b + S_{b+1}L_2 - P_2L_2(1 - k_2) = M_c \quad (49)$$

Therefore, by substituting for S_{b+1} its value from this latter equation, we obtain

$$t_{b+1} = \frac{L_2}{6EI}[P_2L_2(1 - k_2)(k_2 - 2)k_2 - 2M_b - M_c] \quad (50)$$

When $x = L_2$, $dy/dx = t_c$; therefore, the value of t_c may be obtained from (45) by placing $x = L_2$, and substituting for C_3 its value as obtained from (47) after substituting for t_{b+1} its value from (50), and for S_{b+1} its value from (49).

The result thus obtained is given by the following equation:

$$t_c = \frac{L_2}{6EI}[M_b + 2M_c + P_2L_2k_2(1 - k_2^2)] \quad (51)$$

By working in a similar manner in span L_1 an equation identical to (51) would be obtained with the indices reduced to correspond to the nomenclature of span L_1. The equation for this span may therefore be written at once and will be as follows:

$$t_{b-1} = \frac{L_1}{6EI}[M_a + 2M_b + P_1L_1k_1(1 - k_1^2)] \tag{52}$$

The two values, t_{b+1} given by (50) and t_{b-1} given by (52), are identical, since they equal the tangents to the slope of the neutral axis at two sections an infinitesimal distance apart; hence, they may be placed equal to each other, thereby enabling the following equation to be written:

$$M_aL_1 + 2M_b(L_1 + L_2) + M_cL_2 = P_1L_1^2(k_1^3 - k_1) + P_2L_2^2(3k_2^2 - k_2^3 - 2k_2) \tag{53}$$

Equation (53) is the three-moment equation in its general form and is applicable to structures of constant cross section and homogeneous material and supported on level supports.

To obtain the corresponding equation for uniform loads, substitute the following values and integrate between proper limits:

$$P_1 = w_1dx, \qquad P_2 = w_2dx, \qquad k_1L_1 = x_1 \qquad \text{and} \qquad k_2L_2 = x_2$$

In these equations, w_1 and w_2 are the loads per foot on the two spans.

For the case where the load extends over the entire length of one or more spans and is uniform over each span to which it is applied, this gives

$$\begin{aligned} M_aL_1 + 2M_b(L_1 + L_2) + M_cL_2 &= L_1^2\int_0^{L_1} w_1dx\left[\left(\frac{x_1}{L_1}\right)^3 - \frac{x_1}{L_1}\right] + \\ &\quad L_2^2\int_0^{L_2} w_2dx\left[3\left(\frac{x_2}{L_2}\right)^2 - \left(\frac{x_2}{L_2}\right)^3 - 2\left(\frac{x_2}{L_2}\right)\right] = \\ &\quad L_1^2w_1\left(\frac{L_1^4}{4L_1^3} - \frac{L_1^2}{2L_1}\right) + L_2^2w_2\left[\frac{L_2^3}{L_2^2} - \frac{L_2^4}{4L_2^3} - \frac{L_2^2}{L_2}\right] = \\ &\quad -\tfrac{1}{4}L_1^3w_1 - \tfrac{1}{4}L_2^3w_2 \end{aligned} \tag{54}$$

For the case of a level beam and nonlevel supports the constants of integration would contain values of h that would not cancel; hence, the final form of the equation would include such

terms.[1] If the original axis of the girder is slightly curved before loads are applied and if the supports fit the curve of this unloaded girder, Eq. (54) is applicable.

Girders with fixed ends, *i.e.*, ends so constructed that the change of slope under load is zero, may be computed by assuming that a span of zero length may be added at each of the fixed ends and then applying the three-moment equation to the hypothetical case. For certain interesting properties of the three-

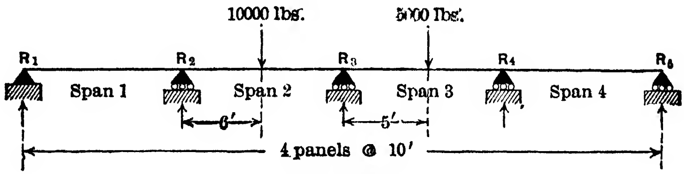


FIG. 316.

moment equation and for information concerning its historical development the reader is referred to Merriman.[2]

192. Application of the Three-moment Equation.—For purpose of illustration the following example of the application of the three-moment equation is given:

Problem: Compute by the three-moment equation the girder reactions for the concentrated loads shown in Fig. 316.

Solution: First apply Eq. (53) to spans 1 and 2.
For these two spans, M_a = moment at $R_1 = 0$, since this is at the end of the girder

$$
\begin{aligned}
M_b &= \text{moment at } R_2 = M_2 \\
M_c &= \text{moment at } R_3 = M_3 \\
P_1 &= 0 \\
P_2 &= 10{,}000 \text{ lb.} \\
L_1 &= L_2 = 10 \text{ ft.} \\
k_2 &= 6/10
\end{aligned}
$$

Hence,

$$2M_2(20) + 10M_3 = 10{,}000 \times 100(1.08 - 0.216 - 1.2) = -336{,}000 \text{ ft.-lb.}$$

[1] See Theory of Continuous Structures and Arches, Spofford, McGraw-Hill Book Company, Inc., New York, 1937, for such equations.

[2] Text Book on Mechanics of Materials, Merriman, John Wiley & Sons, Inc., New York, 1905.

Now, apply Eq. (53) to spans 2 and 3.

For these spans, M_a = moment at $R_2 = M_2$
M_b = moment at $R_3 = M_3$
M_b = moment at $R_4 = M_4$
P_1 = 10,000 lb.
P_2 = 5,000 lb.
$L_1 = L_2$ = 10 ft.
$k_1 = 6/10$
$k_2 = 5/10$

Hence,

$$\begin{aligned}10M_2 + 40M_3 + 10M_4 &= 10{,}000 \times 100(0.216 - 0.600) \\ &\quad + 5{,}000 \times 100(0.75 - 0.125 - 1.00) \\ &= -384{,}000 - 187{,}500 = -571{,}500 \text{ ft.-lb.}\end{aligned}$$

Finally, apply Eq. (53) to spans 3 and 4.

For these spans, M_a = moment at $R_3 = M_3$
M_b = moment at $R_4 = M_4$
M_c = moment at $R_5 = 0$
P_1 = 5,000 lb.
$P_2 = 0$
$L_1 = L_2$ = 10 ft.
$k_1 = 5/10$

Hence,

$$10M_3 + 40M_4 = 5{,}000 \times 100(0.125 - 0.500) = -187{,}500 \text{ ft.-lb.}$$

Solving the three equations thus derived for the three unknowns M_2, M_3, and M_4 gives the following values:

$$\begin{aligned}M_2 &= -\ \ 5{,}250 \text{ ft.-lb.} \\ M_3 &= -12{,}600 \text{ ft.-lb.} \\ M_4 &= -\ \ 1{,}537 \text{ ft.-lb.}\end{aligned}$$

But $M_2 = R_1 \times 10$.

Therefore,

$$R_1 = -525 \text{ lb. (acting down)}$$

and

$$M_4 = R_5 \times 10$$

Therefore,

$$R_5 = -154 \text{ lb. (acting down)}$$

Moreover,

$$M_3 = 20R_1 + 10R_2 - 40{,}000 = 20R_5 + 10R_4 - 25{,}000$$

Hence,

$$10R_2 = 40{,}000 - 12{,}600 + 10{,}500 = 37{,}900$$

Therefore,

$$R_2 = (+)3{,}790 \text{ (acting up)}$$

and

$$10R_4 = 25{,}000 - 12{,}600 + 3{,}080 = 15{,}480$$

Therefore,

$$R_4 = (+)1{,}550 \text{ (acting up)}$$

Application of $\Sigma V = 0$ gives $R_3 = (+)10{,}340$ (acting up).

In a similar manner the reactions for any number of spans, whether equal or unequal in length, and for any loading may be readily computed.

193. Reactions, Shears, and Moments for Common Cases of Continuous Girders.—In order to simplify the determination

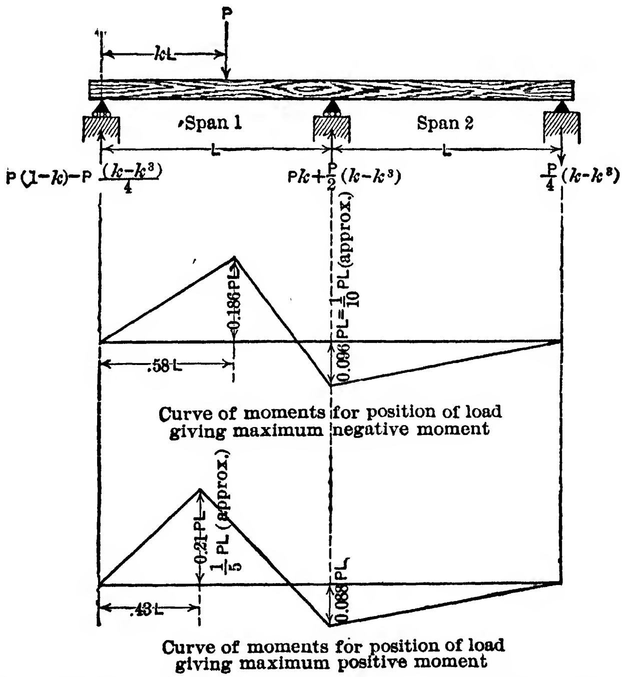
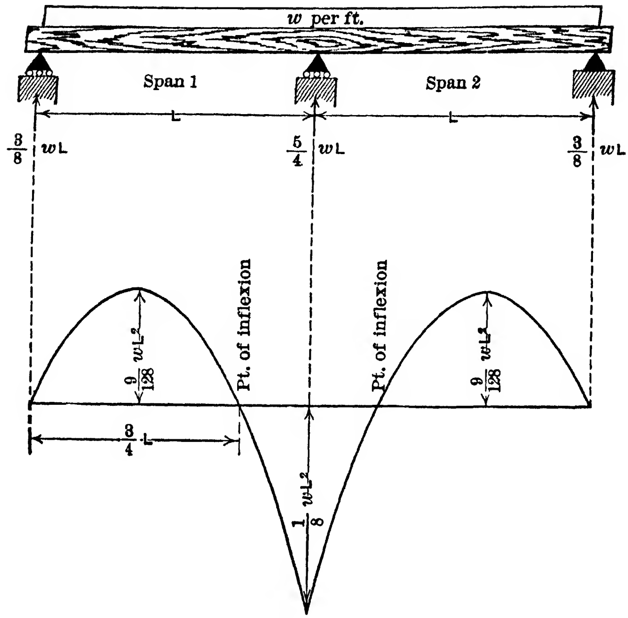
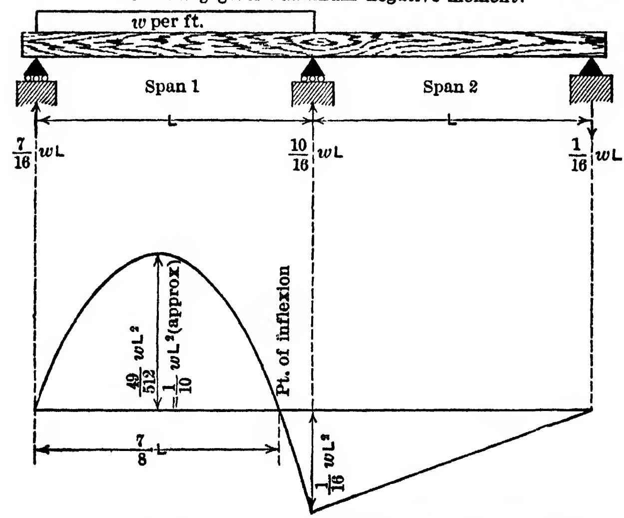


FIG. 317.—Continuous girder. Reactions and moments for a single concentrated load.

of reactions, shears, and moments for certain common cases of continuous girders, the diagrams of Figs. 317 to 325 have been prepared. Inspection of these diagrams shows that for a continuous girder of either two or three equal spans loaded with a

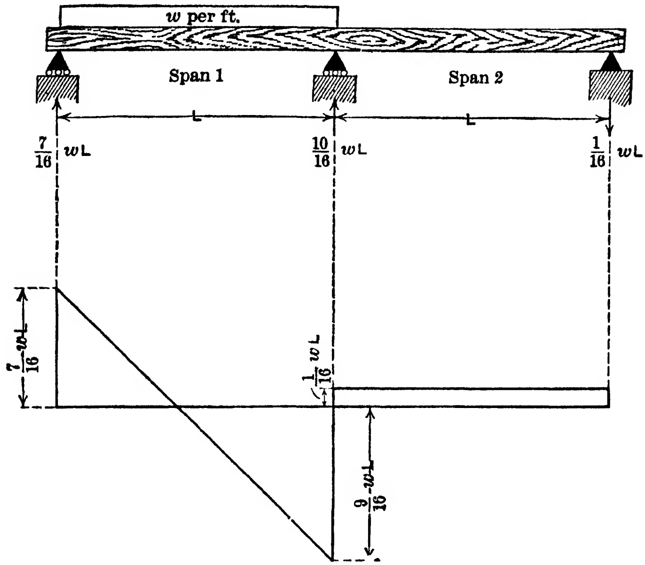


Curve of moments for uniform load $= w$ per ft. over entire structure,
This loading gives maximum negative moment.

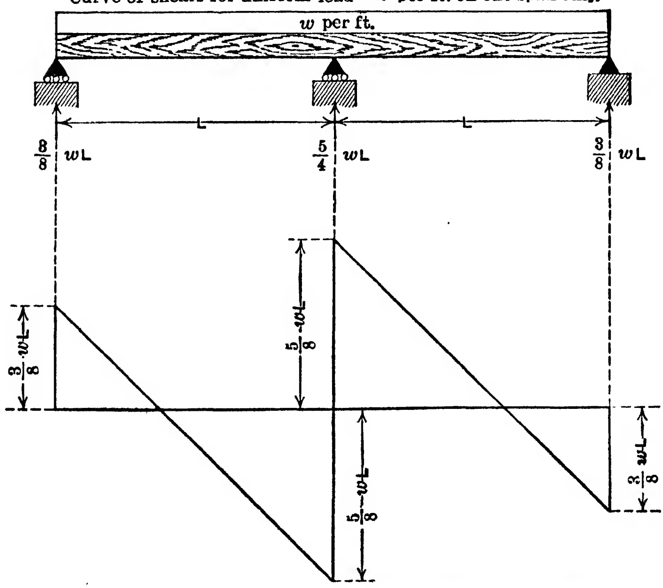


Curve of moments for uniform load $= w$ per ft. on one span only.
This loading gives maximum positive moment.

FIG. 318.—Continuous girder. Curves of moment for uniform load.

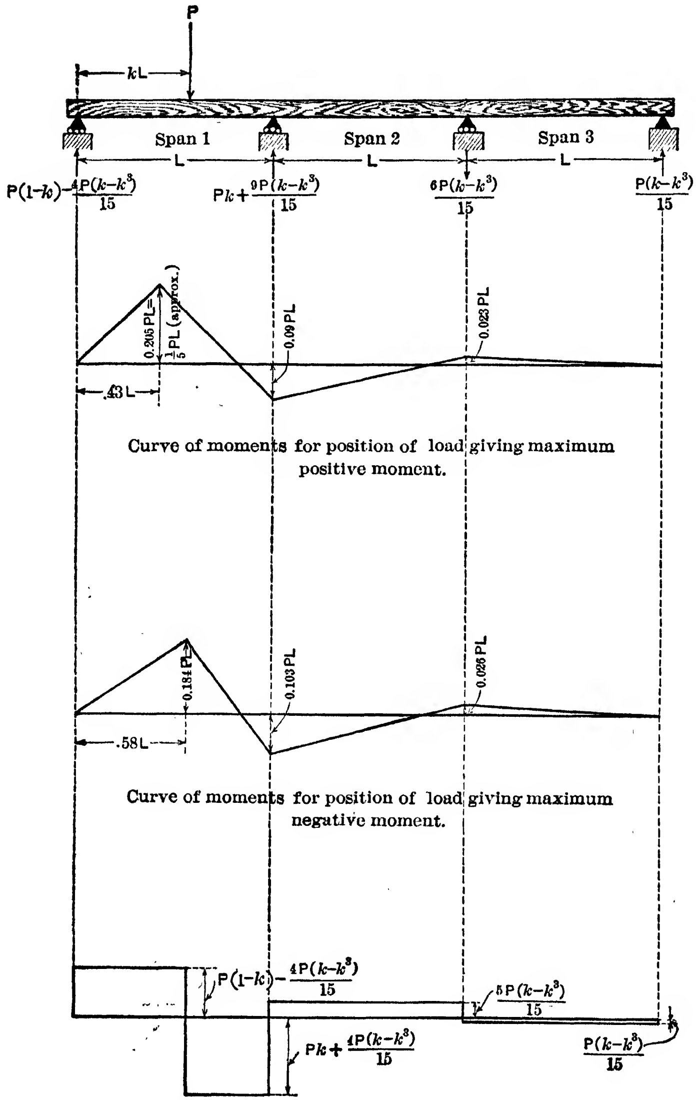


Curve of shears for uniform load $= w$ per ft. on one span only.



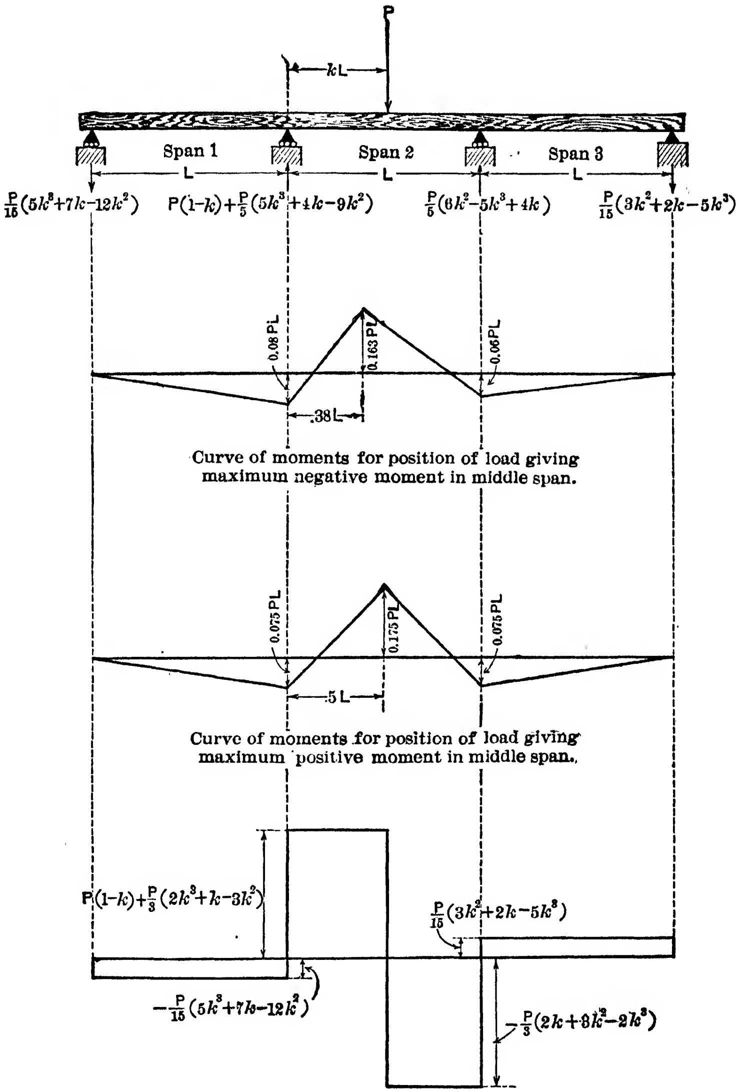
Curve of shears for uniform load $= w$ per ft. over both spans, This loading gives maximum shear.

FIG. 319.—Continuous girder. Curves of shears for uniform load.



Curve of shears for load on side span.
Maximum shear would equal P with load at either end.

FIG. 320.—Continuous girder. Curve of shears and moments. Single concentrated load.



Curve of shears for concentrated load in middle span.
Maximum shear would equal P with load at either end of middle span.

FIG. 321.—Continuous girder. Curves of moments and shears. Single concentrated load.

uniform live load w per foot the maximum live moment occurs at a support and is negative, its value equaling, for the two-span girder, that of the positive live moment on an end-supported span and being slightly less than this for the three-span girder.

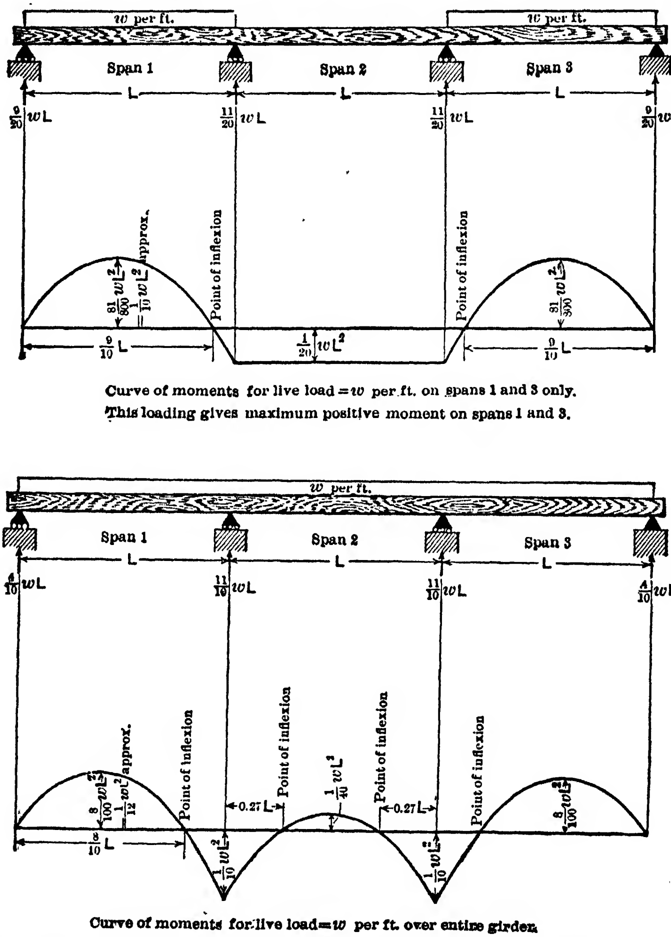
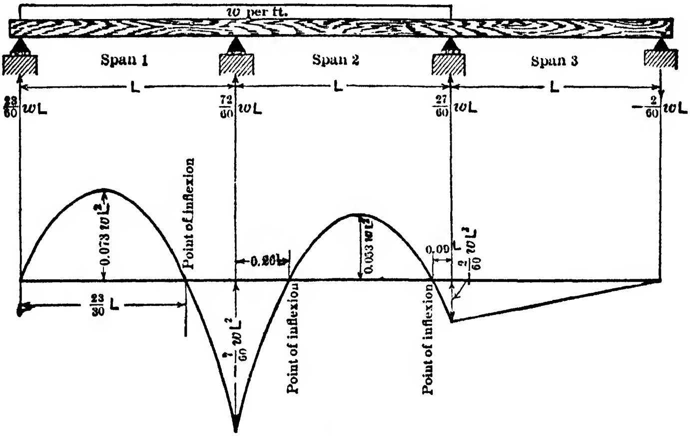


FIG. 322.—Continuous girder. Curves of moments for uniform load.

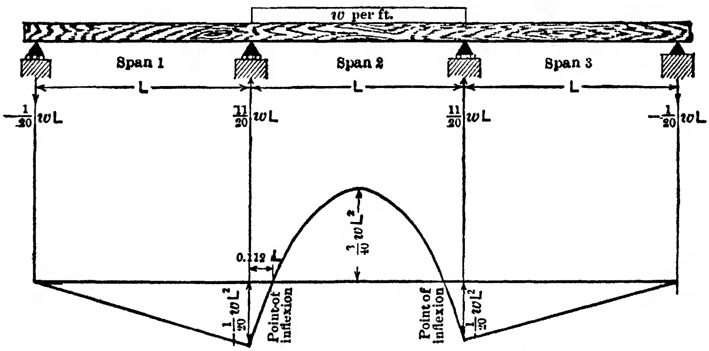
The maximum positive moment equals $\frac{1}{10}wL^2$ for both cases, or about three-quarters of the value it would have for an end-supported span.

Additional data are given for reactions for continuous girders having both two and three spans. For cases where the number

of spans is greater than for those given in the diagrams or for spans of varying lengths, the various functions may be obtained by the student in a manner similar to that used here, or reference



Curve of moments for live load = w per ft. on spans 1 and 2.
This loading gives maximum negative moment on structure.



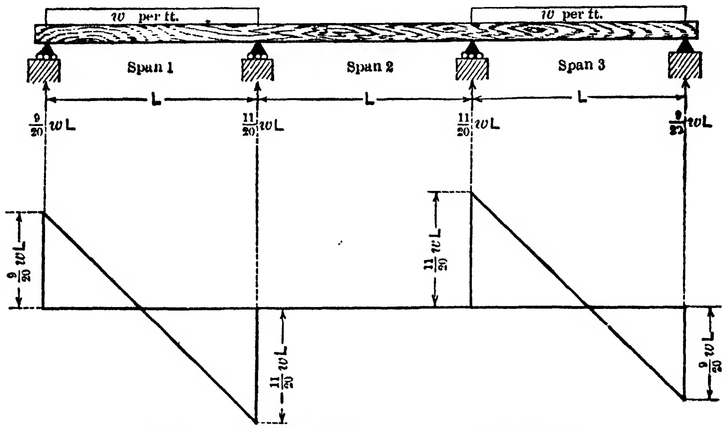
Curve of moments for live load = w per ft. on span 2 only.

FIG. 323.—Continuous girder. Curves of moments for uniform loads.

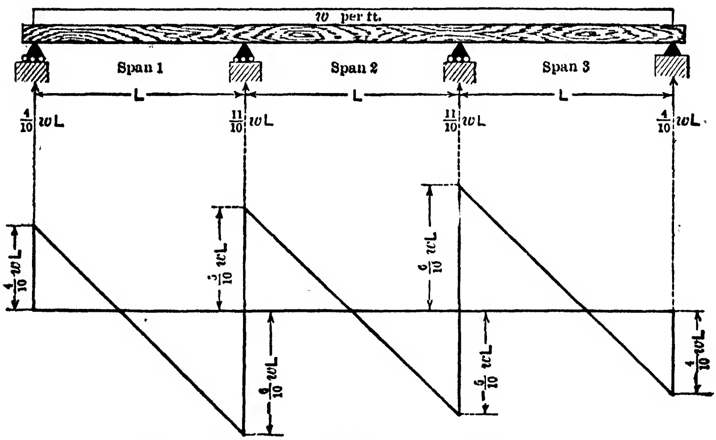
may be made to standard handbooks such as the Carnegie Pocket Companion[1] or to Griot's Kontinuierliche Trager Tabellen.[2] The latter book contains influence tables for moments at

[1] 24th ed., pp. 173 *et seq.*, Carnegie Steel Co., Pittsburgh, Pa. 1934.
[2] Aschmann and Scheller, Zurich, Switzerland, 1916.

sections located at frequent intervals throughout the length of continuous beams having two spans of varying ratios of length and of symmetrical continuous beams having three and four



Curve of shears for uniform load=w per ft. on spans 1 and 3 only.

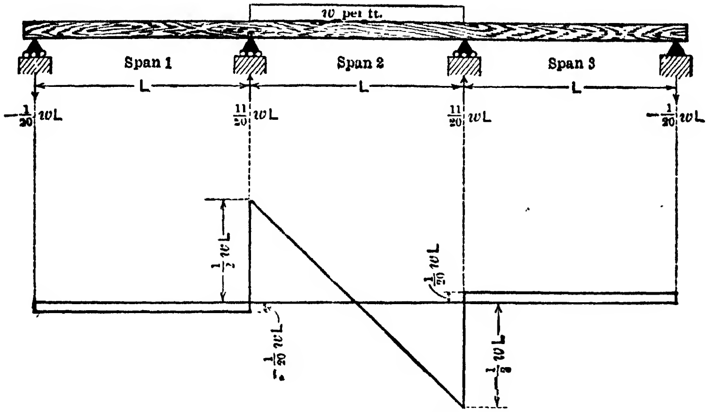


Curve of shears for uniform load=w per ft. over entire structure.

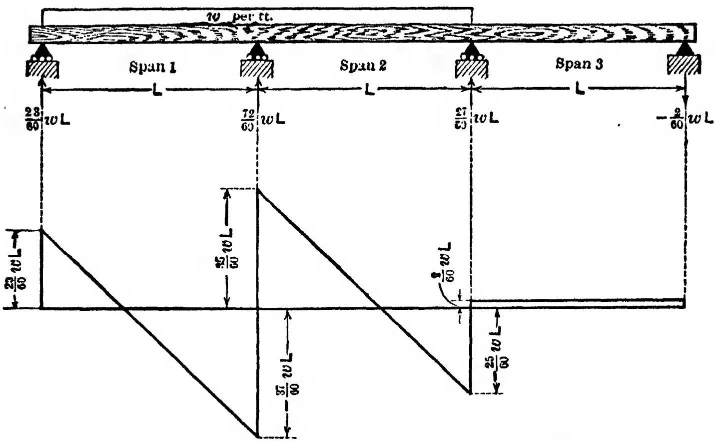
FIG. 324.—Continuous girder. Curves of shears for uniform load.

spans the length of the center span or spans varying from one to two times that of the end span. A series of influence lines for reactions, shear, and moment at frequent intervals throughout

the length of continuous beams of two and three equal spans is also given in Hool and Johnson's The Concrete Engineers' Handbook.[1]



Curve of shears for uniform load$=w$ per ft. on span 2 only.



Curve of shears for uniform load$=w$ per ft. on spans 1 and 2 only.
This loading gives maximum shear in middle and end spans.

FIG. 325.—Continuous girder. Curves of shears for uniform load.

The direction of any reaction on a straight continuous beam resting on level supports, due to a concentrated load on any

[1] McGraw-Hill Book Company, Inc., New York, 1918.

span, may be readily determined by applying the following theorem:

A downward force in any span causes upward reactions at either end of that span, and the other reactions are alternately downward and upward, proceeding from the center to either end of the span.

The truth of this theorem may be readily seen by noting that a beam supported on two adjoining supports a and b and subjected to a vertical downward force between these supports will take the position shown by the dotted line in Fig. 325A. Hence, downward forces will be required at c and d to bring the

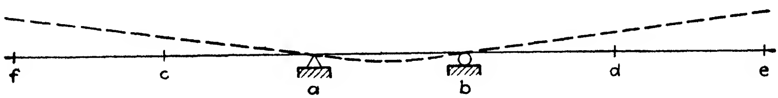


FIG. 325A.

beam back to the level of these supports. Also, if brought down to the levels c and d it will then be deflected below points e and f and will require upward force at these points to bring it back to the level of a and b.

An actual influence line for any reaction can be readily constructed by using a steel spline fastened to a drawing board by steel pins which hold it transversely but not longitudinally. By removing these pins at any one point and raising or lowering the spline at this point and plotting its position, an influence line for the reaction at that point is at once established.

194. Reactions upon Continuous Girders by Method of Deflections.—Reactions upon continuous girders may also

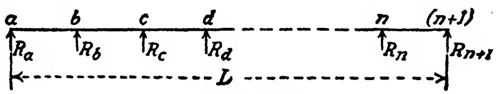


FIG. 326.

by computed by assuming the girder to be supported at the ends only, or at any other two points of support, and equating the deflections at points of application of the other reactions due to the applied loads, with the deflections due to the unknown reactions.

Consider a continuous beam such as that shown in **Fig. 326** with $n + 1$ supports and reactions numbered as shown.

This structure is statically indeterminate to the $(n - 1)$ degree; *i.e.*, there are $(n - 1)$ reactions which cannot be determined by statics. The necessary equations for their solution may be obtained by placing the expression for the deflection at each intermediate support equal to zero.

Let M_0 = moment at any section due to applied loads on the assumption of a beam supported at ends only. Let m_b, m_c, m_d, etc. = numerical coefficient of moment at same section due to upward forces of 1 lb. at b, c, d, etc., on the assumption of a beam supported at ends only. Note that these values of m_b, etc., will all be negative and that M_0 will be positive for downward loads.

It follows that

$$M = M_0 - m_bR_b - m_cR_c \cdots -m_nR_n$$

Let δ_b, δ_c, etc. = upward deflection of points b, c, etc.

By applying Eq. (29), $(n - 1)$ independent equations, of which the following are typical, may now be written and the unknown reactions determined by their solution.

Therefore,

$$\delta_b = 0 = \int\left(\frac{M_0 - m_bR_b - m_cR_c - m_dR_d \cdots -m_nR_n}{EI}\right) \times (-m_b)dx \quad (55)^*$$

$$\delta_c = 0 = \int\left(\frac{M_0 - m_bR_b - m_cR_c - m_dR_d \cdots -m_nR_n}{EI}\right) \times (-m_c)dx \quad (56)^*$$

$$\delta_n = 0 = \int\left(\frac{M_0 - m_bR_b - m_cR_c - m_dR_d \cdots -m_nR_n}{EI}\right) \times (-m_n)dx \quad (57)^*$$

The foregoing equations are identical with the similar equations derived later for this case by the method of least work. They are not necessary for solving a numerical example since a solution can readily be obtained by writing expressions for the deflections at the various points of support separately for the load and for each reaction, the beam being considered as end supported, and then combining their values.

* Note that $\delta M/\delta R_b = -m_b$; $\delta M/\delta R_c = -m_c$, etc.; hence,

$$\int \frac{M}{EI} \cdot \frac{\delta M}{\delta R_b} = \int \frac{M}{EI} \cdot \frac{\delta M}{\delta R_c} = \int \frac{M}{EI} \cdot \frac{\delta M}{\delta R_d} = 0$$

It should be observed that this method can be readily applied to girders where it is desired to allow for the settlement of any support, by placing the deflection at the support in question equal to the assumed deflection instead of zero. It is also possible to apply this method to a girder of varying cross section, in which case the integration of any portion of the girder in which the moment of inertia is not constant must be divided into successive steps, each covering a portion of the girder in which the cross section is constant. This method is approximate to the same degree as the three-moment equation in that the distortion due to shear is neglected; the resulting error is, however, very small.

The vertical deflection at any point q located a distance k_qL from left support of an end-supported beam of constant moment of inertia I and span L due to a load P located at a distance k_pL from same support is given by the following equation when k_q is greater than k_p:

$$\delta_{yq} = \frac{Pk_pL^3}{6EI}(k_q{}^3 - 3k_q{}^2 + 2k_q + k_qk_p{}^2 - k_p{}^2)$$

The deflection when k_p is greater than k_q can be obtained by interchanging the suffixes.[1]

195. Application of the Method of Deflections to Determination of Girder Reactions.—The application of this method is illustrated by the following example:

Problem: Determine by the method of deflections the reactions on the beam shown in Fig. 327, due to the applied load P. E and I are constant through the length of beam.

Solution: Application of Eq. (55) gives

$$EI\delta_b = -\int_0^5 (\tfrac{5}{6}Px)(\tfrac{2}{3}x)dx - \int_5^{10} [\tfrac{5}{6}Px - P(x-5)](\tfrac{2}{3}x)dx - \int_0^{20} (\tfrac{1}{6}Px)(\tfrac{1}{3}x)dx$$

$$+\int_0^{10} R_b(\tfrac{2}{3}x)^2dx + \int_0^{20} R_b(\tfrac{1}{3}x)^2dx$$

$$+\int_0^{10} R_c(\tfrac{2}{3}x)(\tfrac{1}{3}x)dx + \int_{10}^{20} R_c(\tfrac{1}{3}x)[\tfrac{2}{3}x - (x-10)]dx$$

$$+\int_0^{10} R_c(\tfrac{1}{3}x)(\tfrac{2}{3}x)(dx) = 0$$

[1] See Theory of Continuous Structures and Arches, Spofford, McGraw-Hill Book Company, Inc., New York, 1937, Chap. VII.

Application of Eq. (56) gives

$$EI\delta_c = -\int_0^5 (\tfrac{5}{6}Px(\tfrac{1}{3}x)dx - \int_5^{20} [\tfrac{5}{6}Px - P(x - 5)]\tfrac{1}{3}xdx$$

$$-\int_0^{10} [\tfrac{1}{6}Px(\tfrac{2}{3}x)dx] + \int_0^{10} R_b(\tfrac{2}{3}x)(\tfrac{1}{3}x)dx$$

$$+\int_{10}^{20} (R_b)[\tfrac{2}{3}x - (x - 10)](\tfrac{1}{3}x)dx + \int_0^{10} (R_b)(x/3)(\tfrac{2}{3}x)dx$$

$$+\int_0^{10} R_c(\tfrac{2}{3}x)^2dx + \int_0^{20} R_c(\tfrac{1}{3}x)^2dx = 0$$

Performing the indicated integrations and inserting the various limits give the following expressions:

$$-19P + 32R_b + 28R_c = 0$$
$$-31P + 56R_b + 64R_c = 0$$

Solving gives the following values:

$$R_b = \tfrac{29}{40}P, \qquad R_c = -\tfrac{9}{40}P$$

which values agree with those given by the three-moment equation (see Fig. 320).

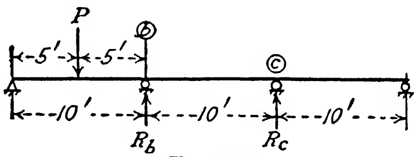


FIG. 327.

The deflection for this case can be found more easily from the equation on the previous page provided that the cross-sectional areas and other properties of the various members are known.

196. Statically Indeterminate Trusses.—The correct stresses in statically indeterminate trusses may be readily computed by either the method of deflections or the method of least work. The method of deflection in the case of a girder has already been described, and its application to trusses follows. The method of least work is given in Arts. 199 *et seq.* Mechanical devices such as the moment indicator described by Ruge and Schmidt in the *Proceedings* of the American Society of Civil Engineers for October, 1938, or the Beggs apparatus mentioned in Art. 185 are also sometimes used in verifying the unknown quantities. In all indeterminate trusses, the cross-sectional areas of the members must be known before the correct stresses in them can be determined; hence, the design must be made by a series of

approximations. In case the truss is a continuous truss and statically indeterminate with respect to outer forces only, the reactions may be approximately determined by the three-moment equation, the structure designed, and the stresses corrected later if it seems desirable by the method of least work or by the method of deflections.

For symmetrical continuous trusses with an odd number of spans and symmetrical loading the shear at center equals zero; hence, for such a condition the web stresses in the center span can be figured by statics if the chord bars are horizontal.

197. Continuous Trusses. *Method of Deflection.*—The application to a continuous truss of Eqs. (55) to (57) requires the determination of the moment of inertia of the truss, which is not readily found and is not constant throughout its length. The method that follows is simpler for such structures.

Let the structure shown in Fig. 326 represent a continuous truss instead of a girder, the truss being statically determinate with respect to the inner forces, and apply to it the general principles explained in Art. 194.

Let S = actual stress in any member of the truss.

S_0 = stress in any member of the truss due to the applied loads, it being assumed that the truss is supported at *ends only*.

S_b, S_c, etc. = stress in any member of the end-supported truss due to application of upward load of unity at points b, c, etc.

A = area of any bar in truss.

It follows that the actual stress in any bar due to applied loads and reactions is given by the following expression:

$$S = S_0 + S_bR_b + S_cR_c \cdot \cdot \cdot + S_nR_n$$

The expressions for upward deflection of points b, c, etc., may now be written as follows:

$$\delta_b = \sum \frac{S_bSL}{AE}, \qquad \delta_c = \sum \frac{S_cSL}{AE}, \text{ etc.}$$

Since the actual deflection of any point of support must be zero unless the point of support settles under the loading, we may now write the following typical equations for the deflection of points b, c, etc.

$$\delta_b = 0 = \sum (S_0 + S_bR_b + S_cR_c \cdots + S_nR_n)\frac{S_bL}{AE} \quad (58)$$

$$\delta_c = 0 = \sum (S_0 + S_bR_b + S_cR_c \cdots + S_nR_n)\frac{S_cL}{AE} \quad (59)$$

$$\delta_n = 0 = \sum (S_0 + S_bR_b + S_cR_c \cdots + S_nR_n)\frac{S_nL}{AE} \quad (60)$$

The foregoing equations agree with the typical equations for the reactions determined by the method of least work derived later. In these equations, any bar in which the stress can be found by statics, as, for example, an end-supported hanger, should be omitted.

Solution of the equations thus obtained enable the unknown reactions to be determined. If it is desired to take account of the settlement of certain of the points of support, the deflection of each of these points should be placed equal to their settlement and not equal to zero.

198. Trusses with Redundant Bars. *Method of Deflection.*—Consider a truss with two redundant members such as that shown in Fig. 328 and loaded in any desired manner. The stresses in the redundant members may be determined by the method of deflections in the following manner:

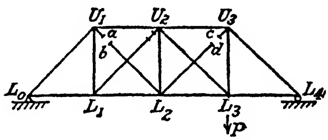


FIG. 328.

a. Assume any two of the redundant members, such as U_1L_2 and U_3L_2, to be cut, and the cut ends *a* and *b* and *c* and *d* in the respective bars to be an infinitesimal distance apart. Let the unknown stress in bar $U_1L_2 = +X$, and the unknown stress in bar $U_3L_2 = +Y$. The stresses in the cut bars may then be replaced by a tension of X acting at both *a* and *b* and a tension of Y acting at both *c* and *d*. Since, with the two redundant members cut, the structure is statically determined, the stresses in all other bars can now be determined by statics in terms of the applied load and the two unknowns X and Y.

Let S = actual stress in any bar.

S_0 = stress in any bar due to any applied load P, X and Y being assumed to equal zero, *i.e.*, bars U_1L_2 and U_3L_2 being assumed to be omitted.

S_b = numerical coefficient of stress in any bar due to tension of unity in bar U_1L_2, *i.e.*, to two forces of unity applied simultaneously at a and b and acting toward each other.

S_c = numerical coefficient of stress in any bar due to tension of unity in bar U_3L_2, *i.e.*, to two forces of unity applied simultaneously at c and d and acting toward each other.

Then $S = S_0 + S_bX + S_cY$.

b. Let δ_a = deflection of point a in direction U_1L_2.
δ_b = deflection of point b in direction L_2U_1.
δ_c = deflection of point c in direction U_3L_2.
δ_d = deflection of point d in direction L_2U_3.

It follows that $\delta_a + \delta_b = 0$ and $\delta_c + \delta_d = 0$.*

Compute the values of the terms $\delta_a + \delta_b$ and $\delta_c + \delta_d$ by the ordinary methods for truss deflection, observing that the values of these combined deflections can be secured by using a tension of unity in the respective bars under consideration instead of two separate loads of unity. To illustrate: In determining the value of δ_a by the ordinary method, the bar stresses are determined due to a load of unity applied at a acting in the direction U_1L_2, and the deflection computed by getting the summation of the products of the stresses due to this load and the changes in bar length, all bars of the truss being considered. Similarly for δ_b the load of unity is taken at b but acting in the opposite direction to that at a. It follows that for $\delta_a + \delta_b$ we have a stress condition corresponding to a tension of unity in the bar U_1L_2.

The resulting expressions for the deflections are as follows:

$$\delta_a + \delta_b = 0 = \sum \frac{(S_0 + S_bX + S_cY)}{AE}(S_bL) \qquad (61)$$

$$\delta_c + \delta_d = 0 = \sum \frac{(S_0 + S_bX + S_cY)}{AE}(S_cL) \qquad (62)$$

Solution of these two equations gives the unknowns X and Y.

* The deflection of a = deflection of b in same direction since these are identical points; but since δ_b is assumed as in opposite direction to δ_a, the expression $\delta_a = -\delta_b$ expresses this relation. Hence, $\delta_a + \delta_b = 0$. A similar reasoning applies to δ_c and δ_d.

The foregoing equations are similar to those found for reactions with the unknown bar stresses substituted for the unknown reactions.

In applying these equations, bars U_1L_0, L_0L_1, U_3L_4, and L_3L_4 should be omitted from the summations since the stresses in these bars are fixed by statics; consequently, they have no influence on the stresses in the other members of the truss. The bars U_1L_2 and U_2L_3 must be included, the stresses in these bars being X and Y respectively.

The foregoing method can be applied to any number of redundant bars.

It will be noted that these equations are identical with those obtained by the method of least work (see Art. 204).

199. Theorem of Least Work.—This theorem was first brought to the attention of engineers by Castigliano.[1] It may be stated as follows:

The internal work done in any stationary structure by the application of outer forces will be the least possible, consistent with equilibrium.

The correctness of this theorem for members subject either to bending or to direct stress, *i.e.*, all cases occurring in the design of girders and trusses, is shown by the identity of the equations derived later by the method of least work with those already deduced by the method of deflections.

The theorem may be readily used to determine the reactions or bar stresses in statically indeterminate structures, provided that the cross sections of the various members have previously been approximately determined by other methods or can be assumed within reasonable limits of precision. As usual in a statically indeterminate structure the final values of the stresses must be obtained by revising the first approximate design one or more times.

The method of application of this theorem is as follows:

Write an expression for the total work in the structure in terms of as many independent unknowns as exist in excess of the number that can be determined by the equations of statics, differentiate the expression with respect to each of these unknowns, place each of these partial derivatives equal to zero, and solve the resulting equations to determine the value of each

[1] See References on p. 453.

unknown. For example, in an end-supported planar structure having five independent reaction components, any three of the components can be expressed in terms of the other two by application of the equations of equilibrium; it therefore follows that any two of these reaction components may be considered as independent variables in terms of which the work in the structure may be expressed and with respect to each of which the expression for work should be differentiated.

200. Expressions for Internal Work.—The application of the theorem of least work involves, in general, expressions for internal work in members due both to direct stress and to flexure, and such expressions follow:

Case 1. Bar subjected to direct axial stress only (no bending).

Let S = total direct axial stress, lb., applied to the bar.
A = area of cross section of bar, sq. in.
E = modulus of elasticity, lb. per sq. in.
L = length of bar, ft.
W = total work in bar, ft.-lb., due to force S.
δ = change in length of bar due to application of force S.

From mechanics, it is known that the total internal work done in a bar by a gradually applied force equals one-half the product of the force by the change in length of the bar if the stress does not exceed the elastic limit and that practically all forces to which structures are subject may be considered to be gradually applied.

It is also known that

$$\delta = \frac{SL}{AE}$$

Hence,

$$W = \frac{1}{2}\frac{S^2L}{AE} \tag{63}$$

Case 2. Bar subjected to bending only.

We may determine the expression for this case in the following manner, referring to Fig. 286, Art. 178, but assuming the depth and width of the bar to be variable in order to obtain a general solution.

Let δ and f_2 be the same as in Art. 178.

w_0 = internal work in a prism of length dx, depth dy, and width b with its center at a distance y from the neutral axis

of the bar, this work being due to the application of outer forces.

I = moment of inertia, in., about the neutral axis with respect to flexure, of the cross section of the bar, at any section such as fg.

M = bending moment in bar, in.-lb., at the same cross section, due to gradually applied loads.

W = total internal work in the bar, in.-lb., due to applied loads.

Then

$$\delta = \frac{f_2 dx}{E}$$

and

$$w_0 = \frac{1}{2}(f_2 b dy)\left(f_2 \frac{dx}{E}\right)$$

But

$$f_2 = \frac{My}{I}$$

Therefore,

$$w_0 = \frac{1}{2} f_2{}^2 \frac{bdydx}{E} = \frac{1}{2} \frac{M^2 y^2}{I^2} \cdot \frac{bdydx}{E}$$

Therefore,

$$W = \int\int \frac{bM^2y^2dydx}{2EI^2}$$

But

$$\int y^2 b dy = I$$

Therefore,

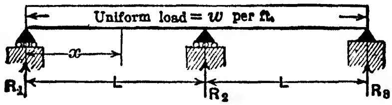
$$W = \int \frac{M^2 dx}{2EI} \tag{64}$$

The expression for the total work in a bar subjected to both direct stress and bending may evidently be obtained by combining Eqs. (63) and (64).

It should be noted that in the expression for work due to bending the work resulting from the distortion due to shear is neglected. The effect of this, however, is not large and the stress due to it may be classed with secondary stresses which may usually be ignored in design.

201. Examples of Applications of Theorem of Least Work.—The following example illustrates clearly the application of the theorem to a simple case.

Problem: Determine by the theorem of least work the reactions on the beam shown in Fig. 329.



Moment of inertia of beam $= I$.
Modulus of elasticity $= E$.

FIG. 329.

Solution: For this case, there are four unknown reactions and three statical equations; hence, only one unknown need be obtained by the theorem of least work. Let this unknown be taken as R_1. By statics,

$$R_3 = R_1$$
$$R_2 = 2wL - 2R_1$$

The internal work may now be expressed in terms of R_1. For the case under consideration the total work in the beam will be double that in the left span.
Hence,

$$W = \int_0^L \frac{M^2dx}{EI} \qquad \text{and} \qquad \frac{\delta W}{\delta R_1} = \int_0^L \frac{2Mdx}{EI} \cdot \frac{\delta M}{\delta R_1}$$

But

$$M = R_1x - \frac{wx^2}{2}, \qquad \text{and} \qquad \frac{\delta M}{\delta R_1} = x$$

Hence,

$$\frac{\delta W}{\delta R_1} = 0 = \frac{2}{EI}\int_0^L \left(R_1x - \frac{wx^2}{2}\right)xdx$$

Integrating and solving the preceding equation give

$$\frac{R_1L_3}{3} - \frac{wL_4}{8} = 0 \qquad \text{and} \qquad R_1 = \frac{3wL}{8}$$

202. Reactions in Continuous Girders. Method of Least Work.—General expressions for the reactions in continuous girders may be readily developed by the method of least work as follows:

Consider the continuous girder shown in Fig. 326.

Let W = total work in the girder.

M = moment at any section of the girder.

Then, from equations of least work,

$$\frac{\delta W}{\delta R_b} = \int \frac{M dx}{EI} \cdot \frac{\delta M}{\delta R_b} = 0, \qquad \frac{\delta W}{\delta R_c} = \int \frac{M dx}{EI} \cdot \frac{\delta M}{\delta R_c} = 0,$$

$$\frac{\delta W}{\delta R_n} = \int \frac{M dx}{EI} \cdot \frac{\delta M}{\delta R_n} = 0$$

Now let M_0, m_b, m_c, etc., have the same value as in Art. 194. Then $M = M_0 - m_b R_b - m_c R_c \cdot \cdot \cdot - m_n R_n$. Therefore,

$$\frac{\delta M}{\delta R_b} = -m_b, \qquad \frac{\delta M}{\delta R_c} = -m_c, \text{ etc.}$$

Hence,

$$\frac{\delta W}{\delta R_b} = 0 = \int \left(\frac{M_0 - m_b R_b - m_c R_c \cdot \cdot \cdot - m_n R_n}{EI} \right) (-m_b) dx$$

$$\frac{\delta W}{\delta R_c} = 0 = \int \left(\frac{M_0 - m_b R_b - m_c R_c \cdot \cdot \cdot - m_n R_n}{EI} \right) (-m_c) dx$$

$$\frac{\delta W}{\delta R_n} = 0 = \int \left(\frac{M_0 - m_b R_b - m_c R_c \cdot \cdot \cdot - m_n R_n}{EI} \right) (-m_n) dx$$

The foregoing equations are the same as Eqs. (55) to (57), showing the identity of the method of least work with the method of deflection for the case of a girder.

203. Reactions in Continuous Trusses. Method of Least Work.—The reactions in continuous trusses, made up of bars the

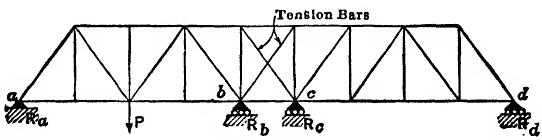


FIG. 330.

cross sections of which have been previously determined by approximate methods, such as the three-moment equation, may be readily determined with accuracy by the method of least work. To illustrate the method, equations for the typical continuous truss shown in Fig. 330 will be developed in this article. This truss is indeterminate to the second degree, and the two central reactions R_b and R_c will be taken as the independent variables entering into the expression for work.

Now, consider the truss as a simple end-supported structure subjected to the applied load P and the two unknown forces R_b and R_c; under these conditions and considering only one of center diagonals in action for each loading:

Let S_0 = stress in any member due to applied load P.

S_b = numerical coefficient of stress in any member due to an upward force of unity acting at b.

S_c = numerical coefficient of stress in any member due to an upward force of unity acting at c.

Then,

R_bS_b = stress in any member due to the unknown upward reaction R_b

and

R_cS_c = stress in any member due to the unknown upward reaction R_c

Hence, the actual stress in any member of the truss when considered as continuous $= S = S_0 + R_bS_b + R_cS_c$.

The work in each bar $= S^2L/2AE$; hence, the work in the entire truss equals $W = \sum \frac{S^2L}{2AE}$.

Also,

$$\frac{\delta W}{\delta R_b} = \sum \frac{SL}{AE} \cdot \frac{\delta S}{\delta R_b} = 0, \qquad \frac{\delta W}{\delta R_c} = \sum \frac{SL}{AE} \cdot \frac{\delta S}{\delta R_c} = 0$$

The values of the foregoing expressions for any given truss can readily be obtained and the unknowns determined at once.

Equations (69) to (72), the development of which follows, may be used in place of the foregoing expressions by those who lack facility in the application of the calculus.

Substituting for S its value, we obtain

$$W = \sum \frac{(S_0 + R_bS_b + R_cS_c)^2L}{2AE}$$

Hence,

$$\frac{\delta W}{\delta R_b} = \sum (S_0 + R_bS_b + R_cS_c)\frac{S_bL}{AE} = 0^*$$

* Note that in problems of this character the work is ordinarily considerably simplified by differentiating before performing the summation or integration.

and

$$\frac{\delta W}{\delta R_c} = \sum (S_0 + R_b S_b + R_c S_c)\frac{S_c L}{AE} = 0$$

These equations are similar to Eqs. (58), (59), and (60), the identity of the method of least work and the method of deflection being thus shown.

These equations may be expressed thus:

$$\sum \frac{S_0 S_b L}{AE} + R_b \sum \frac{S_b{}^2 L}{AE} + R_c \sum \frac{S_b S_c L}{AE} = 0 \qquad (65)$$

and

$$\sum \frac{S_0 S_c L}{AE} + R_b \sum \frac{S_b S_c L}{AE} + R_c \sum \frac{S_c{}^2 L}{AE} = 0 \qquad (66)$$

The solution of these equations gives the following values for the two unknowns:

$$R_b = \frac{\left(\sum \frac{S_0 S_b L}{AE}\right)\left(\sum \frac{S_c{}^2 L}{AE}\right) - \left(\sum \frac{S_0 S_c L}{AE}\right)\left(\sum \frac{S_b S_c L}{AE}\right)}{\left(\sum \frac{S_b S_c L}{AE}\right)^2 - \left(\sum \frac{S_c{}^2 L}{AE}\right)\left(\sum \frac{S_b{}^2 L}{AE}\right)} \qquad (67)$$

$$R_c = \frac{\left(\sum \frac{S_0 S_c L}{AE}\right)\left(\sum \frac{S_b{}^2 L}{AE}\right) - \left(\sum \frac{S_0 S_b L}{AE}\right)\left(\sum \frac{S_b S_c L}{AE}\right)}{\left(\sum \frac{S_b S_c L}{AE}\right)^2 - \left(\sum \frac{S_c{}^2 L}{AE}\right)\left(\sum \frac{S_b{}^2 L}{AE}\right)} \qquad (68)$$

E is usually constant and may be canceled from the previous equations, giving the following equations:

$$R_b = \frac{\left(\sum \frac{S_0 S_b L}{A}\right)\left(\sum \frac{S_c{}^2 L}{A}\right) - \left(\sum \frac{S_0 S_c L}{A}\right)\left(\sum \frac{S_b S_c L}{A}\right)}{\left(\sum \frac{S_b S_c L}{A}\right)^2 - \left(\sum \frac{S_c{}^2 L}{A}\right)\left(\sum \frac{S_b{}^2 L}{A}\right)}$$

$$R_c = \frac{\left(\sum \frac{S_0 S_c L}{A}\right)\left(\sum \frac{S_b{}^2 L}{A}\right) - \left(\sum \frac{S_0 S_b L}{A}\right)\left(\frac{S_b S_c L}{A}\right)}{\left(\sum \frac{S_b S_c L}{A}\right)^2 - \left(\sum \frac{S_c{}^2 L}{A}\right)\left(\sum \frac{S_b{}^2 L}{A}\right)}$$

The denominators of the two preceding reaction equations are alike and are independent of the position of the applied

load. They also contain factors that occur in the numerators and are likewise independent of the applied load; hence, these equations may be written in the following simple forms:

$$R_b = \frac{m\sum\frac{S_0S_bL}{A} - n\sum\frac{S_0S_cL}{A}}{n^2 - mp} \tag{69}$$

$$R_c = \frac{p\sum\frac{S_0S_cL}{A} - n\sum\frac{S_0S_bL}{A}}{n^2 - mp} \tag{70}$$

in which n, m, and p are constants for any particular structure, their values being as follows:

$$n = \sum\frac{S_bS_cL}{A}, \qquad m = \sum\frac{S_c{}^2L}{A}, \qquad p = \sum\frac{S_b{}^2L}{A}$$

It should be particularly noted that bars in which both S_b and S_c are zero may be entirely ignored in applying these equations. This evidently includes each bar, the actual stress in which may be determined by statics.

If there are more than four supports, additional equations may be obtained in a similar manner. For example, if the number of unknown reaction components is $n + 2$, the n equations necessary for solution may be derived from the equation of least work by expressing the stress in each member in terms of the n unknowns and differentiating n times, placing each derivative equal to zero.

If there is but one intermediate support, the equation for R_b may be obtained by placing the last term in Eq. (65) equal to zero and solving the resulting expression, giving the following result:

$$R_b = -\frac{\sum\frac{S_0S_bL}{AE}}{\sum\frac{S_b{}^2L}{AE}} \tag{71}$$

which for a constant value of E gives

$$R_b = -\frac{\sum\frac{S_0S_bL}{A}}{\sum\frac{S_b{}^2L}{A}} \tag{72}$$

As before, the denominator is a constant that is independent of the applied load, and any bar in which S_b equals zero may be ignored in the computation; this includes each bar the stress in which may be computed by statics.

The equations deduced by this method give more accurate values than the three-moment equation which is based upon a constant moment of inertia whereas the method of least work takes due account of the variation in the moment of inertia due to the shape of the truss and the areas of the members. On the other hand, the application of this method requires the predetermination of the areas of all the members. A proper mode of procedure for such trusses is, therefore, to make a preliminary design based upon the determination of the reactions by the application of the three-moment equation and then to check this by the determination of the reactions by the more accurate method of least work. As will be shown later, a preliminary determination of the stresses in a statically indeterminate structure may also be made by the method of least work by assuming a constant value of L/A for all bars.

For the application of the equations, a table should be prepared in which the various constants for each bar should be placed in vertical columns so that the summations may be obtained by adding the columns. Such a table should have the following headings for the case where E is constant and where there are two independent variables; *i.e.*, provided that the formulas are used.

Bar	L, ft.	A, sq. in.	S_0, lb.	S_b	S_c	$\frac{L}{A}$	$\frac{S_0S_bL}{A}$	$\frac{S_0S_cL}{A}$	$\frac{S_bS_cL}{A} = n$	$\frac{S_b^2L}{A} = p$	$\frac{S_c^2L}{A} = m$
U_0L_1		Note that bars in which both S_b and S_0 are zero, may be omitted									
U_1L_2											
Etc.											
Summations											

In obtaining the summations, due attention should be given to the signs of the various terms; *e.g.*, if, for any bar, S_0 is tension and S_b is compression, the value of their product is negative.

To determine the maximum value of any one of the reactions or bar stresses, influence lines may be drawn or influence tables

constructed from which the position of the load for maximum values may readily be determined.

The application of the method on the previous page to the solution of an actual case is illustrated by numerical examples in the following articles.

204. Stresses in Trusses with Redundant Members by Method of Least Work.—The method used for reactions may be applied equally well to trusses that are statically indeterminate

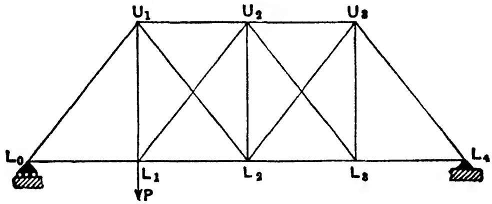


FIG. 331.

with respect to the inner forces (*i.e.*, to trusses with redundant members) by substituting for the unknown reactions the unknown stresses in the redundant members. The equations for such cases may be derived as follows:

Consider the truss shown in Fig. 331 which has two redundant members. Assume these two redundant members to be U_1L_2 and L_2U_3, although any other two superfluous members could be used equally well.

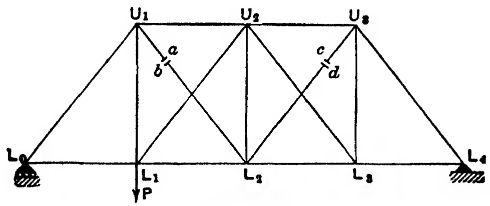


FIG. 332.

Now consider the same truss with bars U_1L_2 and L_2U_3 cut, such a truss being shown in Fig. 332, the cut ends a and b and c and d of the respective bars being an infinitesimal distance apart. For this latter truss,

Let S_0 = stress in any bar due to applied load P.

S_b = stress in any bar due to two forces of unity applied simultaneously at a and b in direction U_1L_2 and acting toward each other.

S_c = stress in any bar due to two forces of unity applied simultaneously at c and d in direction L_2U_3 and acting toward each other.

For the truss in Fig. 331 (the actual truss),

Let X = stress (assumed as tension) in redundant member U_1L_2.

Y = stress (assumed as tension) in redundant member L_2U_3.

S = actual stress in any member.

W = total work in structure.

Then,

$$W = \sum \frac{S^2L}{2AE}$$

Also,

$$\frac{\delta W}{\delta X} = \sum \frac{SL}{AE} \cdot \frac{\delta S}{\delta X}$$

$$\frac{\delta W}{\delta Y} = \sum \frac{SL}{AE} \cdot \frac{\delta S}{\delta Y}$$

The values of the foregoing terms can be determined directly, as is done in one of the following illustrative examples, or the equations of which the derivations follow may be used.

In these summations the work in the cut bars should be included with that in the other bars.

Evidently,

$$S = S_0 + S_bX + S_cY$$

Hence,

$$W = \sum \frac{(S_0 + S_bX + S_cY)^2L}{2AE}$$

Differentiating the previous equation with respect to the two variables X and Y gives the following results:

$$\frac{\delta W}{\delta X} = \sum \left(\frac{S_0 + S_bX + S_cY}{AE}\right) S_bL = 0$$

$$\frac{\delta W}{\delta Y} = \sum \left(\frac{S_0 + S_bX + S_cY}{AE}\right) S_cL = 0$$

Whence

$$\sum \frac{S_0 S_b L}{AE} + X \sum \frac{S_b{}^2 L}{AE} + Y \sum \frac{S_b S_c L}{AE} = 0 \tag{73}$$

and

$$\sum \frac{S_0 S_c L}{AE} + X \sum \frac{S_b S_c L}{AE} + Y \sum \frac{S_c{}^2 L}{AE} = 0 \tag{74}$$

The only difference between these equations and Eqs. (65) and (66) lies in the substitution of X for R_b and Y for R_c; hence, the values for X and Y obtained by solving these equations will be identical in form with the values for R_b and R_c as given by Eqs. (67) and (68), *viz.:*

$$X = \frac{\left(\sum \frac{S_0 S_b L}{AE}\right)\left(\sum \frac{S_c{}^2 L}{AE}\right) - \left(\sum \frac{S_0 S_c L}{AE}\right)\left(\sum \frac{S_b S_c L}{AE}\right)}{\left(\sum \frac{S_b S_c L}{AE}\right)^2 - \left(\sum \frac{S_c{}^2 L}{AE}\right)\left(\sum \frac{S_b{}^2 L}{AE}\right)} \tag{75}$$

$$Y = \frac{\left(\sum \frac{S_0 S_c L}{AE}\right)\left(\sum \frac{S_b{}^2 L}{AE}\right) - \left(\sum \frac{S_0 S_b L}{AE}\right)\left(\sum \frac{S_b S_c L}{AE}\right)}{\left(\sum \frac{S_b S_c L}{AE}\right)^2 - \left(\sum \frac{S_c{}^2 L}{AE}\right)\left(\sum \frac{S_b{}^2 L}{AE}\right)} \tag{76}$$

If the value of E is constant in all cases, it may and should be canceled for all terms, the labor of computation thus being greatly reduced and it being made possible to write the equations in the same form as Eqs. (69) and (70), *viz.:*

$$X = \frac{m \sum \frac{S_0 S_b L}{A} - n \sum \frac{S_0 S_c L}{A}}{n^2 - mp} \tag{77}$$

$$Y = \frac{p \sum \frac{S_0 S_c L}{A} - n \sum \frac{S_0 S_b L}{A}}{n^2 - mp} \tag{78}$$

in which n, m, and p have the values previously given.

The application of the foregoing formulas should be made by means of a table, as previously shown. Bars in which both S_b and $S_c = 0$ should be omitted; this includes in the truss shown by

Fig. 331 all bars such as L_0U_1, L_0L_1, etc., the stresses in which can be computed by statics.

To determine the position of the live load for maximum stress in any member, influence lines or influence tables may be used.

Equations for a truss with more than two redundant members may be derived in a similar manner as previously suggested in the article dealing with reactions.

For the case of a truss with one redundant member, the value of the stress in this member may be obtained by placing $y = 0$ in Eq. (73), giving the following result:

$$X = -\frac{\sum \frac{S_0 S_b L}{AE}}{\sum \frac{S_b^2 L}{AE}} \tag{79}$$

which takes the following form for a structure in which E is constant:

$$X = -\frac{\sum \frac{S_0 S_b L}{A}}{\sum \frac{S_b^2 L}{A}} \tag{80}$$

The illustrative problems that follow show solutions for trusses with redundant members made both with and without the use of the detailed equations.

Problem: Compute the stresses in the redundant members of the truss shown by Fig. 333, due to the loads shown. Gross areas of various members

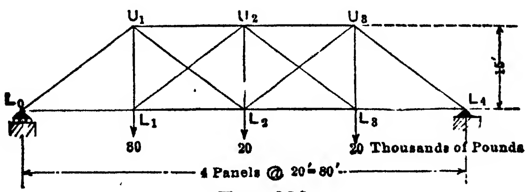


FIG. 333.

are given in the table accompanying the solution. The modulus of elasticity is assumed to be constant throughout.

Solution: This truss is evidently indeterminate to the second degree; bars L_1U_2 and U_2L_3 will be considered the redundant members, although any two bars except L_0U_1, L_0L_1, U_3L_4, and L_3L_4 might be used. S_b is the stress caused by a force of unity in bar L_1U_2, and S_c the stress caused by a force of unity in bar U_2L_3. X = stress in L_1U_2; Y = stress in U_2L_3.

The following table contains all the necessary values, and the equations beneath it give the final results.

$$W = \frac{S^2L}{2AE}; \quad \frac{\delta W}{\delta X} = \sum \frac{SL}{AE}\frac{\delta S}{\delta X}; \quad \frac{\delta W}{\delta Y} = \sum \frac{SL}{AE}\frac{\delta S}{\delta Y}.$$

Bar	L, ft.	A, sq. in.	S, 1,000-lb. units	$\frac{L}{A}$	$\frac{\delta S}{\delta X}$	$\frac{\delta S}{\delta Y}$	$\frac{SL}{A}\cdot\frac{\delta S}{\delta X}$	$\frac{SL}{A}\cdot\frac{\delta S}{\delta Y}$
U_1U_2	20	12	$-\frac{280}{3} - 0.8X$	$\frac{20}{12}$	-0.8	0.0	$+124.4 + 1.07X$	
U_2U_3	20	12	$-\frac{280}{3} - 0.8Y$	$\frac{20}{12}$	0.0	-0.8	....	$+124.4 + 1.07Y$
L_1L_2	20	8	$+100.0 - 0.8X$	$\frac{20}{8}$	-0.8	0.0	$-200.0 + 1.60X$	
L_2L_3	20	8	$+60.0 - 0.8Y$	$\frac{20}{8}$	0.0	-0.8	....	$-120.0 + 1.60Y$
U_1L_2	25	4	$-8\frac{1}{3} + X$	$\frac{25}{4}$	$+1.0$	0.0	$-52.1 + 6.25X$	
L_2U_3	25	4	$+\frac{125}{3} + Y$	$\frac{25}{4}$	0.0	$+1.0$	....	$+260.4 + 6.25Y$
U_1L_1	15	6	$+80.0 - 0.6X$	$\frac{15}{6}$	-0.6	0.0	$-120.0 + 0.90X$	
U_3L_3	15	6	$+20.0 - 0.6Y$	$\frac{15}{6}$	0.0	-0.6	....	$-30.0 + 0.90Y$
L_1U_2	25	4	$0.0 + 1.0X$	$\frac{25}{4}$	$+1.0$	0.0	$+6.25X$	
U_2L_3	25	4	$0.0 + 1.0Y$	$\frac{25}{4}$	0.0	$+1.0$	....	$+6.25Y$
U_2L_2	15	4	$0.0 - 0.6X - 0.6Y$	$\frac{15}{4}$	-0.6	-0.6	$1.35X + 1.35Y$	$+1.35X + 1.35Y$

Summations: $\sum \frac{SL\delta S}{A\delta X} = -247.7 + 17.42X + 1.35Y = 0,$

$$\sum \frac{SL\delta S}{A\delta Y} = 234.8 + 17.42Y + 1.35X = 0,$$

Therefore $1.35(-247.7 + 17.42X + 1.35Y) = 0,$
and $17.42(234.8 + 1.35X + 17.42X) = 0.$

Solving the foregoing equations results in the following values:

$$X = +15.3$$
$$Y = -14.7$$

It should be observed that the stress in any bar may be expressed by a constant C plus terms in X and Y; hence, we may write $S = C + S_bX + S_cY$.

It follows that $\delta S/\delta X = S_b$ and $\delta S/\delta Y = S_c$; hence, $\sum \frac{SS_bL}{AE} = \sum \frac{SS_cL}{AE} = 0$ which equations are identical with those obtained by method of deflections as given in Art. 197.

The solution of the same truss by the application of Eqs. (77) and (78) follows and illustrates the use of the equations. The final results are of course the same as those previously found.

Bar	L, ft.	A, sq. in.	S_0, 1,000-lb. units	S_b	S_c	$\frac{S_0S_bL}{A}$	$\frac{S_0S_cL}{A}$	$\frac{S_bS_cL}{A}$	$\frac{S_b^2L}{A}$	$\frac{S_c^2L}{A}$
U_1U_2..........	20	12	− 93.3	−0.8	0.0	+124.4	0.0	0.0	+1.1	0.0
U_2U_3.	20	12	− 93.3	0.0	−0.8	0.0	+124.4	0.0	0.0	+1.1
L_1L_2.	20	8	+100.0	−0.8	0.0	−200.0	0.0	0.0	+1.6	0.0
L_2L_3...	20	8	+ 60.0	0.0	−0.8	0.0	−120.0	0.0	0.0	+1.6
U_1L_2..........	25	4	− 8.3	+1.0	0.0	− 52.1	0.0	0.0	+6.2	0.0
L_2U_3	25	4	+ 41.7	0.0	+1.0	0.0	+260.4	0.0	0.0	+6.2
U_1L_1..........	15	6	+ 80.0	−0.6	0.0	−120.0	0.0	0.0	+0.9	0.0
U_3L_3	15	6	+ 20.0	0.0	−0.6	0.0	− 30.0	0.0	0.0	+0.9
L_1U_2	25	4	0.0	+1.0	0.0	0.0	0.0	0.0	+6.2	0.0
U_2L_3..........	25	4	0.0	0.0	+1.0	0.0	0.0	0.0	0.0	+6.2
U_2L_2..........	15	4	0.0	−0.6	−0.6	0.0	0.0	+1.3	+1.4	+1.4
Summations..	..	..	....	.....	.....	−247.7	+234.8	+1.3	+17.4	+17.4

$$X = \text{stress in } L_1U_2 = \frac{-247.7 \times 17.4 - 234.8 \times 1.3}{1.3^2 - 17.4^2} = 15.3$$

$$Y = \text{stress in } U_2L_3 = \frac{+234.8 \times 17.4 + 247.7 \times 1.3}{-301} = -14.6$$

With X and Y determined, stresses in all other bars may be readily computed.

Problem: Compute stress in redundant member U_3U_4 of the truss shown by Fig. 334, due to load shown. Areas of all members are given in table

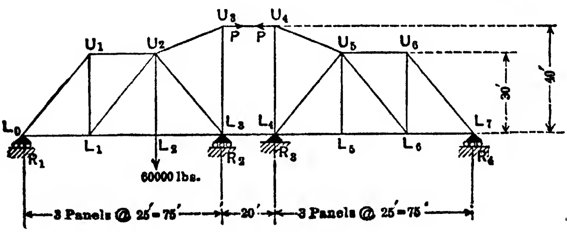


FIG. 334.

accompanying the solution and are selected arbitrarily as factors of bar lengths to save numerical work, the truss being a hypothetical one. E is constant for all members.

Solution: Since the truss has but one redundant member, Eq. (80) is applicable. The following table gives all necessary values; the determination of the stress follows the table.

Bar	L, ft.	A, sq. in. gross	S_0 1,000-lb. units	S_b	$\frac{L}{A}$	$\frac{S_0S_bL}{A}$*	$\frac{S_b^2L}{A}$
L_0U_1	39	13.0	−26.0	+0.69	3.0	− 53.8	+ 1.43
U_2L_3	39	13.0	−52.0	−0.17	3.0	+ 26.5	+ 0.09
L_0L_1	25	10.0	+16.7	−0.44	2.5	− 18.5	+ 0.48
L_1L_2	25	10.0	+33.3	−0.89	2.5	− 74.2	+ 1.98
L_2L_3	25	10.0	+33.3	−0.89	2.5	− 74.2	+ 1.98
U_1L_1	30	10.0	+20.0	−0.53	3.0	− 31.8	+ 0.85
L_1U_2	39	13.0	−26.0	+0.69	3.0	− 53.8	+ 1.43
U_1U_2	25	12.5	−16.7	+0.44	2.0	− 14.7	+ 0.39
U_2U_3	27	9.0	......	+1.08	3.0	....	+ 3.50
U_3L_3	40	8.0	......	−0.40	5.0	....	+ 0.80
Σ for half truss neglecting bars U_3U_4 and L_3L_4	..	....	......	......	...	....	+12.93
Σ for entire truss neglecting bars U_3U_4 and L_3L_4	..	....	......	......	...	....	+25.86
L_3L_4	20	8.0	......	−1.00	2.5	....	+ 2.50
U_3U_4	20	10.0	......	+1.00	2.0	....	+ 2.00
Σ for entire truss	..	....	......	......	...	−294.5	+30.36

* S_0 or S_b equals zero for all bars except those for which numerical values are given; hence, S_0S_bL/A = zero for same bars.

$$X = \text{stress in } U_3U_4 = +\frac{294.5}{30.36} = +9.7$$

The reaction can now be readily found, *e.g.*, the value $R_4 = -9.7 \times {}^{40}\!/_{75}$; and the other bar stress computed.

It is important to note that since L/A appears in each term of numerator and denominator of the fraction giving the stress in a redundant member or reaction, a slight approximation in the value of A would not affect the final result greatly. In the illustrative examples, it makes no serious difference in the final result if L/A should be taken as constant in all members; *e.g.*, in the truss of Fig. 333 the values obtained for X and Y, if L/A is taken as constant, differ but little from the values previously obtained, as is shown by the following solution, and this method may be used for approximate computation.

Table giving value of summations for trusses shown in Fig. 333 on the assumption that L/A = unity. Other terms are the same as in the previous solution.

Bar	$\frac{A}{SL}\frac{ds}{dX}$	$\frac{A}{SL}\frac{ds}{dY}$
U_1U_2	$+74.7 + 0.64X$	
U_2U_3		$74.7 + 0.64Y$
L_1L_2	$-80.0 + 0.64X$	
L_2L_3		$-48.0 + 0.64Y$
U_1L_2	$-8.3 + 1.00X$	
L_2U_3		$+41.7 + 1.00Y$
U_1L_1	$-48.0 + 0.36X$	
U_3L_3		$-12.0 + 0.36Y$
L_1U_2	$+1.00X$	
U_2L_3		$+1.00Y$
U_2L_2	$+0.36X + 0.36Y$	$+0.36Y + 0.36X$
Summation	$-61.6 + 4X \quad + 0.36Y$	$+56.4 + 4Y \quad + 0.36X$

Placing the equations above equal to zero and solving give approximate values of X and Y, *viz.*: $X = 16.8$ and $Y = -15.6$. These values are nearly the same as the values given by the more exact solution. If the last term, which is small, is neglected in each equation, the values of the unknown are $x = 15.4$ and $y = 14.1$, which are very close to the correct values.

If L/A is taken as unity for each member of the truss shown in Fig. 334, the stress in U_3U_4 will be found to be 10.0 instead of 9.7 as found by using correct value of L/A.

205. Influence Lines and Tables for Indeterminate Structures.—The construction of influence lines or tables for reactions or bar stresses for indeterminate trusses is comparatively simple since S_0 is the only variable for any particular truss. Considering, for example, the first set of computations for truss shown in Fig. 334, it is evident that in obtaining data for an influence line, S_0 should first be computed for a load of unity at any convenient point, and the products in the last column but one of the table computed for that value. If we now let S_0' = bar stress due to load unity at any other panel point, the value of the stress X in U_3U_4 for this latter position can be determined by multiplying each separate value in the previously mentioned column by S_0'/S_0 for that particular member, a new summation being

obtained for that column and the other summations being used without modification.

206. Stresses in Indeterminate Structures Due to Changes of Temperature.—To determine bar stresses or reactions due to changes of temperature in any or all members of a statically indeterminate structure, the equations previously deduced may be used, the values for S_0 being not those due to an applied load but instead the forces required to change the lengths of the various bars an amount equal to the change due to variations in temperature. For example, if for the truss shown by Fig. 334 it is desired to compute the stress in bar U_3U_4 due to an increase in temperature of the entire top chord, it is necessary to use for S_0 for each separate bar of the top chord the axial force which would cause a change in length equal to that due to the change in temperature.

In considering the effect of a change of temperature in statically indeterminate trusses, it should be noted that a uniform change in temperature of the entire truss, if the truss reactions are vertical and all at the same level, will not cause stresses in any member but that a similar change will cause material stresses in many if not most of the bars of a structure such as an arch which is rigidly restrained against horizontal movement. It should also be observed that a change in temperature of some but not all bars of a statically indeterminate truss with vertical reactions will also stress many members.

207. Stresses in Framed Bents by Method of Least Work.—Application of the method of least work to framework bents of the portal type is clearly illustrated by the following problems.

Problem: Solve by the method of least work the framed bent shown in Fig. 335.

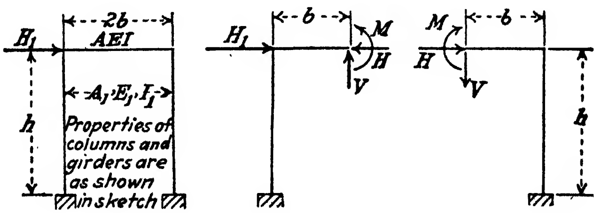


FIG. 335.

Solution: This structure is statically indeterminate to the third degree; *i.e.*, there are three unknowns, the values of which cannot be determined

by statics. Let these three unknowns be taken as the unknown moment, shear, and direct stress acting at center of girder as shown.

Let M_L = moment in left column due to bending.
M_R = moment in right column due to bending.
m_L = moment in left half of girder due to flexure.
m_R = moment in right half of girder due to flexure.
W = total work in structure.
x = horizontal distance from center of girder, or vertical distance down from top of column.

$$W = \int_0^h \frac{M_L^2dx}{2E_1I_1} + \int_0^h \frac{M_R^2dx}{2E_1I_1} + \int_0^b \frac{m_L^2dx}{2EI} + \int_0^b \frac{m_R^2dx}{2EI} + 2\left(\frac{V^2h}{2A_1E_1}\right) + 2\left(\frac{H^2b}{2AE}\right)$$

Let

$$\frac{E_1I_1}{EI} = C, \qquad \frac{2hI_1}{A_1} = C_1, \qquad \frac{2bE_1I_1}{AE} = C_2$$

Substituting these values and differentiating give the following partial derivatives:

$$E_1I_1\frac{\partial W}{\partial M} = \int_0^h \frac{M_L\partial M_Ldx}{\partial M} + \int_0^h \frac{M_R\partial M_Rdx}{\partial M} + \int_0^b \frac{Cm_L\partial m_Ldx}{\partial M} + \int_0^h \frac{Cm_R\partial m_Rdx}{\partial M}$$

$$E_1I_1\frac{\partial W}{\partial V} = \int_0^h \frac{M_L\partial M_Ldx}{\partial V} + \int_0^h \frac{M_R\partial M_Rdx}{\partial V} + \int_0^b \frac{Cm_L\partial m_Ldx}{\partial V} + \int_0^b \frac{Cm_R\partial m_Rdx}{\partial V} + C_1V$$

$$E_1I_1\frac{\partial W}{\partial H} = \int_0^h \frac{M_L\partial M_Ldx}{\partial H} + \int_0^h \frac{M\partial_R M_Rdx}{\partial H} + \int_0^b \frac{Cm_L\partial m_Ldx}{\partial H} + \int_0^b \frac{Cm_R\partial m_Rdx}{\partial H} + C_2H$$

But

$$M_L = M + Vb + (H - H_1)x \qquad \text{and} \qquad M_R = M - Vb + Hx$$

Therefore,

$$\frac{\partial M_L}{\partial M} = 1, \quad \frac{\partial M_L}{\partial V} = b, \quad \frac{\partial M_L}{\partial H} = x, \quad \frac{\partial M_R}{\partial M} = 1, \quad \frac{\partial M_R}{\partial V} = -b, \quad \frac{\partial M_R}{\partial H} = x$$

Also,

$$m_L = M + Vx, \qquad \text{and} \qquad m_R = M - Vx$$

Therefore,

$$\frac{\partial m_L}{\partial M} = 1, \quad \frac{\partial m_L}{\partial V} = x, \quad \frac{\partial m_L}{\partial H} = 0, \quad \frac{\partial m_R}{\partial M} = 1, \quad \frac{\partial m_R}{\partial V} = -x, \quad \frac{\partial m_R}{\partial H} = 0$$

Substituting the values of the partial derivatives in the preceding equations gives

$$E_1I_1\frac{\partial W}{\partial H} = \int_0^h\Big[M + Vb + (H - H_1)x\Big]xdx + \int_0^h\Big[M - Vb + Hx\Big]xdx + 0 + 0 + C_2H = 0$$

$$= \frac{2Mh^2}{2} + \frac{2Hh^3}{3} - \frac{H_1h^3}{3} + C_2H = 0$$

Hence,

$$3Mh^2 + 2Hh^3 - H_1h^3 + 3C_2H = 0$$

$$E_1I_1\frac{\partial W}{\partial V} = \int_0^h\Big[M + Vb + (H - H_1)x\Big]bdx - \int_0^h\Big[M - Vb + Hx\Big]bdx + C\int_0^b(M + Vx)xdx - C\int_0^b(M - Vx)xdx + C_1V$$

$$= 2Vb^2h - \frac{H_1h^2b}{2} + \frac{2CVb^3}{3} + C_1V = 0$$

Hence,

$$12Vb^2h - 3H_1h^2b + 4CVb^3 + 6C_1V = 0$$

$$E_1I_1\frac{\partial W}{\partial M} = \int_0^h\Big[M + Vb + (H - H_1)x\Big]dx + \int_0^h\Big[M - Vb + Hx)\Big]dx + C\int_0^b(M + Vx)dx + C\int_0^b(M - Vx)dx$$

$$= 2Mh + \frac{2Hh^2}{2} - \frac{H_1h^2}{2} + 2CMb = 0$$

Hence,

$$4Mh + 2Hh^2 - H_1h^2 + 4CMb = 0$$

Solving the equations thus obtained gives the following values for the unknowns:

$$V = \frac{3H_1h^2b}{2(6b^2h + 2Cb^3 + 3C_1)}$$

$$H = \frac{H_1h^4 + 4CH_1bh^3}{2(h^4 + 4Cbh^3 + 6C_2h + 6CC_2b)}$$

$$M = \frac{3C_2H_1h^2}{2(h^4 + 4Cbh^3 + 6C_2h + 6CC_2b)}$$

If the effect of direct work is neglected, then $C_1V = 0$, and $C_2H = 0$; therefore, all terms in C_1 and C_2 disappear from all equations, and we obtain the following values:

$$V = \frac{3H_1h^2}{2(6bh + 2Cb^2)}$$

$$H = \frac{H_1}{2}$$

$$M = 0$$

The relative values of the expressions with and without the effect of direct work can be readily computed for a definite case to determine the effect of neglecting the work due to direct stress. For example, consider a one-story two-column steel bent having the following properties: height, 20 ft.; span, 15 ft.; girder, one 18-in. 93-lb. Bethlehem girder beam; each column, one 8-in. 32-lb. Bethlehem beam.

The values for the various unknowns resulting from the application of a horizontal force of H_1 to the top of the bent are as follows:

Including direct stress, $V = 0.6602H_1$; $H = 0.4998H_1$; $M = 0.00138H_1$.
Neglecting direct stress, $V = 0.6667H_1$; $H = 0.5000H_1$; $M = 0.0000H_1$.

In view of the extremely small influence of the work due to direct stress, the terms representing direct stress may evidently be ignored for the case in question and are commonly neglected in all cases.

208. Slope-deflection Theorem.[1]—The so-called *slope-deflection theorem* may sometimes be used to advantage in determining

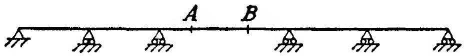


FIG. 336.

the stresses in certain forms of statically indeterminate structures, particularly in framed bents of the portal type employed in high-building construction. The theorem is based upon the relation between the change of slope of a beam axis, the bending moment to which the beam is subjected, and the deflection of one point on the beam axis as compared with another point due to settlement of one of the supports from some influence outside that of the bending of the beam itself.

Before proceeding to the development of the slope-deflection equation, the conditions existing in the portion of a continuous beam between any two sections of any span will be considered.

Let the beam shown in Fig. 336 be a continuous beam with an indefinite number of spans, and let AB represent a section of this beam in any span and of any length. Assume for the present that the beam is loaded in such a manner that the moment and shear at A each equals zero and that no load is applied between A and B.

[1] The application of this theorem is set forth at length in a paper published in *Univ. Illinois, Eng. Expt. Sta., Bull.* 108, by Wilson, Richart, and Weiss, who state that the principle was first developed by Mohr in 1868.

From the relation between bending moment and shear, the loading above evidently causes a bending moment of zero in all portions of the beam between A and B, and hence the axis of this portion of the beam will not be changed in curvature by the assumed loading; if originally straight, it will remain straight. The *position* of this portion of the beam axis will, however, not necessarily remain unchanged since both the deflection at A and the deflection at B are functions of the bending moment in the entire beam as indicated in the development of Eq. (29), Art. 178, the continuous beam being subject to the same laws as an end-supported beam if the intermediate reactions are substituted for some of the loads.

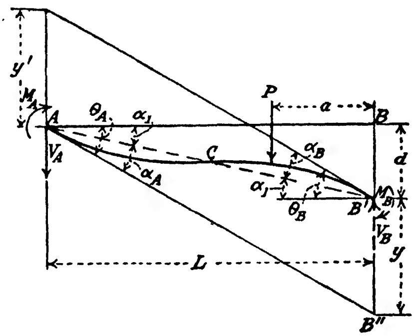


FIG. 337.

It is evident from the foregoing that the portion AB of the beam is in exactly the same condition under the assumed loading as a simple beam supported at points A and B, although not necessarily with A and B at same level.

From the fundamental equation of mechanics for the elastic curve of any beam axis, *viz.*, $d^2y/dx^2 = M/EI$, it is evident that the shape of the beam axis between any two points is a direct function of the moment applied to the beam between these points, that is, if we neglect, as is usual, the distortion due to shear. Hence, if additional loads are applied to the beam either between A and B or elsewhere so as to cause moment in the portion AB, the shape of the axis between A and B will be changed and this change will be due entirely to the moment acting upon the portion of the beam between A and B; it follows that the expression deduced in Art. 186 for slope of an end-supported

beam will be applicable to any portion such as AB of any beam, whether end-supported, continuous, or with fixed ends.

Now consider the portion of any beam having E and I constant throughout lying between two points A and B originally at the same elevation as shown in Fig. 337, and let d represent deflection of one end with respect to the other due to conditions external to the beam itself; *e.g.*, if point B is supported on one of the columns of a structural framework, d represents, for the case shown, the shortening of that column due to forces applied not only to the beam between A and B, but also to any other portion of the framework. It should be observed that d is an elastic deflection of the given structure and hence has a very small value compared with the length of span L; hence, the expressions derived in this article apply only to cases where d is very small compared with L.

Let d at either end be assumed to be positive when it corresponds to a clockwise rotation of the beam about the other end.

Let ACB' represent the beam axis after application of an assumed bending moment to portion AB of the beam.

Let the external forces acting on beam be M_A and V_A acting at A, both assumed to act as shown in Fig. 337, and the load P acting at any point between A and B. These forces cause reactions and moments at B which are designated as M_B and V_B and are shown dotted.

It is immaterial which way these forces are assumed to act. For convenience, however, the moments are considered clockwise at both ends; a negative value for either of them will show that the moments in question is counterclockwise.

Let α_A = angle, radians, through which tangent to axis of beam at A rotates due to application of assumed moment to portion of beam between A and B.

α_B = angle, radians, between tangent to axis of stressed beam at B and axis AB.

α_1 = angle, radians, through which beam axis rotates due to deflection d of B.

θ_A = angle, radians, made by tangent to axis of stressed beam at A with axis AB.

θ_B = angle, radians, between tangent to axis of stressed beam at B and axis BA.

Then, $\theta_A = \alpha_A + \alpha_1$.

Since both d and y are extremely small compared with L, we may now write

$$y + d = L\theta_A, \qquad y = L\alpha_A, \qquad d = L\alpha_1$$

The moment areas due to the forces M_a, V_A, and P are respectively, as follows:

$$M_aL, \qquad -\frac{V_AL^2}{2}, \qquad \frac{Pa^2}{2}$$

The horizontal distances x_b of their centroids from B are, respectively, as follows:

$$\frac{L}{2}, \qquad \frac{L}{3}, \qquad \frac{a}{3}$$

Using these values in Eq. (32) for angular movement of axis at A due to moment applied to beam between A and B, *viz.*, $\alpha_A = Fx_b/LEI$, gives

$$\alpha_A = \frac{1}{EI}\left(\frac{M_AL^2}{2L} - \frac{V_AL^3}{6L} - \frac{Pa^3}{6L}\right)$$

Now,

$$y = (L\alpha_A) = \frac{1}{EI}\left(\frac{M_AL^2}{2} - \frac{V_AL^3}{6} - \frac{Pa^3}{6}\right) \tag{81}$$

Also, from Eq. (36),

$$(\alpha_A - \alpha_B) = \frac{1}{EI}\left(M_AL - \frac{V_AL^2}{2} - \frac{Pa^2}{2}\right) \tag{82}$$

Substituting value of $-\frac{V_AL^2}{2}$ from Eq. (82) in Eq. (81) gives

$$EI(L\alpha_A) = \frac{M_AL^2}{2} + \frac{LEI(\alpha_A - \alpha_B)}{3} - \frac{M_AL^2}{3} + \frac{Pa^2L}{6} - \frac{Pa^3}{6}$$

Therefore,

$$\frac{M_AL^2}{6} = LEI\left(\alpha_A - \frac{\alpha_A - \alpha_B}{3}\right) - \frac{Pa^2}{6}(L - a)$$

whence

$$M_A = \frac{6EI}{L}\left(\frac{2}{3}\alpha_A + \frac{\alpha_B}{3}\right) - \frac{Pa^2}{L^2}(L - a)$$
$$= \frac{2EI}{L}(2\alpha_A + \alpha_B) - \frac{Pa^2}{L^2}(L - a)$$

But

$$\alpha_A = \theta_A - \alpha_1, \qquad \alpha_B = \theta_B - \alpha_1, \qquad \text{and} \qquad \alpha_1 = \frac{d}{L}$$

Therefore,

$$2\alpha_A + \alpha_B = 2\theta_A + \theta_B - 3\alpha_1$$

Hence,

$$M_A = \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3d}{L}\right) - \frac{Pa^2}{L^2}(L - a) \tag{83}$$

A similar expression may be obtained for the moment M_B at the end B of the same beam as follows:
For this case

$$-\alpha_B = \frac{Fx_a}{LEI}$$

Therefore,

$$EIL(\alpha_B) = \frac{M_BL^2}{2} - \frac{V_BL^3}{6} + \frac{P(L - a)^3}{6}$$

Also,

$$EI(\alpha_A - \alpha_B) = -M_BL + \frac{V_BL^2}{2} - \frac{P(L - a)^2}{2}$$

Therefore,

$$-\frac{V_BL^3}{6} = -\frac{EIL}{3}(\alpha_A - \alpha_B) - \frac{M_BL^2}{3} - \frac{P(L - a)^2L}{6}$$

Therefore,

$$EIL\alpha_B = \frac{M_BL^2}{2} - \frac{EIL}{3}(\alpha_A - \alpha_B) - \frac{M_BL^2}{3} - \frac{P(L - a)^2a}{6}$$

Therefore,

$$\frac{M_BL^2}{6} = \frac{2}{3}EIL\alpha_B + \frac{EIL}{3}\alpha_A + \frac{P(L - a)^2a}{6}$$

Hence,

$$M_B = \frac{4EI}{L}\alpha_B + \frac{2EI\alpha_A}{L} + \frac{P(L-a)^2a}{L^2}$$
$$= \frac{2EI}{L}(2\alpha_B + \alpha_A) + \frac{P(L-a)^2a}{L^2}$$

But

$$2\alpha_B + \alpha_A = 2\theta_B + \theta_A - 3\alpha_1$$

Therefore,

$$M_B = \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3d}{L}\right) + \frac{P(L-a)^2a}{L} \qquad (84)$$

If a clockwise moment M is applied to the beam instead of the force P, the last term in Eq. (83) becomes $-\frac{Ma(3a-2L)}{L^2}$ and the last term in Eq. (84) becomes $+\frac{M(3a-L)(L-a)}{L^2}$.

The last terms in Eqs. (83) and (84) equal, respectively, the moments at A and B in a fixed-ended beam due to load P. For a set of concentrated loads, or for a uniform load, these terms should evidently be the fixed-ended beam moments at A and B, respectively, due to all such loads, since the load P is a typical load and may be placed anywhere; hence, we may express these terms by constants C_A and C_B, giving the following equations in which C_A and C_B equal, respectively, the moments at A and B in a single-span fixed-ended beam on unyielding supports due to all forces applied to the beam.

$$M_A = \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3d}{L}\right) - C_A \qquad (85)$$

$$M_B = \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3d}{L}\right) + C_B \qquad (86)$$

The signs in Eqs. (85) and (86) correspond to clockwise moments, and slopes at A and B, clockwise moments at ends of fixed-ended beams, and downward deflection of B with respect to A.

A negative value for any of the unknown terms in the application of the equation indicates the opposite direction to that assumed.

It is common in practice to substitute K for I/L and $3R$ for $3d/L$, equations in the following form being thus produced:

$$M_A = 2EK(2\theta_A + \theta_B - 3R) - C_A \tag{87}$$
$$M_B = 2EK(2\theta_B + \theta_A - 3R) + C_B \tag{88}$$

For a fixed-ended beam on level supports, θ_A, θ_B, and R each equals zero; hence, $M_A = -C_A$ and $M_B = C_B$ as should be the case.

In case one end of the beam is free to bend, *i.e.*, pin-ended or end-supported, Eqs. (87) and (88) should be modified to correspond. For example, if the end B is free, Eq. (88) becomes

$$M_B = 0 = 2EK(2\theta_B + \theta_A - 3R) + C_B \tag{89}$$

Solving foregoing equation for θ_B and substituting the value thus obtained in Eq. (87) give the following result:

$$M_A = 2EK\left(1.5\theta_A - \frac{3}{2}R\right) - \left(C_A + \frac{C_B}{2}\right) \tag{90}$$

The foregoing equation may, for convenience, be written as follows:

$$M_A = 2EK\left(1.5\theta_A - \frac{3R}{2}\right) - H_A \tag{91}$$

An expression for M_b for the case where M_A is free can be found in a similar manner and is as follows:

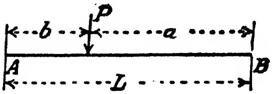
$$M_B = 2EK\left(1.5\theta_B - \frac{3R}{2}\right) + \left(C_B + \frac{C_A}{2}\right) \tag{92}$$

which may be written

$$M_B = 2EK\left(1.5\theta_B - \frac{3R}{2}\right) + H_B \tag{93}$$

The values of C_A and C_B can be found in standard books on mechanics, or they can be determined by the three-moment equation for any given case. For certain frequent cases, these values are as shown by the following table. It should be noted that for any symmetrical loading about center line of beam, $C_A = C_B = F/L$ in which F = moment area for an end-supported beam.

TABLE GIVING NUMERICAL VALUES OF C_A, C_B, H_A, AND H_B



$$C_A = \frac{Pa^2b}{L^2}, \qquad C_B = \frac{Pab^2}{L^2}$$

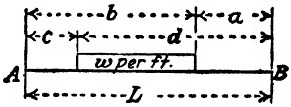
$$H_A = C_A + \frac{C_B}{2} = \frac{Pba}{2L^2}(2a + b)$$

$$H_B = C_B + \frac{C_A}{2} = \frac{Pba}{2L^2}(2b + a)$$

It should be observed that if the value of P is replaced by the moment which would exist at the point of application of P in an end-supported beam of span L, the values above of C_A and C_B equal, respectively, the reactions at A and B due to this hypothetical load.

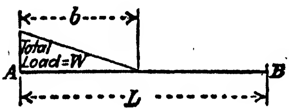
$$C_A = \frac{w}{12L^2}[d^3(4L - 3d) - a^3(4L - 3a)]$$

$$C_B = \frac{w}{12L^2}[b^3(4L - 3b) - c^3(4L - 3c)]$$



$$H_A = C_A + \frac{C_B}{2} = \frac{w}{12L^2}\left[2L(b^3 - 2a^3 - c^3 + 2d^3) - 3\left(\frac{b^4}{2} - a^4 - \frac{c^4}{2} + d^4\right)\right]$$

$$H_B = C_B + \frac{C_A}{2} = \frac{w}{12L^2}\left[2L(2b^3 - a^3 - 2c^3 + d^3) - 3\left(b^4 - \frac{a^4}{2} - c^4 + \frac{d^4}{2}\right)\right]$$



$$C_A = \frac{Wb}{30L^2}(3b^2 - 10bL + 10L^2)$$

$$C_B = \frac{Wb^2}{30L^2}(5L - 3b)$$

$$H_A = C_A + \frac{C_B}{2} = \frac{Wb}{60L^2}(3b^2 - 15bL + 20L^2)$$

$$H_B = C_B + \frac{C_A}{2} = \frac{Wb}{60L^2}(10L^2 - 3b^2)$$

The application of the slope-deflection theorem to a simple case is illustrated by the following problem.

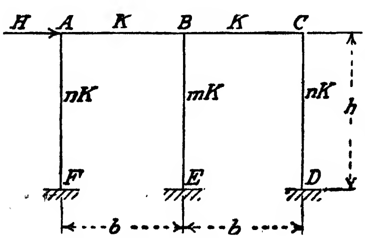


FIG. 338.

Problem: Determine moments and shears in the members of the structure shown in Fig. 338.

Symmetrical structure. All members have same value of E. K of side columns $= nK$ of girders. K of center column $= mK$ of girders. All columns fixed at base. All joints rigid.

Solution: Neglect change of length of girder and columns due to direct stresses, since application of method of least work to structure demonstrates that effect of work due to direct stress as compared with work due to moments is negligible (see Art. 207).

This structure is statically undetermined to the sixth degree. The six unknowns that permit its solution are θ_A, θ_B, θ_C, and $3R$ for each of the three columns, the values of $3R$ all being equal since by hypothesis the girder lengths remain unchanged and hence the deflections of the tops of all columns are identical. (Since change in length of columns is considered equal to zero, the value of $3R$ for each of the top girders is zero.)

Now let M_{AB} = moment at A in girder AB.

M_{AF} = moment at A in column AF.

Assume all moments, slopes and deflections positive, *i.e.*, clockwise.

Also, let θ_{AB} = slope at end A of girder AB.

θ_{AF} = slope at end A of column AF.

Use similar nomenclature for moments and slopes of other members.

Since all joints are rigid by hypothesis, we get

$$\theta_{AB} = \theta_{AF}, \qquad \theta_{BA} = \theta_{BC} = \theta_{BE}, \qquad \theta_{CB} = \theta_{CD}$$

These slopes will hereafter be called

$$\theta_A, \qquad \theta_B \qquad \text{and} \qquad \theta_C$$

Also since columns have fixed ends at bases,

$$\theta_F = \theta_E = \theta_D = 0$$

Applying $\Sigma M = 0$ at each joint at top of columns gives

$$M_{AB} + M_{AF} = 0, \qquad M_{BA} + M_{BC} + M_{BE} = 0, \qquad M_{CB} + M_{CD} = 0$$

Applying $\Sigma M = 0$ to that portion of the columns below a horizontal section passing through the bent at an infinitesimal distance below top of columns at which section the combined shear in all columns $= H$ gives

$$Hh + M_{AF} + M_{BE} + M_{CD} + M_{FA} + M_{EB} + M_{DC} = 0$$

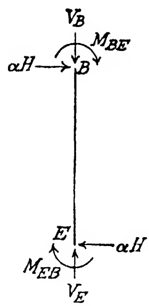


FIG. 339.

The correctness of this equation is evident if we consider the equilibrium of that portion of one of the columns, say BE, below the horizontal section. The forces acting on this portion of the column are shown in Fig. 339, in which αH = portion of shear carried by this particular column. Evidently for this case $\alpha H(h) + M_{BE} + M_{EB} = 0$, the vertical forces causing no moment. Similar expressions may be deduced for each of the columns, and the H terms may all be combined in one term since the sum of the separate terms equals Hh.

Substituting in foregoing moment equations the values of the various moment terms as obtained from the fundamental slope-deflection Eqs. (87) and (88), noting that the constant terms C are zero since no forces are applied to members between ends, gives the following four equations for the solution of the four unknowns θ_A, θ_B, θ_C, and $3R$:

$$M_{AB} + M_{AF} = 0 = 2EK(2\theta_A + \theta_B) + 2EKn(2\theta_A - 3R) \qquad \text{(I)}$$

$$M_{BA} + M_{BC} + M_{BE} = 0 = 2EK(2\theta_B + \theta_A) + 2EK(2\theta_B + \theta_C) + 2mEK(2\theta_B - 3R) \qquad \text{(II)}$$

$$M_{CB} + M_{CD} = 0 = 2EK(2\theta_C + \theta_B) + 2nEK(2\theta_C - 3R) \quad \text{(III)}$$

$$\begin{aligned} M_{AF} + M_{FA} + M_{BE} + M_{EB} + M_{CD} + M_{DC} + Hh = 0 \\ = 2nEK(2\theta_A - 3R) + 2nEK(\theta_A - 3R) + 2mEK(2\theta_B - 3R) \\ + 2mEK(\theta_B - 3R) + 2nEK(2\theta_C - 3R) + 2nEK(\theta_C - 3R) \\ + Hh = 0 \end{aligned} \quad \text{(IV)}$$

The solution of these equations may most readily be accomplished by arranging them in tabular form as shown below, after first dividing through by $2EK$.

Equation	θ_A	θ_B	θ_C	$3R$	Total
I	$2 + 2n$	$+1$	0	$-n$	$= 0$
II	1	$4 + 2m$	1	$-m$	$= 0$
III	0	1	$2 + 2n$	$-n$	$= 0$
IV	$3n$	$3m$	$3n$	$-4n - 2m$	$= -\dfrac{Hh}{2EK}$

It is evident that $\theta_A = \theta_C$ and that the equations above may be readily solved for numerical values of n and m. The solution follows for the case of $n = 1$ and $m = 3$, the various tabular values being obtained by obvious methods.

Equation	θ_A	θ_B	$3R$	Total
I	4	1	-1	$= 0$
II	2	10	-3	$= 0$
IV	6	9	-10	$= -\dfrac{Hh}{2EK}$
I_a	120	30	-30	$= 0$
II_a	20	100	-30	$= 0$
IV_a	18	27	-30	$= -\dfrac{3}{2}\dfrac{Hh}{EK}$
I_a—II_a = V	100	-70	0	$= 0$
I_a—IV_a = VI	102	3	0	$= +\dfrac{3}{2}\dfrac{Hh}{EK}$
V	100×3	-70×3	0	$= 0$
VI	102×70	$+70 \times 3$	0	$= \dfrac{210}{2}\dfrac{Hh}{EK}$
V—VI	$+7,440$			$= \dfrac{210}{2}\dfrac{Hh}{EK}$

Inspection of equations in the foregoing table shows the following relations to exist:

$$\theta_A = \theta_C = \frac{7}{10}\theta_B = \frac{7}{496}\frac{Hh}{EK}, \qquad \theta_B = \frac{10}{496}\frac{Hh}{EK}$$

$$3R = \theta_B + 4\theta_C = \frac{38}{496}\frac{Hh}{EK}$$

Substituting the values of θ_A, θ_B, θ_C, and $3R$ in the fundamental slope-deflection equations (87) and (88) gives the following values of the various moments:

$$\begin{aligned}
M_{AB} &= 2EK(2\theta_A + \theta_B) = +{}^{24}/_{248}Hh = -M_{AF} \\
M_{BA} &= 2EK(2\theta_B + \theta_A) = +{}^{27}/_{248}Hh \\
M_{BC} &= 2EK(2\theta_B + \theta_C) = +{}^{27}/_{248}Hh
\end{aligned} \Bigg\} M_{BA} + M_{BC} = -M_{BE} = {}^{54}/_{248}Hh$$

$$\begin{aligned}
M_{CB} &= 2EK(2\theta_C + \theta_B) = +{}^{24}/_{248}Hh = -M_{CD} \\
M_{FA} &= 2EK(0 + \theta_A - 3R) = -{}^{31}/_{248}Hh \\
M_{EB} &= 2E(3K)(0 + \theta_B - 3R) = -{}^{84}/_{248}Hh \\
M_{DC} &= 2EK(0 + \theta_C - 3R) = -{}^{31}/_{248}Hh
\end{aligned}$$

In the foregoing values, positive signs indicate clockwise moments in accordance with the convention adopted in the development of the slope-deflection equation.

With the values of the moments at the ends of all members known, the moments and shears at any section of any of the members can be readily determined; *e.g.*, consider column EB, shown in Fig. 339 and apply $\Sigma M = 0$. This gives

$$M_{BE} + M_{EB} + \alpha Hh = 0$$

Substituting numerical values gives

$$-({}^{54}/_{248} + {}^{84}/_{248})Hh + \alpha Hh = 0$$

Therefore,

$$\alpha = {}^{138}/_{248}$$

Similarly, α for each side column is found to equal ${}^{55}/_{248}$.

With α known the moment at any section of the column can be readily figured.

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209. Moment Distribution.—Consideration of Eqs. (87) and (88) shows that if a beam is fixed against rotation and vertical deflection at both A and B and if no load is applied to it between these sections, θ_A, θ_B, R, C_A, and C_B each equals zero. Therefore, by hypothesis $M_A = M_B = 0$.

Also, if θ_A, R, C_A, and C_B each equals zero but θ_B has a definite value, *i.e.*, if the beam is only partially restrained against rotation at B, then

$$M_A = 2EK\theta_B \qquad \text{and} \qquad M_B = 4EK\theta_B \tag{94}$$

Hence,

$$M_A = \frac{M_B}{2}$$

Also, if, under the foregoing conditions, end A is merely supported, *i.e.*, if M_A, R, C_A, and C_B each equals zero, then

$$\theta_A = \frac{-\theta_B}{2} \qquad \text{and} \qquad M_B = 3EK\theta_B \tag{95}$$

The term K in the foregoing equations is generally called the *stiffness factor*.

Equations (94) and (95) form the basis of the moment-distribution method developed by Prof. Hardy Cross and are frequently used in the solution of complicated rectangular framed structures.

Equation (94) shows that, if end A of a beam is fixed and a moment M_B is applied at end B sufficient to cause a rotation of θ_B at B, it will develop a moment at A of the same character and one-half the magnitude of the moment applied at B whereas if end A is merely supported, *i.e.*, not restrained against bending, the moment applied at B to cause a rotation there of θ_B will be only three-fourths of that required when end A is fixed.

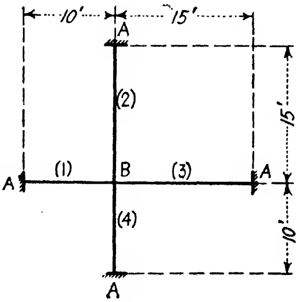


FIG. 340.

It is clear from the foregoing that, if several beams each fixed at end A meet at a point B as illustrated in Fig. 340 and if that point is then rotated through the angle θ by the application of a moment M, the portion of the moment carried by each of the beams at point B will be in accordance with the respective values of EK for the various beams, since M_B for each beam will be given by the expression $M_B = 4EK\theta_B$; also that the moment at end A of each beam will be of the same character and half the magnitude of that carried at end B since M_A for each beam is given by the expression $M_A = 2EK\theta_B$.

It is also evident that, if any one of the beams is supported but not fixed at end A, the moment carried at end B of any such beam will equal three-fourths of that which it would carry if end A were fixed; *i.e.*, L for an end-supported beam may be assumed to equal $\frac{4}{3}L$ for an identical beam with end A fixed.

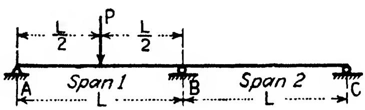


FIG. 341.

210. Application of Moment-distribution Method to Vertical Loads.—This method may be applied by a series of approximations to the determination of the moments in continuous beams or at the ends of all members meeting at a joint of a rigid framed structure and is illustrated by the examples that follow.

Consider first the determination of the moment at B in a simple continuous beam having a constant value of EI, such as the beam shown in Fig. 341.

Assume for the time being that the beam is fixed at A, B, and C, the moments at the supports will then be as follows:[1]

In span 1, At A, $-\frac{pL}{8}$, at B, $+\frac{pL}{8}$

In span 2, At C, 0, at B, 0

Now apply an external moment at $B = -\frac{1}{8}PL$. This moment will be divided equally between spans 1 and 2, giving

$$-\tfrac{1}{16}PL \text{ in span 1}$$
$$-\tfrac{1}{16}PL \text{ in span 2}$$

Each of these moments will cause a moment equal to one-half its value at the other end of the appropriate span; hence, the moment at each point of support will now have the following values:

At A, $-\frac{1}{8}PL - \frac{1}{32}PL = -\frac{5}{32}PL$

At B, 0

At C, $-\frac{1}{32}PL$

Now apply a moment at $A = +\frac{5}{32}PL$ and at $C = +\frac{1}{32}PL$. These moments will cause moments at B as follows:

In span 1, $+\frac{5}{64}PL$

In span 2, $+\frac{1}{64}PL$

The moments at points of support will then have the following values:

At A and C, 0

At B, span 1, $(\frac{1}{8} - \frac{1}{16} + \frac{5}{64})PL = +\frac{9}{64}PL$

At B, span $2(-\frac{1}{16} + \frac{1}{64})PL = -\frac{3}{64}PL$

The foregoing process may now be continued by applying $-\frac{6}{64}PL$ at B, distributing it between the two spans, and then applying end moments to counterbalance the end moments resulting from the distribution, which will give the following results:

Moment at A and C $= 0$

Moment at B, span 1, $= PL[+\frac{9}{64} - \frac{1}{2}(\frac{6}{64}) + \frac{1}{8}(\frac{6}{64})] = +\frac{27}{256}PL$

[1] Signs of moments and slopes to be the same as in the slope-deflection theory, *viz.*, positive when clockwise.

See Art. 208 for moments at ends of fixed-ended beams; also Carnegie Pocket Companion, p. 167, Carnegie Steel Co., Pittsburgh, Pa., 1934; also Continuity in Building Frames, No. St.40 Part II, published by Portland Cement Association, Chicago, Ill., April, 1938.

Moment at B, span 2, $= PL[-3/64 - 1/2(6/64) + 1/8(6/64)] = -21/256 PL$

Total moment at $B = (27/256 - 21/256)PL = +1/4(6/64)PL$

Continuing as before by applying $-1/4(6/64)PL$ at B, distributing and eliminating end moments, gives the following values:

Moment at A and C $= 0$

Moment at B, span 1, $= PL[+27/256 - 1/8(6/64) + 1/32(6/64)]$

$$= \frac{99}{1{,}024}PL$$

Moment at B, span 2, $= PL[-21/256 - 1/8(6/64) + 1/32(6/64)]$

$$= -\frac{93}{1{,}024}PL$$

$$\text{Total moment at } B = \frac{6}{1{,}024}PL = \frac{1}{16}\left(\frac{6}{64}PL\right)$$

Evidently for this particular problem the residual error at each stage is one-fourth that of the previous stage. We can, therefore, place the resulting moment at the center at the end of the next operation as $1/64(6/64)PL$.

The reactions at the end of any stage may now be computed from the moments as follows:

For the third stage,

$$R_A L - \frac{PL}{2} + \frac{99PL}{1{,}024} = 0$$

Therefore,

$$R_A = \frac{P}{2} - \frac{99P}{1{,}024} = 0.403P$$

which agrees very closely with the correct value of $0.406P$ as determined by the three-moment equation.

Had the moments of inertia, or the span lengths, or both, differed in the two spans, the moments at B would not have been equally distributed between the spans, but otherwise the method would be unchanged.

Since the ends are neither of them fixed, it was not necessary to assume span 2 fixed at C; this would have affected the distribution between the two spans of the moment applied at B, which in the case of equal value of I and L would have been distributed

in the ratio of 4 to 3, *i.e.*, 4/7 to span 1 and 3/7 to span 2. In this particular case the arithmetical operations would not be simplified by this assumption, but they would be in some cases.

This method is not recommended as a method of solving continuous girders of constant cross section, but the example furnishes a simple illustration of the moment-distribution method of solving the slope-deflection equations by a series of approximations instead of by solving a number of simultaneous equations.

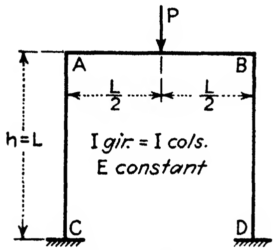


FIG. 342.

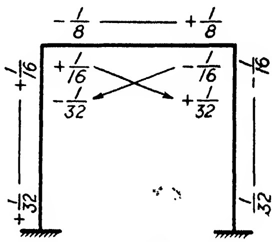


FIG. 343.

The application of the foregoing method to the solution of the simple one-story bent shown in Fig. 342 may be carried out as follows:

Assume the girder to be fixed at the ends.

Then

$$M_A = -\frac{PL}{8}, \qquad M_B = +\frac{PL}{8}$$

To balance these moments, apply $+PL/8$ at A and $-PL/8$ at B.

These moments will be distributed in proportion to the stiffness factors of the members meeting at the joints, which in this example are equal. Hence, the counterbalancing moments applied at A and B will each be divided equally between girder and column, each carrying $\frac{1}{16}PL$, and one-half of each such counterbalancing moment will be carried over to the other end of the member. Figure 343 shows the application at this stage in which the column moments are written to be read from the right and the girder moments to be read from below, and the factor PL is omitted from each moment. The moments transferred from one end of a member to the other are indicated by

diagonal lines with arrowheads showing the moment that is carried over, usually called the *carry-over* moment.

Evidently the foregoing operation results in an unbalanced moment at A in column and girder combined of $-1/32$ and at B of $+1/32$; hence, apply new counterbalancing moments at each joint, *viz.*, $+1/32$ at A and $-1/32$ at B, distribute as before, and continue the process until the unbalanced moments at A and B are reduced as low as seems necessary.

Figure 344 shows the effect of three distributions which have reduced the remaining unbalanced moments to such small values that further adjustments are evidently unnecessary. The results in all cases are to be multiplied by PL.

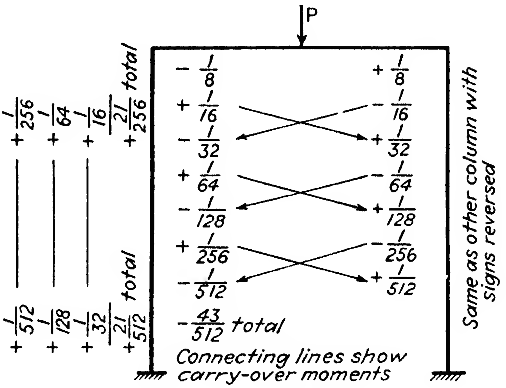


FIG. 344.

The column shear H may be found from the following equation which refers to the left column:

$$21/512\,PL + 21/256\,PL - HL = 0$$

Therefore,

$$H = 63/512\,P$$

By the method of least work, the effect of direct stress being neglected, the moment at the bottom of the left column $= \dfrac{PL}{24} = \dfrac{21\frac{1}{3}}{512}PL$ and the shear $= P/8 = 64/512\,P$, showing that the values obtained by the moment-distribution method differ very slightly from the correct values. The determination of these values by the method of least work may be readily accomplished by

noting that points of inflection in the columns occur at one-third the height from the bottom, since for each moment applied to top of column the carry-over moment is one-half thereof.

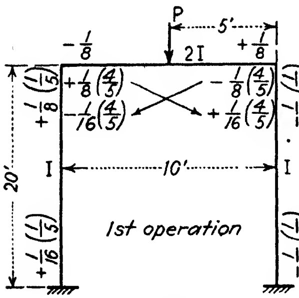


FIG. 345.

Evidently the number of operations that were applied to this problem give results to a greater degree of precision than is necessary in practice.

Had the last operation been omitted, the moment in the left column would have had the following value:

Bottom of column $= +PL(\frac{1}{32} + \frac{1}{128}) = +\frac{5}{128}PL$

Top of column $= +PL(\frac{1}{16} + \frac{1}{64}) = +\frac{5}{64}PL$

Left end of girder $PL(-\frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128}) =$
$-\frac{11}{128}PL$

Shear H in columns equals thrust in girder $= (\frac{5}{128} + \frac{5}{64})P$

None of above values is greatly in error.

Had members had different stiffness factors, the counterbalancing moments should have been divided between girders and columns in proportion to stiffness factors as illustrated by Fig. 345 in which K for girders $= 4K$ for columns.

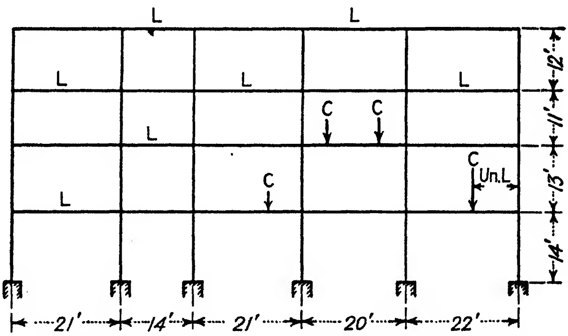


FIG. 346.—All spans are loaded over entire length with uniform dead load. Spans marked L are also loaded over entire length with a uniform live load. Other spans are loaded with live concentrated loads marked C and one span is loaded with both a concentrated load and a uniform live load extending over a portion only of its length.

211. Application of Moment-distribution Method to a Building.—The application of the moment-distribution method to the determination of stresses due to vertical loads in a building is illustrated by the table on pages 462–463, which applies to the upper portion of the four-story building shown in Fig. 346. The constants in this building were taken by permission of the Portland Cement Association, from an example given in *Bulletin* ST40, part II, of their Structural Bureau. Enough of the work is given here to illustrate the method clearly. The direction of the reactions for loading in any span may be determined by the method given in Art. 193.

212. Application of Moment-distribution Method to Transverse Forces.—The foregoing methods may also be applied to a bent subjected to a transverse force as shown in Fig. 347. In

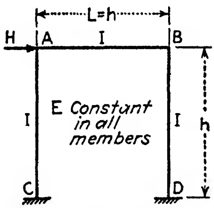


FIG. 347.

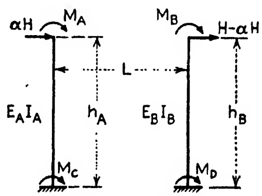


FIG. 348.

this case, if the change in length of the girder is neglected, the tops of all columns will move equal distances horizontally under the influence of the force H.

The forces actually applied at the tops and bases of the two columns will be as shown in Fig. 348.

Let δ_A = horizontal deflection of top of left column.

δ_B = horizontal deflection of top of right column.

From hypothesis,

$$\delta_A = \delta_B$$

Also, the horizontal deflection of A with respect to C equals numerically the horizontal deflection of C with respect to A; a similar relation exists between the deflections of B and D.

Now from Eq. (29),

$$\delta_A = \int_0^{h_A} \frac{Mmdx}{E_AI_A} = \int_0^{h_A} \frac{(\alpha Hx + M_A)(x)dx}{E_AI_A}$$

	Column 1	Column 2	Column 3	Column 4	Column 5	Column 6
$K/\Sigma K$ Fourth floor	0.74	0.38 0.49	0.49 0.38	0.36 0.51	0.46 0.43	0.79
	0.26	0.13	0.13	0.13	0.11	0.21
Fixed-ended beam moments	+53	−53 +33	−33 +53	−53 +68	−68 +58	−58
First moment distribution beams	−39	+8 +10	−10 −8	−5 −8	+5 +4	+46
First moment distribution columns	−14	+2	−2	−2	+1	+12
ΣM (check)	0	0	0	0	0	0
Carry-over moments from beams	+4	−20 −5	+5 −3	−4 +3	−4 +23	+2
Carry-over moments from columns	−10	+4	−4	+3	−4	+9
Second moment distribution beams	+4	+8 +10	+1 +1	−1 −1	−7 −6	−9
Second moment distribution columns	+2	+3	−0	0	−2	−2
ΣM (check)	0	+0	0	0	0	0
Total moments at end of second distribution:						
Beams	+22	−57 +48	−37 +43	−63 +62	−74 +79	−19
Columns	−22	+9	−6	+1	−5	+19
$K/\Sigma K$ Third floor	0.17	0.09	0.09	0.09	0.09	0.15
	0.64	0.34 0.40	0.40 0.34	0.35 0.38	0.35 0.40	0.67
	0.19	0.17	0.17	0.18	0.16	0.18
Fixed-ended beam moments	+112	−112 +27	−27 +112	−112 +44	−44 +123	−123
First moment distribution:						
Columns above floor	−19	+8	−8	+6	−7	+18
Beams	−72	+29 +34	−34 −29	+24 +26	−28 −32	+83
Columns below floor	−21	+14	−14	+12	−12	+22
ΣM (check)	0	0	0	0	0	0
Carry-over moments:						
From columns in story above	−7	+1	−1	−1	+0	+6
From beams	+15	−36 −17	+17 +12	−15 −14	+13 +42	−16
From columns in story below	−5	−0	−1	−3	+3	+6

	Column 1	Column 2	Column 3	Column 4	Column 5	Column 6
Second moment distribution:						
Columns above floor............	0	+5	−2	+3	−5	0
Beams........................	−2	+17 +21	−11 −9	+12 +12	−20 −23	+3
Columns below floor............	−1	+9	−5	+6	−10	+1
ΣM (check).....................	0	0	0	0	0	0
Total moments at end of second distribution:						
Columns above floor............	−26	+14	−11	+8	−12	+24
Beams........................	+53	−102 +65	−55 +86	−91 +68	−79 +110	−53
Columns below floor............	−27	+23	−20	+15	−19	+29
ΣM (check).....................	0	0	0	0	0	0
		0.16	0.16	0.17	0.16	0.18
K/ΣK Second floor.............	0.17 0.58	0.33 0.37	0.37 0.33	0.33 0.36	0 33 0.38	0.64
	0.25	0.14	0.14	0.14	0.13	0.18
Fixed-ended beam moments......	+53	−53 +54	−54 +66	−66 +97	−97 +65	−65
First moment distribution:						
Columns above floor............	−9	0	−2	−6	+5	+12
Beams........................	−31	0 −1	−4 −4	−10 −11	+11 +12	+41
Columns below floor............	−13	0	−2	−4	+4	+12
ΣM (check).....................	0	0	0	0	0	0

Carry-over beam moments indicated by dotted diagonal lines.
Carry-over column moments are shown by dotted vertical and horizontal lines.

and

$$\delta_B = \int_0^{h_B} \frac{[(H - \alpha H)x + M_B)]x\,dx}{E_B I_B}$$

Therefore,

$$\frac{\alpha H h_A^3}{3E_A I_A} + \frac{M_A h_A^2}{2E_A I_A} = (H - \alpha H)\frac{h_B^3}{3E_b I_b} + \frac{M_B h_B^2}{2E_B I_B} \qquad (96)$$

The bottoms of the columns are fixed in direction as well as position; if the tops of the columns are, however, temporarily assumed to be fixed in direction but not in position,

$$M_A = \frac{2E_A I_A}{h_A}\left(0 + 0 - \frac{3\delta_A}{h_A}\right)$$

$$M_C = \frac{2E_A I_A}{h_A}\left(0 + 0 - \frac{3\delta_A}{h_A}\right)$$

But δ_A = movement of either end with respect to the other end; hence, $M_A = M_C$ and also $M_A + M_C + \alpha H h_A = 0$.
Therefore,

$$\alpha H h_A = -(M_A + M_C) = -2M_A$$

Therefore,

$$\alpha H = -\frac{2M_A}{h_A}$$

Similarly,

$$M_B = M_d \qquad \text{and} \qquad H - \alpha H = -\frac{2M_b}{H_B}$$

Note that the foregoing equations do not apply to the actual structure but only to a structure in which the tops as well as the bottoms of the columns are fixed in direction.

Inserting the foregoing values of αH and $H - \alpha H$ in Eq. (96) gives

$$-\frac{2M_A h_A^2}{3E_A I_A} + \frac{M_A h_A^2}{2E_A I_A} = -\frac{2M_b h_B^2}{3E_B I_B} + \frac{M_b h_B^2}{2E_B I_B}$$

Therefore,

$$\frac{M_A h_A^2}{6E_A I_A} = \frac{M_B h_B^2}{6E_B I_B}$$

Hence,

$$\frac{M_A}{M_B} = \frac{E_A I_A h_B^2}{E_B I_B h_A^2}$$

If E is the same in both columns, as would generally be the case,

$$\frac{M_A}{M_B} = \frac{I_A h_B^2}{I_B h_A^2}$$

Now,

$$M_A + M_C + M_B + M_D + \alpha H h_A + (H - \alpha H)h_B = 0$$

But

$$M_A = M_C \qquad \text{and} \qquad M_B = M_D$$

Therefore,

$$2M_A + 2M_B + Hh_B + \alpha H(h_A - h_B) = 0$$

Hence,

$$2M_A + 2M_A \frac{I_B h_A^2}{I_A h_B^2} = -[Hh_B + \alpha H(h_A - h_B)]$$

Therefore,

$$M_A = -\frac{[Hh_B + \alpha H(h_A - h_B)]}{2\left(1 + \frac{I_B h_A^2}{I_A h_B^2}\right)}$$

If $h_A = h_B = h$, as is common in a two-column bent,

$$M_A = -\frac{HhI_A}{2(I_A + I_B)}$$

If I_A also $= I_B$, as is not unusual, $M_A = M_C = M_B = M_D = -\frac{Hh}{4}$ = moments at column tops and bases if columns are fixed in direction at both ends.

By application of the foregoing conclusions, the structure shown in Fig. 347 may be solved in a manner similar to that used for a bent subjected to a vertical force, *i.e.*, by first assuming each column rigidly fixed in direction at top and bottom, determining the moments applied at the column ends under this condition, applying counterbalancing moments, distributing them, revising the moments thus obtained through applying the

additional moments necessary for equilibrium, and continuing the process until the approximations in the results are so small as to be negligible. It should be observed that, in this case, counterbalancing moments have to be applied at both top and bottom of each column.

A solution follows for the case when $h_a = h_b = h$; $I_A = I_B$, and E is constant.

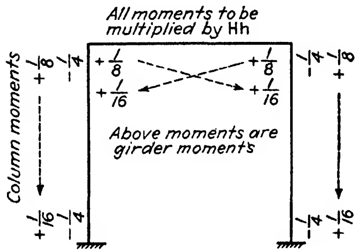


FIG. 349.

The application of counterbalancing moments $= +\frac{Hh}{4}$ at column tops gives the results shown in Fig. 349.

Evidently the moments resulting from the first operation are not final since the combined *column moments* after distribution $= -2(\frac{3}{16} + \frac{1}{8})Hh = -\frac{5}{8}Hh$ instead of $-Hh$; hence, an additional similar operation must be performed by applying an additional moment $= \frac{1}{4}(-\frac{3}{8}Hh)$ at top and bottom of each column, applying counterbalancing moments, and distributing them as before. The moments at the end of this operation are shown in Fig. 350 in circles, and the other figures show the

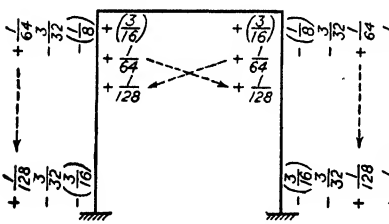


FIG. 350.

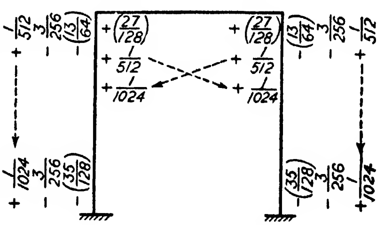


FIG. 351.

counterbalancing and distributed moments. The reader should note that the application of $-\frac{3}{32}Hh$ at top of the left column leaves a resultant about the joint at the top of the column equal to $(-\frac{1}{8} - \frac{3}{32} + \frac{3}{16})Hh = -\frac{1}{32}Hh$; hence, $+\frac{1}{32}Hh$ must be applied at this joint and divided equally between the girder and the column.

The combined column moments at the end of the second operation $= -2(\frac{13}{64} + \frac{35}{128})Hh = -\frac{122}{128}Hh$ instead of $-Hh$.

The previous operation may now be repeated by applying moments equal to $-3/256$ at column tops and bases. Figure 351 shows, in circles, the moments before counterbalancing, and the other figures show the counterbalancing and distributed moments.

The sum of moments at tops and bases of columns now equals

$$-2\left(\frac{109}{512} + \frac{291}{1{,}024}\right)Hh = -\frac{1{,}018}{1{,}024}Hh$$

which is so nearly in agreement with the correct value that no further adjustments need be made on account of this term. Also, the combined moment in column and girder at point A and at point B equals in each case

$$+\left(\frac{1 + 2 + 216 + 2 - 12 - 208}{1{,}024}\right)Hh = \frac{1}{1{,}024}Hh$$

which is practically equal to zero; hence, no further corrections are necessary.

The shear at girder center, computed by the method of least work, neglecting the work due to direct stress in members as determined in Art. 207 by placing $b = h/2$ and $C = 1$

$$= \frac{3Hh^2}{2\left(3h^2 + 2\frac{h^2}{4}\right)} = \frac{3Hh^2}{7h^2} = \frac{3}{7}H$$

and the direct stress in the girder $= H/2$, hence, M at bottom column $= \frac{3H}{7} \cdot \frac{h}{2} - \frac{Hh}{2} = -\frac{2}{7}Hh = -0.286Hh$. The value of this moment as shown in Figure (351) =

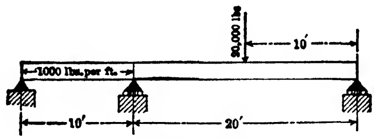
$$\frac{-280 - 12 + 1}{1{,}024} = -\frac{291}{1{,}024}Hh = -0.284Hh$$

showing that the results obtained by the approximate method are almost exact.

The simple structures shown in Figs. 345 and 347 can both be solved more readily by the slope-deflection method than by the moment-distribution method, the latter being valuable only in more complicated structures.

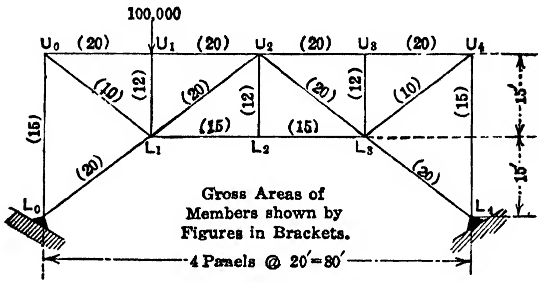
Problems

74. Determine reactions on this beam by application of the three-moment equation; also by method of deflections.



Prob. 74.

75. *a.* Compute by method of least work the horizontal reaction component on this two-hinged arch due to load shown.



Prob. 75.

b. Compute by method of deflections the horizontal reaction component on same structure due to an increase of 60°F. in temperature of all members.

76. Compute reactions on a beam continuous over four equal spans of 10 ft. each, due to the application of a clockwise moment of unity acting at the second support from the left.

77. *a.* Compute by method of least work the moments and the horizontal and vertical reactions at column bases of a two-story two-column framed bent. Story height = 20 ft. Column spacing = 20 ft. I of each cross-girder = 1,500 in.^{4} I of each column = 500 in.^{4} E same for all members. Neglect direct stresses in members.

b. Same as (a) but use method of slope deflection.

CHAPTER XVI

SPACE FRAMEWORK

213. Definition.—A space framework may be defined as a structure composed of end-connected bars lying in more than one plane, which, owing to the arrangement of its members, cannot be solved by dividing it into several planar structures as can the ordinary framed structure. The Schwedler dome, shown in Fig. 352, represents a type of such structure that has been used in Europe. Transmission-line towers are also sometimes of this type and the principles of this chapter may be applied to their solution.

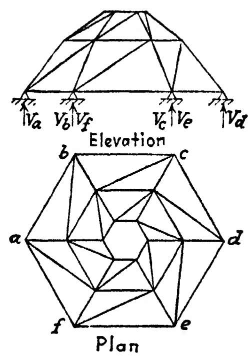


FIG. 352.

214. Statical Conditions. *With Respect to Outer Forces.*—A space framework will be in equilibrium so far as the outer forces are concerned if the following statical conditions are satisfied:

1. The algebraic sum of the components acting parallel to any reference axis of all the forces applied to the structure equals zero.
2. The algebraic sum of the moments of all the forces about any axis equals zero.

Inasmuch as any force is completely determined in magnitude and direction if its components parallel respectively to three independent axes are known, it is evident that the application of the first of these rules will give three and not more independent equations which may be utilized in determining the reactions on such a structure. If each force is resolved into three components parallel respectively to three axes, such for example as *ov*, *ox*, and *oz* (Fig. 353), these three equations may be stated as $\Sigma V = 0$ $\Sigma X = 0$, and $\Sigma Z = 0$. If the outer forces, including reactions, have such values as to satisfy all these equations, the structure will be in equilibrium with respect to translation.

Since any force may be resolved into components lying in not more than three intersecting planes, such, for example, as the three planes shown in Fig. 353, equilibrium against rotation will be assured when the algebraic sum of the moments of all the components in each plane about any axis normal to the plane equals zero.

The application of the second rule will, therefore, also give three equations.

If axes of rotation are taken as the lines of intersection of the three planes, and if we let

ΣM_x = moment of components in plane *efh* about axis *xx*
ΣM_v = moment of components in plane *acd* about axis *vv*
ΣM_z = moment of components in plane *ijk* about axis *zz*

we may express these three equations of rotation as follows:

$$\Sigma M_x = 0, \qquad \Sigma M_v = 0, \qquad \text{and} \qquad \Sigma M_z = 0$$

It follows from the foregoing that there are six independent equations of equilibrium that may be applied to the outer forces acting upon any space framework. Hence, the reactions upon such a structure can, in general, be computed by statics if they have in all six unknown and independent components. With more than six, the structure would be statically indeterminate with respect to outer forces only; with less than six, it would, in general, be unstable. It may, however, also be necessary to impose certain conditions upon the reactions in order to make the problem solvable. For example, in a structure such as that shown in Fig. 354 to which a force, such as P, is applied, it will, in general, be necessary to have at least three of the unknown reaction components act in the plane *abcd* in order to avoid ambiguity. For instance, if, in the structure shown in Fig. 354, rollers or other devices are used so that only two horizontal reaction components exist, *viz.*, one at *d* parallel to *zz* and one at *a* parallel to *xx*, we might obtain inconsistent results in trying to

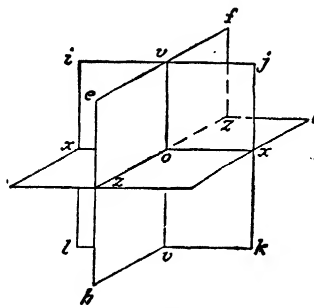


FIG. 353.

determine the horizontal forces, since the application of $\Sigma Z = 0$ would give a definite value for the Z component at d, whereas the application of $\Sigma M = 0$ about a vertical axis through a might give a quite different value for the same component, a manifestly unreasonable result. The occurrence of this condition may be prevented by imposing the following conditions:

The number of unknown reactions acting in any plane should be equal to or greater than the number of statical equations applicable to the plane. For example, for the plane *abcd* (Fig. 354) there should, in general, be at least three unknown reaction components; these may consist of three reactions, each unknown in magnitude but fixed in point of application and direction, the line of action of no two of which coincide, or they may consist, as is usually the case in vertical bridge trusses, of one reaction

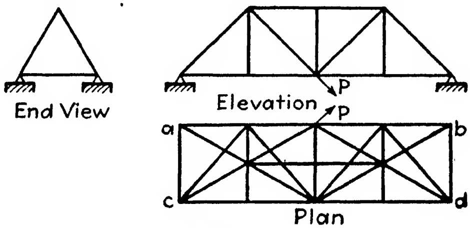


FIG. 354.

unknown in magnitude but fixed in direction and point of application and one reaction unknown in both magnitude and direction but fixed in point of application.

Inasmuch as the statical condition of a framed structure as a whole depends upon the number and arrangement of bars as well as upon the reaction conditions a complete investigation of its condition makes necessary a consideration of the number of bars and joints.

With Respect to Outer and Inner Forces.—The members meeting at a joint of a space framework cannot be connected by a single pin unless they lie in a plane. To obtain a condition analogous to that furnished by a single pin in a planar framework, several pins might have to be used. Riveted joints would, however, be generally employed, and the stress in each bar meeting at the joint may be assumed as a direct stress, the resistance to bending of the connection being ignored, as is done in determining primary stresses in planar structures. There

will, therefore, be three equations of statics applicable at each joint, *viz.*:

$$\Sigma V = 0, \qquad \Sigma X = 0, \qquad \Sigma Z = 0$$

If n equals the number of joints, $3n$ will equal the number of statical equations available for determining the bar stresses and reactions. Now, let b equal the number of bars, r the number of points of support, and s the number of reaction components, if any, that are eliminated by the method of construction, either by fixing the direction of certain reactions or by eliminating certain reactions entirely. It follows that the total number of unknowns $= b + 3r - s$; hence, for complete statical determination of a space framework, the following equation must in general be satisfied:

$$3n = b + 3r - s$$

The foregoing equation is based upon there being bars or forces at each joint acting in two planes. If at some joints there are bars and forces acting only in one plane, there will be only two equations for that joint; hence, the left-hand term in the foregoing equation would have to be correspondingly modified.

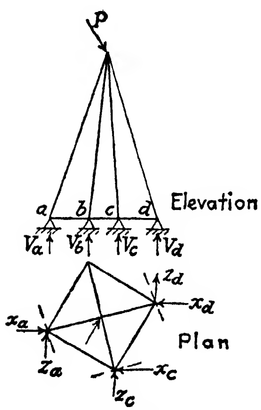


FIG. 355.

For the structure shown in Fig. 355 it being assumed that no reaction components are predetermined by construction,

$$n = 5, \qquad b = 8, \qquad r = 4, \qquad 3n = 15$$

But

$$b + 3r = 20$$

Hence, the structure is indeterminate to the fifth degree.

215. Fixing the Direction of Reactions.—The following illustrates the reduction of the number of unknowns by fixing the direction of reactions. Suppose that the horizontal reactions of the structure shown in Fig. 355 are so fixed that they can act only along the dotted lines shown on the figure. We then have seven unknown reactions, *i.e.*, $s = 5$, and the structure is deter-

minate.[1] The establishment of the direction of the reactions may be advantageous not only theoretically but structurally; *e.g.*, if a steel framework is supported on walls, as in the case of a steel dome of a building, the reactions may be so fixed that they can cause but little outward thrust. A reaction may be determined in direction by the use of rollers restrained against side movement, as in the case of a roller bearing for a bridge, or by rockers, or by planed plates restrained sideways. In either case, the component of the reaction in the direction in which motion is permitted will be zero if frictional resistance is neglected.

216. Types of Space Framework.—Figures 356, 357, and 358 illustrate possible types of simple space frameworks, the direction

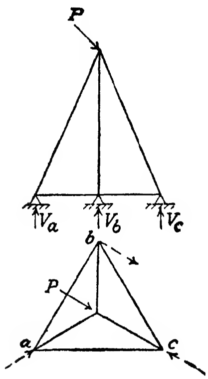


FIG. 356.

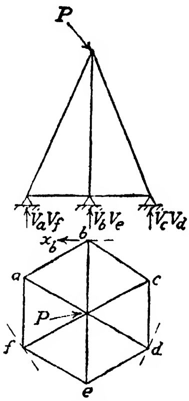


FIG. 357.

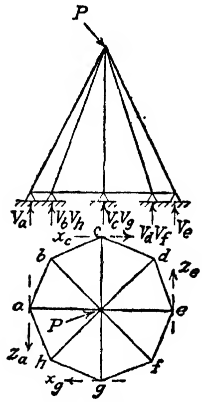


FIG. 358.

of the horizontal reaction at each point of support as determined by the method of construction being shown by a dotted line. The omission of such a line indicates that no horizontal reaction can exist at the given point. The statical condition for each structure is as follows:

Structure Shown in Fig. 356.

$$n = 4, \qquad b = 6, \qquad r = 3, \qquad s = 3$$

Therefore,

$$b + 3r - s = 12 = 3n$$

[1] See Der Eisenkonstruktionen der Ingenieurhochbauten, 3rd ed., Foerster, listed among the References at the end of this chapter, for various methods of fixing the direction of reactions in space frameworks.

Hence, the structure is statically determined.

Structure Shown in Fig. 357.

$$n = 7, \qquad b = 12, \qquad r = 6, \qquad s = 9$$

Therefore,

$$b + 3r - s = 21 = 3n$$

and there are three unknowns in the bottom plane and hence the structure is statically determined.

Structure Shown in Fig. 358.

$$n = 9, \qquad b = 16, \qquad r = 8, \qquad s = 12$$

Therefore, $b + 3r - s = 28$, and, as $3n = 27$, the structure is indeterminate to the first degree. This structure might have been made statically determinate by omitting one more horizontal reaction.

More complicated types of such frameworks are shown in Figs. 352, 359, and 364.

It should be borne in mind that the correct stresses in such structures, as well as in planar structures, even if statically undetermined, can be computed by the method of least work or by other methods commonly used in solving statically indeterminate planar structures.

217. Method of Computation.—The reactions of a statically determined space framework may sometimes be computed by applying the general equations of equilibrium to the outer forces; occasionally, this cannot be done and the reactions must be determined by first determ ning the bar stresses. The bar stresses, in case the structure is statically determined, may be computed by either the analytical or the graphical method of joints. The following theorems are, however, extremely useful in determining bar stresses and should be freely used:

a. If several bars of any framework meet at a joint and all but one lie in the same plane, the component normal to this plane of the stress in that bar which does not lie in the plane will equal the algebraic sum of the components normal to the same plane of all the outer forces which may be applied at the joint under consideration.

b. The moment of any force or bar stress acting in a given plane about an axis lying in that plane equals zero. It follows

that sometimes the stress in a bar of a space framework may be determined by the method of moments in the same general manner as in the case of a planar structure; *i.e.*, if a section can be taken cutting a number of bars, all of which except bar a pass through or are parallel to a *given axis*, the stress in bar a may be found by applying $\Sigma M = 0$ about that axis of all the forces acting on the portion of the structure isolated by the section.

c. At any joint at which no outer force is applied and at which the stresses in all bars but two have been found to be zero, the stress in each of these two bars will also be zero, provided that the two bars do not lie in the same straight line.

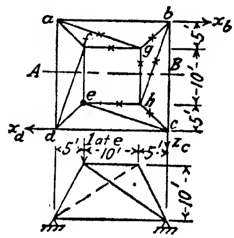


FIG. 359.

Theorems a and b are based upon the fact that none of the forces lying in a given plane, whether external forces or bar stresses, have components normal to that plane; in consequence, their effect may be entirely disregarded in determining the stress in a bar that does not lie in that plane or in computing the moment about an axis lying in that plane.

Theorem c may be proved by applying the ordinary equations of equilibrium to such a joint

The application of these theorems is illustrated by the case shown in Fig. 359 in which the stress in each bar marked with a cross is found to equal zero by applying the rules given in (a) and (c). To determine stress in bar ef, apply theorem b by taking the vertical section AB and applying $\Sigma M = 0$ about axis dc of all forces acting upon portion of structure below AB (toward bottom of page). All the bars cut by the section, except bar ef and those bars in which the stress has been determined by rules a and b to equal zero, pass through cd; hence, stress in ef = $-1 \times 5 \div 10$.

218. Illustration of Methods of Computation.—The examples that follow clearly illustrate the method of solution of simple types of space frameworks. It should be noted that a reaction may be assumed in the computations to act in either direction along its line of action, its actual direction being determined by the sign of its value as finally determined.

Problem: Compute reactions and bar stresses for the tripod shown in Fig. 360 due to the horizontal force of 10,000 lb. applied at d_v in the direction shown, assuming members can carry direct stress only.

Solution: This structure really consists of one joint d and three bars meeting at the joint; hence, the stresses in all bars may readily be determined by the method of joints through the application of the three equations of equilibrium. The stress may also be computed as follows by the application of the rules in the previous paragraph.

Apply theorem a to joint d. Since, at this joint, bars 1 and 2 and the applied force all lie in one plane and the only other bar at the joint is bar 3, the stress in bar 3 $= 0$

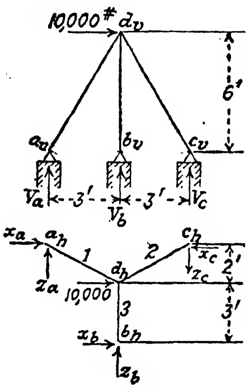


FIG. 360.

Hence,

$$V_b = 0$$

Now, apply $\Sigma M = 0$ about the Z axis passing through a_h.

$$10{,}000 \times 6 - 6V_c = 0$$

Therefore,

$$V_c = 10{,}000$$

Next, apply $\Sigma V = 0$ to entire structure.

$$V_a + V_c = 0$$

Therefore,

$$V_a = -10{,}000$$

From the vertical reactions the stresses in bars 1 and 2 can be readily determined since their vertical components equal, respectively, the vertical reactions already computed. These stresses are, therefore, as follows:

$$\text{Bars 1 and (2)} \rightarrow \text{length} = \sqrt{3^2 + 2^2 + 6^2} = 7$$

$$\text{Bar 1 stress} = +10{,}000 \times \frac{7}{6} = +\frac{70{,}000}{6}$$

$$\text{Bar 2 stress} = -10{,}000 \times \frac{7}{6} = -\frac{70{,}000}{6}$$

The horizontal components of the bar stresses are as follows:

$$\text{Bar 1} + 10{,}000 \times \frac{\sqrt{3^2 + 2^2}}{6}$$

$$\text{Bar 2} - 10{,}000 \times \frac{\sqrt{3^2 + 2^2}}{6}$$

The horizontal components of the reactions at a and c are, therefore

$$X_a = -10{,}000\left(\frac{\sqrt{3^2 + 2^2}}{6}\right)\left(\frac{3}{\sqrt{3^2 + 2^2}}\right) = -5{,}000$$

$$Z_a = -\tfrac{2}{3}X_a = +3{,}333$$
$$X_c = 10{,}000 - 5{,}000 = +5{,}000$$
$$Z_c = Z_a = +3{,}333$$

Note that the values above of the horizontal components are consistent with the reactions at a and c coinciding with the line of action of bars 1 and 2, respectively, as should be the case.

Problem: Compute reactions and stresses in all bars of the structure shown in Fig. 361, the various horizontal reactions being so fixed by construction that they act in the directions shown in the plan. In this structure, $n = 4$, $b = 6$, $r = 3$, $s = 3$; hence, $3n = b + 3r - s$, showing that the structure is statically determined.

Solution: Apply $\Sigma M = 0$ about the horizontal axis a_hc_h. The moment about this axis of the applied force, the horizontal reactions X_a, Z_a, and Z_c, and the vertical reactions V_a and V_c equal zero in each case; hence, V_b equals 0.

Apply $\Sigma V = 0$ at point b.

This gives the vertical component in bar 1 = $V_b = 0$; hence, stress in bar 1 = 0. This result may also be obtained by the application of theorem a of previous paragraph to point d.

Apply $\Sigma M = 0$ about the Z-axis passing through the point a.

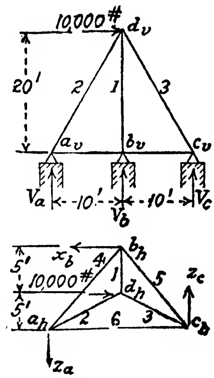


FIG. 361.

$$10{,}000 \times 20 - 20V_c = 0$$

Therefore,

$$V_c = +10{,}000 \text{ lb.}$$

If we now let L_2 = length of bar 2 in feet and L_3 = length of bar 3 in feet,

$$\text{Stress in bar 3} = -10{,}000 \times \frac{L_3}{20}$$
$$= -500L_3$$

Apply $\Sigma V = 0$ to entire structure.

$$V_a + V_c = 0$$

Therefore,

$$V_a = -10{,}000$$

Hence,

$$\text{Stress in bar 2} = +500L_2$$

Apply $\Sigma X = 0$ to entire structure

$$10{,}000 - X_b = 0$$

Hence,

$$X_b = 10{,}000$$

Apply $\Sigma M = 0$ about a vertical axis through intersection of Z_a and X_b

$$-10{,}000 \times 5 - Z_c \times 20 = 0$$

Therefore,

$$Z_c = -2{,}500$$

Apply $\Sigma Z = 0$ *to entire structure.*

$$Z_a - Z_c = 0$$

Therefore,

$$Z_a = Z_c = -2{,}500$$

The X and Z components in bars 2 and 3 may now be found from the stresses in these bars and are as follows:

Bar 2, X component $= 10 \times 500 \times \frac{L_2}{L_2} = 5{,}000$ acting to right

Bar 2, Z component $= 5{,}000 \times 5/10 = 2{,}500$, acting toward top of sheet

Bar 3, X component $= 10 \times 500 \times \frac{L_3}{L_3} = 5{,}000$, acting to right

Bar 3, Z component $= 2{,}500$, acting toward bottom of sheet

The horizontal bottom triangle may now be treated as any truss, all the panel-point loads being known and there being three independent reactions. The forces acting upon it are as shown in Fig. 362.

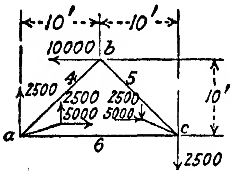


FIG. 362.

Evidently the Z component in bar 4 $= -5{,}000$; therefore, the X component $= -5{,}000$. Hence, the application of $\Sigma X = 0$ at joint a shows zero stress in bar 6. This result may be verified by considering joint c where the Z component in bar 5 $= +5{,}000$. Hence, X component $= +5{,}000$, and the stress in bar 6 $= 0$.

The actual stress in the various bars of the entire structure are, therefore, as follows:

Bars 1 and 6 $= 0$

Bar 2 $= +11{,}450$

Bar 3 $= -11{,}450$

$$\text{Bar 4} = -5{,}000\frac{\sqrt{200}}{10} = -7{,}070$$

$$\text{Bar 5} = +5{,}000\frac{\sqrt{200}}{10} = +7{,}070$$

In a similar manner, the effect of a horizontal force at d parallel to the Z-axis or that of a vertical force may be obtained.

The following example illustrates the computations for a somewhat more complicated structure:

Problem: Compute stresses in all bars of the structure shown in Fig. 363, the horizontal reactions acting at points b, f, and d only and along the line of action indicated by dotted lines. For this structure, $n = 7$, $b = 12$, $3r = 18$, $s = 9$; hence, $3n = b + 3r - s$, and the structure is statically determined.

Solution: **Application of $\Sigma M = 0$ about a vertical axis through m shows at once that**

$$Z_d = 0$$

Application of $\Sigma M = 0$ about a vertical axis through f gives the following equation:

$$10{,}000a - X_b(2a) = 0 \qquad X_b = 5{,}000$$

Application of $\Sigma X = 0$ to entire structure gives

$$X_f = 5{,}000$$

From the foregoing values of X_b and X_f, we obtain from the known direction of the horizontal reactions at these points the following values of Z_b and Z_f:

$$Z_b = Z_f = \frac{5{,}000}{\sqrt{3}}$$

Therefore, the horizontal reactions at b and f each equal

$$+5{,}000 \times \frac{2}{\sqrt{3}}$$

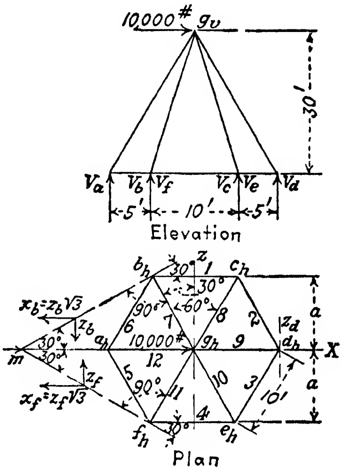


FIG. 363.

If the method of joints is now applied to the bottom ring, the following results may be obtained, tension being assumed in all bars, the bar stresses and components being indicated by S, X, Z, H, and V with sub-numbers indicating the bar in question, and H and V representing the actual horizontal and vertical components in a bar:

Joint c

Component in bar 1 normal to cg = component in bar 2 normal to cg

Hence

$$S_1 = S_2$$

Joint d

$$Z_2 = Z_3 \qquad \text{since,} \qquad Z_d = 0$$

Hence,

$$S_3 = S_2 = S_1$$

Joint e

Component in bar 3 normal to ge = component in bar 4 normal to ge

Hence,

$$S_4 = S_3 = S_2 = S_1$$

Joint f

Component in bar 4 normal to fg = component in bar 5 normal to fg

$$+ 5{,}000 \times \frac{2}{\sqrt{3}}$$

Hence,

$$S_5\frac{\sqrt{3}}{2} + 5{,}000\frac{2}{\sqrt{3}} = S_1 \times \frac{\sqrt{3}}{2}$$

Therefore,

$$S_5 = S_1 - 5{,}000 \times 4/3$$

Joint a

$$Z_6 = Z_5$$

Hence,

$$S_6 = S_5 = S_1 - 5{,}000 \times 4/3$$

Joint b

Component in bar 6 normal to $bg + 5{,}000 \times \frac{2}{\sqrt{3}}$ = component in bar 1 normal to bg

Hence,

$$S_6 \times \frac{\sqrt{3}}{2} + 5{,}000 \times \frac{2}{\sqrt{3}} = S_1 \times \frac{\sqrt{3}}{2}$$

Therefore,

$$S_6 = S_1 - 5{,}000 \times 4/3 = S_1 \text{ (checking previous value)}$$

Collecting these values, we obtain

$$S_2 = S_3 = S_4 = S_1 \qquad \text{and} \qquad S_6 = S_5 = S_1 = -\frac{20{,}000}{3}$$

Hence, the stress in each ring bar has been expressed in terms of S_1, which has been assumed as tension.

By considering point c, it is evident that the horizontal component in bar 8 = component in bar 1 parallel to gc plus component in bar 2 parallel to gc

$$H_8 = -\left(\frac{S_1}{2}\right) - \left(\frac{S_2}{2}\right) = -S_1$$

In a similar manner, it may be shown that

$$H_9 = H_{10} = -S_1$$

It also follows from the conditions existing at joint f that $-H_{11}$ = combined components parallel to fg of S_4 and S_5.
Therefore,

$$-H_{11} = \frac{1}{2}S_1 + \frac{1}{2}\left(S_1 - \frac{20{,}000}{3}\right) = S_1 - \frac{20{,}000}{6}$$

Similarly, from joint a, we get

$$-H_{12} = S_1 - \frac{20{,}000}{3}$$

and from joint b,

$$-H_7 = \frac{1}{2}S_1 + \frac{1}{2}\left(S_1 - \frac{20{,}000}{3}\right) = S_1 - \frac{20{,}000}{6}$$

Collecting these results gives

$$H_8 = H_9 = H_{10} = -S_1$$
$$H_{11} = H_7 = -S_1 + \frac{20{,}000}{6}$$
$$H_{12} = -S_1 + \frac{20{,}000}{3}$$

Note that all these expressions assume tension in the ring bars; hence, if S_1 = tension, H_8 = compression since it has a negative sign.

Considering now joint g and applying $\Sigma X = 0$, assuming all bars to be in tension, give

$$\tfrac{1}{2}(H_8 + H_{10} - H_{11} - H_7) + H_9 - H_{12} + 10{,}000 = 0$$

Substituting values previously found gives

$$\frac{1}{2}\left(-S_1 - S_1 + S_1 - \frac{20{,}000}{6} + S_1 - \frac{20{,}000}{6}\right) - S_1 + S_1 - \frac{20{,}000}{3} + 10{,}000 = 0$$

showing that these values are correct.

We now have all the bar stresses in terms of S_1. It remains to determine the value S_1.

Bar	Stress	Bar	Vertical component of stress
1	$+\frac{20{,}000}{9}$	7	$+\frac{10{,}000}{3}$
2	$+\frac{20{,}000}{9}$	11	$+\frac{10{,}000}{3}$
3	$+\frac{20{,}000}{9}$	8	$-\frac{20{,}000}{3}$
4	$+\frac{20{,}000}{9}$	9	$-\frac{20{,}000}{3}$
5	$\frac{20{,}000}{9} - \frac{20{,}000}{3} = -\frac{40{,}000}{9}$	10	$-\frac{20{,}000}{3}$
6	$\frac{20{,}000}{9} - \frac{20{,}000}{3} = -\frac{40{,}000}{9}$	12	$+\frac{40{,}000}{3}$

Since the vertical component of each rib stress = $^{30}\!/_{10}$ of its horizontal component, we may write

$$V_8 = V_9 = V_{10} = -3S_1$$
$$V_7 = V_{11} = -3S_1 + 10{,}000$$
$$V_{12} = -3S_1 + 20{,}000$$

Now, applying $\Sigma V = 0$ at joint g, we get

$$-9S_1 - 6S_1 + 20{,}000 - 3S_1 + 20{,}000 = 0$$

Therefore,

$$18S_1 = 40{,}000$$
$$S_1 = \frac{20{,}000}{9}$$

The stresses in all the ring bars and the vertical components in the rib bars may now be written and are given in the table on page 481.

Note that the vertical components of the rib stresses equal the vertical reactions, negative signs for the rib stresses corresponding to upward reactions.

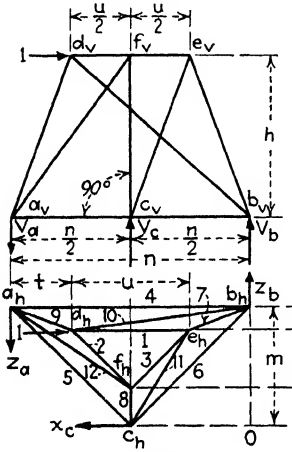


FIG. 364.

Had the applied load while remaining horizontal acted in a different direction, the stresses could have been obtained in the same general manner. Had the load been vertical, the horizontal reactions would all have been zero, and for the symmetrical figure under consideration the stresses in the ring bars would all have been equal; consequently the rib stresses would have also been equal, and the structure could have been easily solved.

The stresses due to an inclined force can be found by resolving the force into horizontal and vertical components and considering the effect of each component separately.

Problem: Compute reactions and bar stresses in structure shown in Fig. 364.

Solution: The horizontal reactions are fixed in direction as shown. Hence $n = 6$, $b = 12$, $r = 3$, and $s = 3$; therefore, $3n = b + 3r - s$, showing that the structure is statically determined.

To determine the reactions, proceed as follows:

Apply $\Sigma X = 0$ to entire structure.

This gives $X_c = 1$.

Apply $\Sigma M = 0$ about horizontal axis a_hb_h.

This gives $V_c = 0$.

Apply $\Sigma V = 0$ to entire structure.

This gives $V_a - V_b = 0$.

Apply $\Sigma M = 0$ about any Z-axis. This gives $1 \times h - V_an$.

Therefore,

$$V_a = V_b = \frac{h}{n}$$

Apply $\Sigma M = 0$ about a vertical axis through O, the point of intersection of Z_b and X_c.

This gives $Z_a = \dfrac{m - r}{n}$.

Apply $\Sigma Z = 0$ to entire structure.

This gives $Z_b = Z_a = \dfrac{m - r}{n}$.

Now consider the various joints, assuming tension in all bars unless otherwise stated.

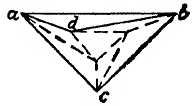


FIG. 365.

Joint e. Application of theorem a gives $S_1 = 0$.
Joint f. Application of theorem a gives $S_3 = 0$.
Joint d. Application of theorem a gives $S_2 = 0$.

Since S_1 and $S_3 = 0$, application of theorem c gives S_{11} and $S_7 = 0$. Similarly, since S_2 and $S_3 = 0$, S_8 and $S_{12} = 0$.

Figure 365 shows by full lines the bars that are brought into action by the given load.

We may now determine the stresses in bars 9 and 10 as follows:

Joint b

Apply $\Sigma V = 0$, assuming tension in unknown bars.

$$V_{10} + V_b = 0$$

Hence

$$V_{10} = -V_b = -\frac{h}{n}$$

and

$$S_{10} = -\frac{h}{n}\frac{\sqrt{r^2 + (n - t)^2 + h^2}}{h}.$$

whence

$$X_{10} = -\frac{h}{n}\cdot\frac{n - t}{h} = -\frac{n - t}{n} \qquad \text{and} \qquad Z_{10} = \frac{r}{n}$$

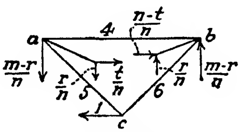


FIG. 366.

Joint a

Apply

$$\Sigma V = 0$$

$$V_9 - \frac{h}{n} = 0$$

Therefore,

$$V_9 = +\frac{h}{n}; \text{ whence } X_9 = \frac{h}{n}\cdot\frac{t}{h} = \frac{t}{n};\ Z_9 = \frac{r}{n}, \text{ and } S_9 = +\frac{h}{n}\frac{\sqrt{r^2 + t^2 + h^2}}{h}.$$

The base triangle abc is a simple triangular truss with three reactions and hence may be solved by the usual methods for planar trusses, the applied forces being as shown in Fig. 366.

Apply $\Sigma Z = 0$ to joint a.

$$Z_5 = -\frac{m}{n}$$

Hence,

$$X_5 = -\frac{m}{n}\frac{n}{2m} = -\frac{1}{2}$$

Apply $\Sigma X = 0$ and $\Sigma Z = 0$ to joint c, giving

$$Z_6 = +\frac{m}{n} \qquad \text{and} \qquad X_6 = +\frac{1}{2}$$

Apply $\Sigma X = 0$ to joint a.

$$X_4 = \frac{1}{2} - \frac{t}{n}$$

Check at joint b by applying $\Sigma X = 0$, giving $\frac{1}{2} + \left(\frac{1}{2} - \frac{t}{n}\right) - \frac{n-t}{n} = 0.$ The actual stresses in the ring bars will be as follows:

$$S_4 = X_4 = \frac{1}{2} - \frac{t}{n}$$

$$S_5 = X_5 \frac{\sqrt{m^2 + \frac{n^2}{4}}}{n/2} = -\frac{1}{n}\sqrt{m^2 + \frac{n^2}{4}}$$

$$S_6 = -S_5 = +\frac{1}{n}\sqrt{m^2 + \frac{n^2}{4}}$$

The same general methods may be applied to a force of unity acting at d in any other direction.

219. Stresses in Symmetrical Polygonal Ring without Radial Bars. *Analytical Method.*—Before investigating other structures

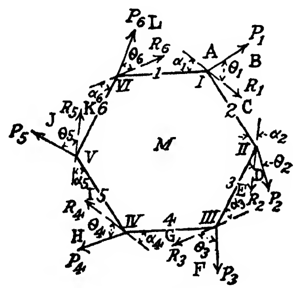


FIG. 367.

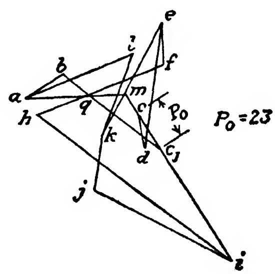


FIG. 368.

of the Schwedler dome type, it is desirable to consider the general case of a symmetrical polygonal ring without radial bars subjected to forces acting at the various apices, and with reactions fixed in one direction acting at each apex, forces and reactions acting in the plane of the ring.

For purposes of study, the ring shown in Fig. 367 has been chosen. The applied forces P_1, P_2, etc., are assumed to act in the directions shown. The lines of action of the reactions are

shown dotted, the reactions being assumed to act in the directions indicated by arrowheads. The value of the internal angle of the polygon is assumed as 2β.

Let the stress in bar 1 $= S_1$, in bar 2 $= S_2$, etc., and assume each bar to be in tension. Now, by resolving the forces at each joint into components parallel and perpendicular, respectively, to the line of action of the reaction at the given joint and applying

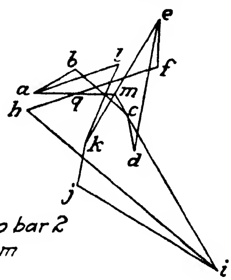


FIG. 370.

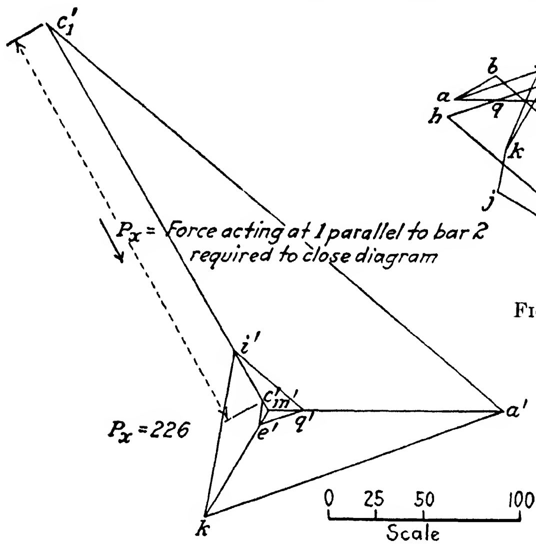


FIG. 369.

the appropriate equation of equilibrium, the following equations may be written:

$$
\begin{aligned}
S_1 \sin \alpha_1 &= P_1 \sin \theta_1 - S_2 \sin (180^\circ - 2\beta - \alpha_1)\\
S_2 \sin \alpha_2 &= P_2 \sin \theta_2 - S_3 \sin (180^\circ - 2\beta - \alpha_2)\\
S_3 \sin \alpha_3 &= P_3 \sin \theta_3 - S_4 \sin (180^\circ - 2\beta - \alpha_3)\\
S_4 \sin \alpha_4 &= P_4 \sin \theta_4 - S_5 \sin (180^\circ - 2\beta - \alpha_4)\\
S_5 \sin \alpha_5 &= P_5 \sin \theta_5 - S_6 \sin (180^\circ - 2\beta - \alpha_5)\\
S_6 \sin \alpha_6 &= P_6 \sin \theta_6 - S_1 \sin (180^\circ - 2\beta - \alpha_6)
\end{aligned}
$$

The foregoing equations may be written as follows:

$$S_1 = \frac{P_1 \sin \theta_1}{\sin \alpha_1} - S_2 \frac{\sin (180^\circ - 2\beta - \alpha_1)}{\sin \alpha_1} = A - aS_2$$

$$S_2 = \frac{P_2 \sin \theta_2}{\sin \alpha_2} - S_3 \frac{\sin (180^\circ - 2\beta - \alpha_2)}{\sin \alpha_2} = B - bS_3$$

$$S_3 = \frac{P_3 \sin \theta_3}{\sin \alpha_3} - S_4 \frac{\sin (180° - 2\beta - \alpha_3)}{\sin \alpha_3} = C - cS_4$$

$$S_4 = \frac{P_4 \sin \theta_4}{\sin \alpha_4} - S_5 \frac{\sin (180° - 2\beta - \alpha_4)}{\sin \alpha_4} = D - dS_5$$

$$S_5 = \frac{P_5 \sin \theta_5}{\sin \alpha_5} - S_6 \frac{\sin (180° - 2\beta - \alpha_5)}{\sin \alpha_5} = E - eS_6$$

$$S_6 = \frac{P_6 \sin \theta_6}{\sin \alpha_6} - S_1 \frac{\sin (180° - 2\beta - \alpha_6)}{\sin \alpha_6} = F - fS_1$$

Solving these equations gives the following:

$$S_1 = A - aB + abC - abcD + abcdE - abcdeF + abcdefS_1 \quad \text{(I)}$$
$$S_2 = B - bC + bcD - bcdE + bcdeF - bcdefS_1 \quad \text{(II)}$$
$$S_3 = C - cD + cdE - cdeF + cdefS_1 \quad \text{(III)}$$
$$S_4 = D - dE + deF - defS_1 \quad \text{(IV)}$$
$$S_5 = E - eF + efS_1 \quad \text{(V)}$$
$$S_6 = F - fS_1 \quad \text{(VI)}$$

Hence,

$$S_1 = \frac{A - aB + abC - abcD + abcdE - abcdeF}{1 - abcdef} \quad \text{(VII)}$$

Evidently if the product $abcdef$ = unity, which occurs when each of the terms a, b, c, d, e, and f equals + unity or − unity, Eq. (VII) does not give a solution, since S_1 either equals infinity, which is impossible, or 0/0, which is indeterminate.

The ring chosen is typical of any regular polygon. Hence, it follows that any ring having the form of a regular polygon with an even number of sides and with horizontal reactions at each joint is unstable when $\frac{\sin (180° - 2\beta - \alpha)}{\sin \alpha}$ = plus or minus unity in each case; *i.e.*, when $\alpha = 90° - \beta$ or $180° - \beta$ which occurs if each reaction is normal to the radius of the polygon at the joint where the reaction is applied or is radial.

Examination of the equations also shows that if, at two adjoining supports, the values of the angles α are, respectively, 0 and $180° - 2\beta$, the equations give, in general, inconsistent results and the ring is therefore unstable. For example, if $\alpha_1 = 0$, we obtain from the first equation in this article

$$S_2 = \frac{P_1 \sin \theta_1}{\sin (180° - 2\beta)}$$

Also, if $\alpha_2 = 180° - 2\beta$, we obtain

$$S_2 = \frac{P_2 \sin \theta_2}{\sin (180° - 2\beta)}$$

Both the foregoing results can be correct only when $P_1 \sin \theta_1 = P_2 \sin \theta_2$; hence, such a ring is, in general, unstable for this case also.

When $\alpha_1 = 0$, R_1 is parallel to bar 1; and when $\alpha_2 = 180° - 2\beta$, R_2 is parallel to bar 3. Hence, the truth of the statement is seen.

For a five-sided polygon, the following equations may be similarly derived:

$$\begin{aligned}
S_1 &= A - aB + abC - abcD + abcdE - abcdeS_1 \\
S_2 &= B - bC + bcD - bcdE + bcdeS_1 \\
S_3 &= C - cD + cdE - cdeS_1 \\
S_4 &= D - dE + deS_1 \\
S_5 &= E - eS_1
\end{aligned}$$

Hence,

$$S_1 = \frac{A - aB + abC - abcD + abcdE}{1 + abcde}$$

This equation is solvable except in the special case where any two adjoining values of α are respectively 0° and $180° - 2\beta$ or where $\alpha = 180° - \beta$ in all cases.

For any other odd-sided polygon, the equations would resemble the foregoing, the sign of the last term in the numerator of the expression for S_1 being always positive.

Note that, in order to have the product $abcde = -1$ for a ring with an odd number of sides and constant value of α, $\sin (180° - 2\beta - \alpha)$ must equal $-\sin (\alpha)$. Therefore, $180° - 2\beta - \alpha = -\alpha$, whence $\beta = 90°$, which is not possible for any regular polygon with an odd number of sides.

Consideration of the foregoing enables us to draw the following conclusions:

An equal-sided polygonal ring with more than three sides loaded with concentrated loads at the various apices, with reactions fixed in direction acting at each or at any apex, is in general *unstable* if:

a. The structure has an even number of sides and each horizontal reaction is either perpendicular or parallel to the radius

acting at the apex where the reaction is applied, since for this case a, b, c, d, e, and f each equals unity and hence their product equals unity.

b. The reactions at two adjoining apices are respectively parallel to alternate bars of the polygon.

c. The structure has an odd number of sides and each reaction is radial.

Otherwise, the ring is stable and determinate provided that the reactions are symmetrical as stated in the hypothesis.

With the ring-bar stresses determined, the reactions may be computed as follows:

Referring again to the hexagonal ring, shown in Fig. 367, we may now write:

$$\begin{aligned}
R_1 &= S_1 \cos \alpha_1 - S_2 \cos (180° - 2\beta - \alpha_1) - P_1 \cos \theta_1 \\
R_2 &= S_2 \cos \alpha_2 - S_3 \cos (180° - 2\beta - \alpha_2) - P_2 \cos \theta_2 \\
R_3 &= S_3 \cos \alpha_3 - S_4 \cos (180° - 2\beta - \alpha_3) - P_3 \cos \theta_3 \\
R_4 &= S_4 \cos \alpha_4 - S_5 \cos (180° - 2\beta - \alpha_4) - P_4 \cos \theta_4 \\
R_5 &= S_5 \cos \alpha_5 - S_6 \cos (180° - 2\beta - \alpha_5) - P_5 \cos \theta_5 \\
R_6 &= S_6 \cos \alpha_6 - S_1 \cos (180° - 2\beta - \alpha_6) - P_6 \cos \theta_6
\end{aligned}$$

For the case where the angles α are all equal, the stresses and reactions computed for a single load at one apex may be readily revised to give corresponding functions for loads at other apices, this method generally involving less work than that of solving the equations for loads at all apices. For this case, the constant $A = \dfrac{P_1 \sin \theta_1}{\sin \alpha_1}$, the constants B, C, D, E, and F each equals zero, while the constants a, b, c, d, e, f are equal and may be expressed by the single constant $a = \dfrac{\sin (60° - \alpha_1)}{\sin \alpha_1}$. It follows that for a hexagonal ring under above conditions

$$\begin{aligned}
S_1 &= \frac{A}{1 - a^6}\,* \\
S_6 &= -aS_1 \\
S_5 &= a^2S_1 = -aS_6 \\
S_4 &= -a^3S_1 = -aS_5 \\
S_3 &= a^4S_1 = -aS_4 \\
S_2 &= -a^5S_1 = -aS_3
\end{aligned}$$

* For a hexagonal ring with P_1 acting outward, S_1 is tension whenever α_1 is greater than 30°.

The following example illustrates the numerical work required in the solution of such a ring.

Problem: Determine the stresses in the bars of the hexagonal ring shown in Fig. 367, assuming that $\alpha_1 = \alpha_2$, etc., $= 40°$, that P_1 is the only force acting, and that $\theta_1 = 60°$.

Solution: For the assumed case the constants have the following values:

$$A = \left(\frac{\sin 60°}{\sin 40°}\right)P_1 = 1.347P_1$$
$$B = C = D = E = F = 0$$
$$a = b = c = d = e = f = \frac{\sin 20°}{\sin 40°} = 0.532$$

Therefore,

$$S_1 = \frac{1.347P_1}{1 - 0.0222} = 1.377$$

Substituting this value in the appropriate equations, we obtain the following results:

$$\begin{aligned}
S_6 &= -0.532S_1 = -0.732P_1 \\
S_5 &= -0.532S_6 = +0.389P_1 \\
S_4 &= -0.532S_5 = -0.207P_1 \\
S_3 &= -0.532S_4 = +0.110P_1 \\
S_2 &= -0.532S_3 = -0.0585P_1
\end{aligned}$$

The following expressions for the reactions may be readily derived from the reaction equations:

$$\begin{aligned}
R_1 &= S_1 \cos 40° - S_2 \cos 20° - P_1 \cos 60° \\
R_2 &= S_2 \cos 40° - S_3 \cos 20° \\
R_3 &= S_3 \cos 40° - S_4 \cos 20° \\
R_4 &= S_4 \cos 40° - S_5 \cos 20° \\
R_5 &= S_5 \cos 40° - S_6 \cos 20° \\
R_6 &= S_6 \cos 40° - S_1 \cos 20°
\end{aligned}$$

Substituting numerical values, we get the following results:

$$\begin{aligned}
R_1 &= 1.377P_1(0.767) + (0.0585P_1)0.940 - \frac{P_1}{2} = +0.612P_1 \\
R_2 &= (-0.0585P_1)(0.767) - (0.110P_1)(0.940) = -0.148P_1 \\
R_3 &= (+0.110\ P_1)(0.767) + (0.207P_1)(0.940) = +0.278P_1 \\
R_4 &= (-0.207\ P_1)(0.767) - (0.390P_1)(0.490) = -0.525P_1 \\
R_5 &= (+0.390\ P_1)(0.767) + (0.734P_1)(0.940) = +0.989P_1 \\
R_6 &= (-0.734\ P_1)(0.767) - (1.377P_1)(0.940) = -1.858P_1
\end{aligned}$$

These reactions may be checked either graphically or analytically by using their vertical and horizontal components. The numerical work involved in checking these results follows.

The angles made by the various forces with the horizontal are as follows:

$$\begin{array}{lll} P_1,\ 20°; & R_3,\ 20°; & R_6,\ 20°; \\ R_1,\ 40°; & R_4,\ 40°; & \\ R_2,\ 80°; & R_5,\ 80°. & \end{array}$$

The various components are therefore as follows, plus signs represent forces upward and to the right, and the actual directions of the forces as determined on previous page being used instead of the assumed directions.

$$\begin{array}{rl} P_1,\ \text{V.C.} &= +P_1 \sin 20° \\ \text{H.C.} &= +P_1 \cos 20° \\ R_1,\ \text{V.C.} &= -R_1 \sin 40° \\ \text{H.C.} &= +R_1 \cos 40° \\ R_2,\ \text{V.C.} &= +R_2 \sin 80° \\ \text{H.C.} &= +R_2 \cos 80° \\ R_3,\ \text{V.C.} &= -R_3 \sin 20° \\ \text{H.C.} &= -R_3 \cos 20° \\ R_4,\ \text{V.C.} &= -R_4 \sin 40° \\ \text{H.C.} &= +R_4 \cos 40° \\ R_5,\ \text{V.C.} &= +R_5 \sin 80° \\ \text{H.C.} &= +R_5 \cos 80° \\ R_6,\ \text{V.C.} &= -R_6 \sin 20° \\ \text{H.C.} &= -R_6 \cos 20° \end{array}$$

Hence,

$$\Sigma V = (P_1 - R_3 - R_6) \sin 20° - (R_1 + R_4) \sin 40° + (R_2 + R_5) \sin 80°$$

and

$$\Sigma H = (P_1 - R_3 - R_6) \cos 20° + (R_1 + R_4) \cos 40° + (R_2 + R_5) \cos 80°$$

Now,

$$\begin{array}{rl} P_1 - R_3 - R_6 &= P_1(1 - 0.278 - 1.858) = -1.136P_1 \\ R_1 + R_4 &= P_1(0.612 + 0.525) = +1.137P_1 \\ R_2 + R_5 &= P_1(0.148 + 0.989) = +1.137P_1 \\ \sin 20° = 0.342, & \cos 20° = 0.940 \\ \sin 40° = 0.643, & \cos 40° = 0.766 \\ \sin 80° = 0.985, & \cos 80° = 0.174 \end{array}$$

Therefore,

$$\Sigma V = [(-1.137 \times 0.342) - (1.137 \times 0.643) + (1.137 \times 0.985)]P_1 = 0$$

Also,

$$\Sigma H = [(-1.137 \times 0.940) + (1.137 \times 0.766) + (1.137 \times 0.174)]P_1 = 0$$

To check the moments, take moments about the apex at which the force P_1 is applied, assuming the sides of the hexagon each to equal unity. The following equations result, clockwise moments being used as plus:

$$
\begin{aligned}
\Sigma M = & -R_2 \sin 80°(\cos 60°) - R_2 \cos 80°(\sin 60°) \\
& +R_3 \cos 20°(2 \sin 60°) - R_4 \cos 40°(2 \sin 60°) \\
& -R_4 \sin 40° + R_5 \sin 80°(1 + \cos 60°) - R_5 \cos 80°(\sin 60°) \\
& -R_6 \sin 20°
\end{aligned}
$$

Now,

$$
\begin{aligned}
\sin 80° \cos 60° &= 0.492 \\
\cos 80° \sin 60° &= 0.150 \\
\cos 20° \sin 60° &= 0.815 \\
\cos 40° \sin 60° &= 0.665 \\
\sin 80°(1 + \cos 60°) &= 1.477
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Sigma M &= -[R_2(0.642) - R_3(1.630) + R_4(1.330 + 0.643) \\
&\quad +R_5(-1.477 + 0.150) + R_6(0.342)]P_2 \\
&= [-0.148(0.642) + 0.278(1.630) - 0.525(1.973) \\
&\quad +0.989(1.327) - 1.858(0.342)]P_1 = 0
\end{aligned}
$$

220. Stresses in a Polygonal Ring without Radial Bars. *Graphical Method.*—While the analytical method of determining stresses and reactions in a polygonal ring is comparatively simple although laborious, the graphical method that follows, based upon a method given by Jasinski in *Schweizerische Bauzeitung,* May 5, 1900, may be quicker, especially if there are outer forces at every point. The solution by this method, however, involves great precision in drafting, as the intersections are likely to be poor, and may consume considerable time. Like many graphical methods, it may often be used more advantageously to supplement the analytical method once the stress in any one bar is determined than for determining the stresses in all bars (see example in the case of the Schwedler dome given later).

Referring to a ring such as that shown in Fig. 367 having loads and reactions in the plane of the ring bars at each vertex, construct a stress diagram, assuming the stress in one of the bars to be known. Such a diagram is shown by Fig. 368 in which the stress in bar 2 is assumed as compression equal to mc, and the diagram is begun by first plotting the forces at joint II and continuing around the polygon in a clockwise direction. Such a diagram will not close unless the assumed stress is the correct stress, which is not likely to be the case. In the case under consideration, the error in closure equals cc_1, since the intersection of the line from b parallel to BC with that from m parallel to MC is at c_1 instead of c.

The diagram would, however, have closed had there been another force P_o equal to c_1c applied at joint I in direction c_1c. In other words, the stresses and reactions given by Fig. 368 are the correct values for the applied forces shown in Fig. 367 *with the addition of another force* acting at joint I and equal and parallel to c_1c. If we can find the bar stresses and reactions due to this single force therefore, we may correct all the values given By Fig. 368 and thereby obtain the correct stresses for the actual forces. To accomplish this, proceed as follows:

Assume the stress in bar 2 to have a known tension $c'm'$; draw a stress diagram (Fig. 369), omitting all the forces P_1, P_2, etc., and determine the force P_x acting at joint I, parallel to bar 2, required to close the diagram. The bar stresses and reactions given by this diagram when multiplied by the ratio between P_o and P_x give the values due to P_o, which may be added or subtracted, as the case may be, to the corresponding values as given by Fig. 368 to obtain the correct stresses and reactions due to the applied forces P_1 to P_6.

If preferred, after determining by the above method the correct stresses in one of the bars, a new stress diagram may be constructed from which the correct stresses may be found. For example, Fig. 368 assumes the stress in bar 2 to be $-mc$; Fig. 369 shows the stress in bar 2, due to P_x acting in the direction shown, to be $+m'c'$. It follows that the stress in bar 2, due to the force P_o which is needed to close the diagram of Fig. 368, and which acts in the opposite direction to P_x, has the following value:

$$-m'c' \times \frac{P_o}{P_x}$$

Hence, a new diagram, Fig. 370, which will give the correct stresses in all bars and the correct values of all reactions, may be drawn by starting with the corrected value of the stress in bar 2, this correction being made by increasing the first assumed value of the stress in bar 2 by the value given by the above equation.

221. Schwedler Dome.—Illustration of the application of the methods previously deduced to the determination of reactions and stresses in all bars of a Schwedler dome due to a single vertical load of unity acting at a given panel point of the upper

ring will now be given. The same methods are also applicable for the case of a horizontal force of unity at any panel point. Conclusions are also drawn for the case of equal loads at each panel point of the upper ring. The reactions and stresses may be computed in a similar manner for a load of unity applied at any panel point of the intermediate ring.

Problem: Compute all reactions and bar stresses for the Schwedler dome shown in Fig. 371 due to a downward load of unity acting at point B.

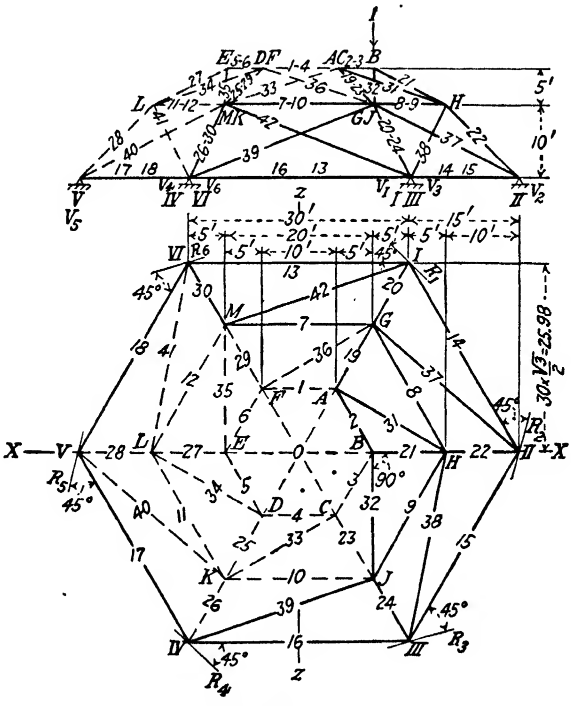


FIG. 371.

The lines of action of the various horizontal reactions R_1 to R_6 are as shown in the figure. The structure is statically determined since the horizontal reactions are properly fixed in direction and number, and the equation $3n = b + 3r - s$ is satisfied.

$$n = 18, \qquad b = 42, \qquad r = 6, \qquad s = 6$$

It should be observed that solution of the Schwedler dome and of space-framework towers is greatly simplified for the case of horizontal forces if the bar stresses are computed separately for the force at each joint,

COORDINATES OF ALL PANEL POINTS AT WHICH STRESSED BARS MEET
(+ signs indicate distances upward or to right referred to center of bottom ring)

Panel point	V ordinate, feet	X ordinate, feet	Z ordinate, feet
B	+15	+10	0
H	+10	+20	0
II	0	+30	0
A	+15	+ 5	+ 8.66
G	+10	+10	+17.32
I	0	+15	+25.98
M	+10	−10	+17.32
J	+10	+10	−17.32
III	0	+15	−25.98
IV	0	−15	−25.98
VI	0	−15	+25.98

From the data in the table above the following table may readily be prepared.

TRUE AND PROJECTED LENGTHS
(Bars in action under vertical load at B, bottom ring bars which are considered later being omitted)

Bar	V projection $= V_0$, feet	X projection $= X_0$, feet	Z projection $= Z_0$, feet	Length of bar, feet
2	0	5	8.66	10.0
7	0	20	0	20.0
8	0	10	17.32	20.0
9	0	10	17.32	20.0
19	5	5	8.66	$11.2 = \sqrt{125}$
20	10	5	8.66	$14.1 = \sqrt{200}$
21	5	10	0	11.2
22	10	10	0	14.1
24	10	5	8.66	14.1
30	10	5	8.66	14.1
31	5	15	8.66	18.0
32	5	0	17.32	$18.0 = \sqrt{325}$
37	10	20	17.32	$28.2 = \sqrt{800}$
38	10	5	25.98	28.2
39	10	25	8.66	28.2
42	10	25	8.66	28.2

this force being resolved into two components each acting in the plane of one of the sides; *e.g.*, if in Fig. 371 the horizontal force at B is resolved into two components, one acting along the line BC and the other along the line AB, and the bar stresses are computed separately for each component and then combined, the work is much reduced. It should be further noted that if at any point the wind force lies in the plane of one of the sides the bars in that side only are stressed and hence that side may be treated as a simple truss.

Solution: Application of theorem *a* shows that the stress equals zero in each bar shown dotted in the figure.

A convenient method of determining the true and projected length of the various stressed bars is to prepare a table of coordinate distances relative to some convenient origin, such, for example, as the center of the bottom ring. Such a table is given on page 494 for the structure under consideration, the origin being point O.

Stresses in Bars above Bottom Ring Bar.—These will be computed by application of the method of joints. All bars in which the stresses are unknown are assumed to be in tension and hence to have a positive stress.

Joint B

$$\Sigma v = 0 \qquad S_{21} \times \frac{5}{11.2} + S_{32} \times \frac{5}{18.0} + 1 = 0$$

$$\Sigma x = 0 \qquad S_{21} \times \frac{10}{11.2} - S_2 \times \frac{5}{10} = 0$$

$$\Sigma z = 0 \qquad S_{32} \times \frac{17.32}{18.0} - S_2 \times \frac{8.66}{10} = 0$$

Therefore,

$$S_2 = -2; \qquad S_{32} = -1.8, \qquad S_{21} = -1.12$$

Joint A

$$\Sigma v = 0 \qquad S_{31} \times \frac{5}{18} + S_{19} \times \frac{5}{11.2} = 0$$

$$\Sigma x = 0 \qquad S_{31} \times \frac{15}{18} + S_{19} \times \frac{5}{11.2} - 2 \times \frac{5}{10} = 0$$

Therefore,

$$S_{19} = -1.12 \qquad \text{and} \qquad S_{31} = +1.8$$

Joint J

$$\Sigma v = 0 \qquad 1.8 \times \frac{5}{18} + S_{39} \times \frac{10}{28.2} + S_{24} \times \frac{10}{14.1} = 0$$

$$\Sigma x = 0 \qquad S_9 \times \frac{10}{20} - S_{39} \times \frac{25}{28.2} + S_{24} \times \frac{5}{14.1} = 0$$

$$\Sigma z = 0 \qquad S_9 \times \frac{17.32}{20} - S_{39} \times \frac{8.66}{28.2} - S_{24} \times \frac{8.66}{14.1} - 1.8 \times \frac{17.32}{18.0} = 0$$

Therefore,

$$S_{39} = +\frac{28.2}{60}, \qquad S_9 = +1\frac{1}{2}, \qquad S_{24} = -\frac{56.4}{60}$$

Joint H

$$\Sigma v = 0 \qquad 1.12 \times \frac{5}{11.2} - 1.8 \times \frac{5}{18.0} + S_{22} \times \frac{10}{14.1} + S_{38} \times \frac{10}{28.2} = 0$$

$$\Sigma x = 0 \qquad 1.12 \times \frac{10}{11.2} - 1.8 \times \frac{15}{18} + S_{22} \times \frac{10}{14.1} - S_{38} \times \frac{5}{28.2} - 1\frac{1}{2} \times \frac{10}{20} - S_8 \times \frac{10}{20} = 0$$

$$\Sigma z = 0 \qquad -1\frac{1}{2} \times \frac{17.32}{20} + 1.8 \times \frac{8.66}{18.0} - S_{38} \times \frac{25.98}{28.2} + S_8 \times \frac{17.32}{20} = 0$$

Therefore,

$$S_{38} = -\frac{28.2}{20}, \qquad S_{22} = +\frac{28.2}{40}, \qquad S_8 = -1$$

Joint G

$$\Sigma v = 0 \qquad 1.12 \times \frac{5}{11.2} + S_{37} \times \frac{10}{28.2} + S_{20} \times \frac{10}{14.1} = 0$$

$$\Sigma x = 0 \qquad 1.12 \times \frac{5}{11.2} + S_{37} \times \frac{20}{28.2} + S_{20} \times \frac{5}{14.1} - 1 \times \frac{10}{20} - S_7 = 0$$

$$\Sigma z = 0 \qquad 1.12 \times \frac{8.66}{11.2} - S_{37} \times \frac{17.32}{28.2} + S_{20} \times \frac{8.66}{14.1} + 1 \times \frac{17.32}{20} = 0$$

Therefore,

$$S_{20} = -\frac{28.2}{20}, \qquad S_{37} = +\frac{28.2}{20}, \qquad S_7 = +\frac{1}{2}$$

Joint M

$$\Sigma v = 0 \qquad S_{30} \times \frac{10}{14.1} + S_{42} \times \frac{10}{28.2} = 0$$

$$\Sigma x = 0 \qquad S_{30} \times \frac{5}{14.1} - \frac{1}{2} - S_{42} \times \frac{25}{28.2} = 0$$

Therefore,

$$S_{42} = -\frac{28.2}{60}, \qquad S_{30} = \frac{14.1}{60}$$

From the bar stresses already determined, the vertical reactions and the horizontal components of the forces applied through the rib and diagonal bars to the apices of the lower ring bars may now be determined.

These components are shown in the following table:

STRESS COMPONENTS IN LOWER RIB AND DIAGONAL BARS OF SCHWEDLER DOME

Bar	Stress	Components parallel to V-axis	Components parallel to X-axis	Components parallel to Z-axis
		Signs indicate character of bar stress		
20	$-\frac{28.2}{20}$	-1.0	$-\frac{1}{2}$	-0.866
22	$+\frac{28.2}{40}$	$+\frac{1}{2}$	$+\frac{1}{2}$	0
24	$-\frac{56.4}{60}$	$-\frac{2}{3}$	$-\frac{1}{3}$	-0.577
30	$+\frac{14.1}{60}$	$+\frac{1}{6}$	$+\frac{1}{12}$	$+0.144$
42	$-\frac{28.2}{60}$	$-\frac{1}{6}$	$-\frac{5}{12}$	-0.144
37	$+\frac{28.2}{20}$	$+\frac{1}{2}$	$+1.0$	$+0.866$
38	$-\frac{28.2}{20}$	$-\frac{1}{2}$	$-\frac{1}{4}$	-1.300
39	$+\frac{28.2}{60}$	$+\frac{1}{6}$	$+\frac{5}{12}$	$+0.144$

The values in the following tables may now be readily computed.

VERTICAL AND HORIZONTAL COMPONENTS OF FORCES APPLIED BY RIB BARS AND DIAGONALS AT APICES OF LOWER RING BAR OF SCHWEDLER DOME

Apex	Vertical component + downward	X component + to right	Z component + to top of sheet
I	$V_1 = -1 - \frac{1}{6} = -1\frac{1}{6}$	$\frac{5}{12} + \frac{1}{2} = +\frac{11}{12}$	$0.144 + 0.866 = +1.010$
II	$V_2 = +\frac{1}{2} + \frac{1}{2} = +1$	$-1 - \frac{1}{2} = -1\frac{1}{2}$	$+0.866$
III	$V_3 = -\frac{1}{2} - \frac{2}{3} = -1\frac{1}{6}$	$-\frac{1}{4} + \frac{1}{3} = +\frac{1}{12}$	$-1.300 - 0.577 = -1.877$
IV	$V_4 = +\frac{1}{6}$	$+\frac{5}{12}$	$+0.144$
V	$V_5 = 0$	0	0
VI	$V_6 = +\frac{1}{6}$	$+\frac{1}{12}$	-0.144
Σ	-1	0	0

The values of the totals in previous table show that for the entire structure $\Sigma V = 0$, $\Sigma X = 0$, and $\Sigma Z = 0$ and furnish to that extent a check on the stresses.

Another check is found by applying ΣM about a vertical axis since the summation of the moments of the horizontal components about any vertical axis should also equal zero. The horizontal components are shown in Fig. 372.

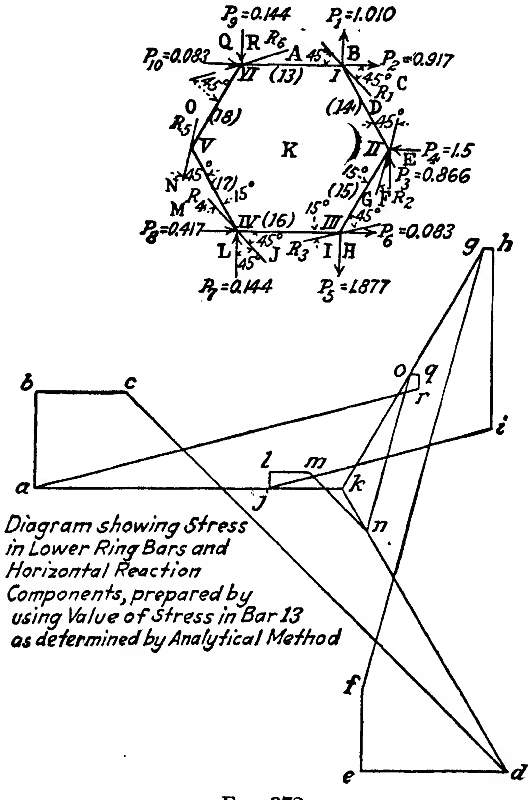


FIG. 372.

For convenience, the moment about a vertical axis passing through I may be taken. The resulting expression is as follows:

$$0.866 \times 15 - 1\tfrac{1}{2} \times 25.98 + \tfrac{1}{2} \times 2 \times 25.98 = 0$$

The vertical forces may also be checked by taking moments of the reactions and the applied load about any horizontal axis such as that parallel to the ZZ-axis passing through 0. This gives $-2\tfrac{2}{3} \times 15 + 1(10 + 30) = 0$.

Stresses in Lower Ring Bar.—The horizontal forces applied to the apices of the lower ring are shown in Fig. 372. The bar stresses and reactions may be computed by the analytical method or by the graphical method, which is quicker, but somewhat less precise.

For illustrative methods the complete analytical computations follow for the case under consideration. The reader will note that the actual compu-

tations are comparatively simple if arranged in the tabular form shown which is applicable to any case.

The graphical method has already been illustrated. A complete graphical stress diagram using the value of the stress in bar 13 as determined by the analytical computations is given in Fig. 372 and will be found to give the same results as the analytical method with considerably less labor, although great care must be used in the drawing to ensure closure.

CONSTANTS USED IN COMPUTATIONS OF SCHWEDLER DOME

$\alpha = 45°$, $\sin \alpha = 0.707$, $a = \dfrac{\sin 60° - \alpha}{\sin \alpha} = 0.366$, $a^6 = 0.0024$.

$1 - a^6 = 0.9976$

Joint	Force*	$\theta°$	$\sin \theta$	$\cos \theta$	$\dfrac{\sin \theta}{\sin \alpha}$	$A = P\dfrac{\sin \theta}{\sin \alpha}$	$\dfrac{A}{1 - a^6}$	$P \cos \theta$
I	$P_1 =$ 1.010	135	0.707	−0.707	1.000	1.010	1.012	−0.714
	$P_2 =$ 0.917	45	0.707	0.707	1.000	0.917	0.920	0.648
II	$P_3 =$ −0.866	15	0.259	0.966	0.366	−0.317	−0.318	−0.836
	$P_4 =$ −1.500	105	0.966	−0.259	1.366	−2.049	−2.052	0.388
III	$P_5 =$ 1.877	75	0.966	0.259	1.366	2.564	2.570	0.486
	$P_6 =$ 0.083	165	0.259	−0.966	0.366	0.031	0.031	−0.081
IV	$P_7 =$ −0.144	135	0.707	−0.707	1.000	−0.144	−0.145	0.102
	$P_8 =$ −0.417	45	0.707	0.707	1.000	−0.417	−0.418	−0.294
V	0	0	0	1.000	0	0	0	0
	0	0	0	1.000	0	0	0	0
VI	$P_9 =$ −0.144	75	0.966	0.259	1.366	−0.197	−0.198	−0.037
	$P_{10} =$ −0.083	165	0.259	−0.966	0.366	−0.031	−0.031	+0.080

* The signs in this column are the signs that should be used in the second terms of the equations for the bar stresses if the various components at the joints are substituted for P_1, P_2, etc., (Fig. 367), their directions being as shown in Fig. 372.

From values in preceding and following tables the following values for the reactions may be readily obtained. Positive signs correspond to reactions causing clockwise moments.

$$
\begin{aligned}
R_1 &= 2.245 + 3.296 + 0.714 - 0.648 = +5.607\\
R_2 &= -2.411 - 2.770 + 0.836 - 0.388 = -4.733\\
R_3 &= 2.027 + 0.721 - 0.486 + 0.081 = +2.343\\
R_4 &= -0.528 - 0.490 - 0.102 + 0.294 = -0.826\\
R_5 &= +0.358 + 1.343 = +1.701\\
R_6 &= -0.983 - 3.067 + 0.037 - 0.080 = -4.093
\end{aligned}
$$

STRESS COMPUTATIONS OF SCHWEDLER DOME

Joint	Force	Stress in bars $a = 0.366$					
		S_{13}	S_{14}	S_{15}	S_{16}	S_{17}	S_{18}
		$\frac{A}{1-a^6}$	$-aS_{15}$	$-aS_{16}$	$-aS_{17}$	$-aS_{18}$	$-aS_{13}$
I	P_1	+1.012	−0.007	+0.018	−0.050	+0.135	−0.370
	P_2	+0.920	−0.006	+0.017	−0.045	+0.124	−0.337
		$-aS_{14}$	$\frac{A}{1-a^6}$	$-aS_{16}$	$-aS_{17}$	$-aS_{18}$	$-aS_{13}$
II	P_3	+0.116	−0.318	+0.002	−0.006	+0.015	−0.042
	P_4	+0.751	−2.052	+0.013	−0.037	+0.101	−0.275
		$-aS_{14}$	$-aS_{15}$	$\frac{A}{1-a^6}$	$-aS_{17}$	$-aS_{18}$	$-aS_{13}$
III	P_5	+0.344	−0.939	+2.570	−0.017	+0.045	−0.126
	P_6	+0.004	−0.011	+0.031	−0.000	+0.000	−0.001
		$-aS_{14}$	$-aS_{15}$	$-aS_{16}$	$\frac{A}{1-a^6}$	$-aS_{18}$	$-aS_{13}$
IV	P_7	+0.007	−0.019	+0.053	−0.145	+0.001	−0.003
	P_8	+0.020	−0.056	+0.152	−0.417	+0.003	−0.007
V	0	0	0	0	0	0	0
	0	0	0	0	0	0	0
		$-aS_{14}$	$-aS_{15}$	$-aS_{16}$	$-aS_{17}$	$-aS_{18}$	$\frac{A}{1-a^6}$
VI	P_9	+0.001	−0.004	+0.010	−0.026	+0.072	−0.198
	P_{10}	+0.000	−0.000	+0.001	−0.004	+0.011	−0.031
Total stresses in bars	....	S_{13}	S_{14}	S_{15}	S_{16}	S_{17}	S_{18}
		+3.175	−3.412	+2.867	−0.747	+0.507	−1.390

REACTION COMPUTATIONS OF SCHWEDLER DOME

Bar	S = stress in bar	$S \cos \alpha$ ($\cos \alpha = 0.707$)	$S \cos (60° - \alpha)$ $\cos (60° - \alpha) = 0.966$
13	+3.175	+2.245	+3.067
14	−3.412	−2.411	−3.296
15	+2.867	+2.027	+2.770
16	−0.747	−0.528	−0.721
17	+0.507	+0.358	+0.490
18	−1.390	−0.983	−1.343

The foregoing reactions may be checked by taking moments about a vertical axis passing through center of structure. Inasmuch as in that case the lever arms are all equal, the moment will equal zero if the algebraic sum of the reactions equal zero, which is seen to be true within the limits of precision warranted by the numerical work.

Stresses Due to Vertical Load of Unity at Each Panel Point of Upper Ring.—Inasmuch as the dome under consideration is symmetrical, the stresses in any bar due to a vertical load of unity at any other upper panel point than that already considered may be readily written. If the stress for each bar is written for vertical loads at each of the upper panel points and these stresses are then combined, the following results will be obtained:

Vertical reaction at each point of support	= unity
Horizontal reaction at each point of support	= zero
Vertical component in each rib bar	= unity
Stress in each diagonal	= zero
Stress in each upper ring bar	= -2
Stress in each intermediate ring bar	= $+1$
Stress in each lower ring bar	= $+1$

The results above are in accordance with what might be expected.

The foregoing example is one in which the computations are simplified by the fact that many of the bar stresses are zero. This is not so in all space frameworks, and in such cases the computations are much more laborious. Were the lower ring bars omitted, the stresses in all other bars and the vertical reactions would not be changed; but the horizontal reactions would be modified by the amount of stress that the lower ring bars would otherwise carry, and the points of support would have to be capable of carrying the outward thrusts of the rib bar stresses, which might involve considerable expense in the case, say, of a dome supported on steel columns.

Problems

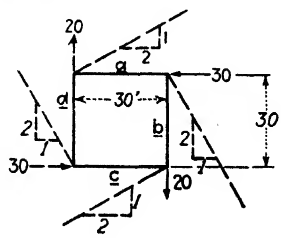
(In all these problems the line of action of a horizontal component is shown by a dotted line.)

78. Compute all reactions and stresses in each bar of this ring due to forces shown.

79. *a.* Show that this structure is statically determined.

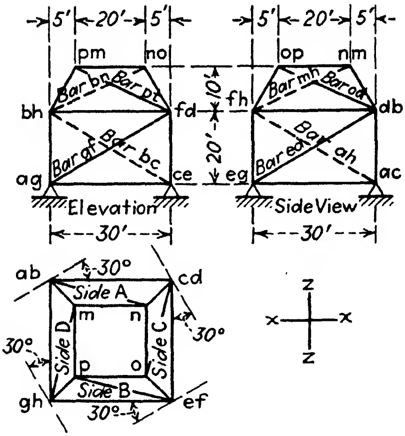
b. Mark with X all bars in which a horizontal load of unity acting at n in direction XX causes zero stress.

c. Determine direction and magnitude of each vertical reaction due to load above.

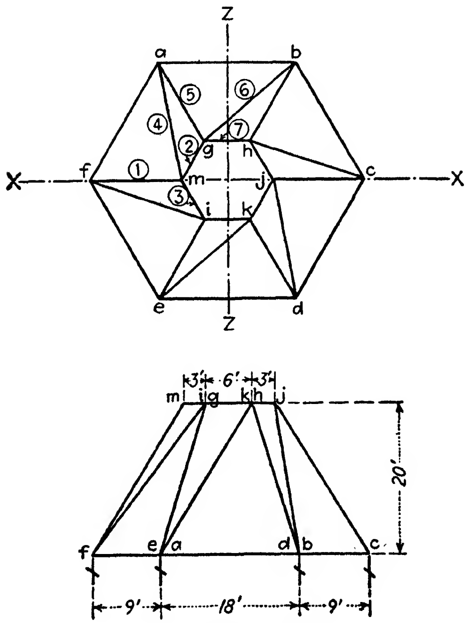


PROB. 78.

80. The structure shown is a steel tower supporting a water tank. The wind pressure on the tank gives a horizontal force of 10,000 lb. acting to the right at each of the vertices of the top ring and in the direction



PROB. 79.



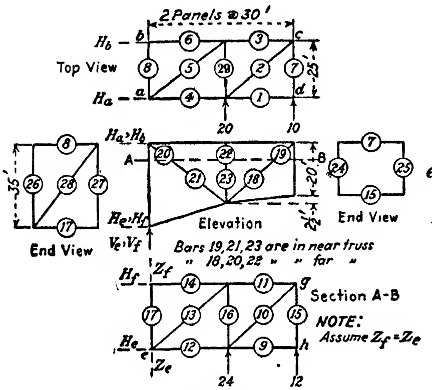
PROB. 80.

XX. Supports a, c, and e are so constructed that the reactions at these points can have no horizontal components. At the other points of support, the horizontal reactions may have components parallel to both XX and ZZ. All points of support are capable of taking vertical forces.

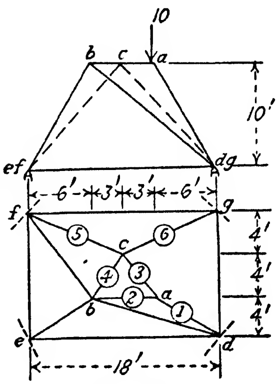
a. Show that the structure is statically determined.

b. Determine the stress or any component thereof in bars 1, 2, 3, 4, 5, 6, and 7.

81. The structure shown in the figure is the projecting end of a cantilever bridge subjected to the horizontal wind forces shown in the top view. Horizontal reactions act along the dotted lines shown in the plan. Show that the structure is statically determinate if $Z_e = Z_f$, and compute the stress in bar 13.



PROB. 81.



PROB. 82.

82. Compute the stresses in bars 1, 2, 3, 4, 5, and 6 of this structure and the vertical components of all reactions. Give character of the stress and direction of the reaction in each case.

83. Determine character and magnitude of stress (either component) as indicated below for bars of structure shown in Fig. 371.

a. Bars 2, 21, and 32 due to a horizontal force of unity acting at at point B in the direction of the Z-axis and toward the top of the sheet.

b. Bars 2, 21, and 32 due to a horizontal force of unity acting to the right at point B.

c. All bars meeting at point H due to a downward vertical load of unity applied at point H.

d. The lower ring bars due to a horizontal force of unity applied at point J and acting parallel to Z-axis toward the top of the sheet (in plan), using a combination of analytical and graphical methods.

e. Make a sketch showing which bars are brought into action by a vertical force of unity acting at point H.

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CHAPTER XVII

MOVABLE BRIDGES

222. Movable Bridges. General.—The term *movable bridge* is applied to all bridges that may be temporarily changed in position to allow passage of vessels in the streams that they span. Such bridges may be classified in the following six general divisions:

1. Swing Bridges.
   a. Center-supported (center-pivoted or rim-bearing).
   b. End-supported.
2. Bascule Bridges.
   (Bridges that may be revolved vertically about one end.)
3. Direct Lift Bridges.
   (Bridges that may be lifted bodily.)
4. Retractile Bridges.
   (Bridges that may be moved in a horizontal plane.)
5. Ferry or Transporter Bridges.
   (Bridges supporting a movable car that transports vehicles or passengers.)
6. Pontoon Bridges.
   (Bridges built on boats.)

It is the purpose of this chapter to deal chiefly with the stresses in swing-bridge trusses. The stresses in lift bridges and single-leaf bascule spans can be computed by the methods used for end-supported spans except as noted in the next article. The other types of movable bridges are too infrequent to require special treatment. The design of the machinery for movable bridges will not be considered as a thorough discussion of this subject would require a large amount of space and it is well covered in the references given at the end of the chapter.

223. Stresses in Bascule Bridges.—Bascule bridges may consist of one leaf which when closed acts as a simple end-supported

span so far as the live load is concerned, or they may be made up of two leaves which are connected at the center, when the bridge is closed, by a lock that can transmit shear but not bending moment, thus giving a structure that acts under live load like a three-hinged arch. For double-leaf bascule spans the reactions at the connecting ends when the bridge is closed can be computed by the methods applied to other statically indeterminate structures.

Counterweights are required to balance the dead weight, in all bascule spans, but exact balance cannot be obtained in all positions of the structure owing to the changing position of the operating strut and other movable parts and to the fact that the bridge floor may carry snow or ice or other temporary weights. It is desirable to have a small upward reaction at the free end, and such a reaction should be allowed for in the design; a hook should also be furnished at this end to carry uplift.

The maximum dead stresses may occur with the bridge partly or fully opened or with the bridge closed. The following rules for determining the maximum dead stress in any member may be applied:

Let S_h = dead stress in any member, bridge closed with reaction at free end equal to zero.

S_v = dead stress in same member, bridge standing at 90° to closed position.

S = maximum dead stress in same member.

θ = angle of opening between closed bridge and position of bridge at which maximum stress occurs in the given member.

a. If S_h and S_v are of the same character, the maximum stress and the angle at which it occurs will be given by the following expressions:

$$S = \sqrt{S_h^2 + S_v^2}$$

$$\tan \theta = \frac{S_v}{S_h}$$

b. If S_h and S_v are unlike in character, the maximum stress equals the larger of the two values S_h and S_v.*

* See article entitled Maximum Stresses in Bascule Bridges, by Pagon, *Trans. Am. Soc. C. E.*, Vol. LXXVI, p. 73.

224. Types of Girders and Trusses for Swing Bridges and Equations for Reactions.—Main girders and trusses of swing bridges are statically determined with respect to the outer forces when the bridge is open but are usually statically undetermined for the condition that exists when the bridge is closed and subjected to live loads. The girders of plategirder bridges under the latter condition generally act as continuous girders supported at three points. Trusses may, however, be constructed according to any one of the following types:

A. Continuous—supported at three points.
B. Continuous—supported at four points.
C. Partially continuous—supported at four points.
D. Discontinuous—supported at four points.

The difference between types B and C lies in the fact that the latter type is so constructed that moment but not shear can be transmitted across the central panel, thus producing a condition similar to that described for cantilever trusses in Art. 125. Type C is illustrated by Fig. 374 and is the type generally employed in the United States for trusses supported at four points. Its principal advantage over type B, also supported at four points, is that it cannot be loaded in such a manner as to cause uplift at either of the central points of support where provision for resisting uplift cannot readily be made. Type D can be readily constructed by providing adjustment in the top-chord eyebars at the connection to the tower posts by means of oblong holes in the eyebars. Such trusses have been used for several important bridges in the United States but have proved uneconomical to operate because of the expense required to raise the ends when closing the bridge. Serious wear may also occur in the adjustable members.

If the girders or trusses correspond to the conditions assumed in developing the three-moment equation, *i.e.*, if their moments of inertia and moduli of elasticity are constant throughout their length, the reactions may be accurately computed for types A and B by the use of the three-moment equation, the labor of calculation being much simplified by the use of the special equations and tables of this chapter. For type C, special formulas are derived in the following article. Type D is statically determined.

225. Points of Support for Swing Bridges.—The points of support of the main girders or trusses are usually upon cross-

girders. In the class A type, one such girder is required, the cross-girder itself being supported on a pivot resting on the central pier and on wedges designed to support the live load and thus relieve the center pivot of all load except the dead load. In types B, C, and D, two cross-girders are needed, these girders being supported either directly upon a circular girder or upon other girders so arranged as to distribute the reactions more uniformly over the circular girder than would otherwise be possible. The circular girder is itself supported upon a ring of conical rollers running upon a track supported upon the central pier.

226. Determination of Reactions on a Partially Continuous Girder.—For this case which is illustrated by Fig. 374 there are

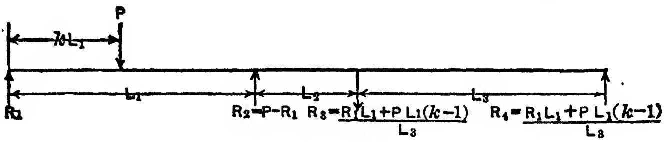


FIG. 373.

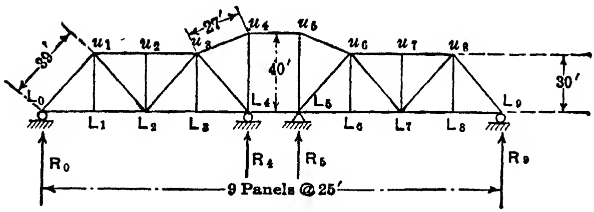


FIG. 374.

five unknown reactions. All these, however, can be expressed in terms of R_1 by applying the equations of statics accompanied by the equation of condition, *viz.*, that the shear in the center panel equals zero. The resulting values of the reactions due to a vertical force are shown in Fig. 373.

The internal work in this girder is given by the following equations, it being assumed that E and I are constant throughout:

$$W = \frac{1}{2EI}\int_0^{kL_1}(R_1x)^2dx + \frac{1}{2EI}\int_0^{L_1(1-k)}[R_1kL_1 + (R_1 - P)x]^2dx + \frac{1}{2EI}\int_0^{L_2}\left[\left(\frac{R_1L_1 + PL_1(k-1)}{L_3}\right)(L_3)\right]^2 dx + \frac{1}{2EI}\int_0^{L_3}\left(\frac{R_1L_1 + PL_1(k-1)}{L_3}\right)^2 x^2dx$$

Determining the value of $\delta W/\delta R_1$ from foregoing equation and equating to zero give the following value for R_1:

$$R_1 = P(1 - k)\frac{\frac{L_1}{6}(1 - k)(k + 2) + L_2 + \frac{L_3}{3}}{\frac{L_1}{3} + L_2 + \frac{L_3}{3}}$$

For the special case in which $L_1 = L_3 = nL_2$ where n equals the number of panels in each arm of the truss, we may substitute L for L_1 and L_3 and L/n for L_2, giving the following expressions for the reactions, the signs indicating the actual direction of the reactions as compared with the directions assumed in Fig. 373:

$$R_1 = P(1 - k) - \frac{Pn(k - k^3)}{4n + 6}$$

$$R_2 = Pk + \frac{Pn(k - k^3)}{4n + 6}$$

$$R_3 = -\frac{Pn(k - k^3)}{4n + 6}$$

$$R_4 = -\frac{Pn(k - k^3)}{4n + 6}$$

In view of the fact that swing-bridge trusses seldom have a constant moment of inertia the methods of Chap. XV may well be applied as a check after a preliminary design has been made by the method given in this and the preceding article.

227. Influence of End Supports upon Swing-bridge Reactions. The continuous and partially continuous girders hitherto considered have been assumed to be level and supported on level supports. It is evident that, if this condition exists when the bridge is closed, upon opening the bridge the trusses will deflect, the ends dropping below the level of the end supports. Consequently, when the bridge is again closed the ends will have to be raised to reach their original level, the force required to accomplish this slightly exceeding the dead reactions that would exist when the truss is closed. If the ends are not raised, there will be no dead reactions at the ends when closed, the dead stresses being the same as when the bridge is open. If this latter condition exists, however, a partial live loading which would tend to produce negative reactions at the ends would cause the ends to rise unless latched down, a serious objection, especially for a multiple-track railroad bridge. It is common,

therefore, in the case of swing bridges to raise the ends when the bridge is closed, this being accomplished by means of levers and wedges, toggle joints, or hydraulic jacks. In case the ends are raised sufficiently, the reactions for the closed bridge for both live and dead loads will be given by the formulas already deduced. If the ends are latched down and not raised, the end dead reactions will be zero and the live reactions will be given by the formulas of this chapter. If the ends are neither latched down nor raised, the end dead reactions will again be zero, and the live reactions will be those given by the formulas, provided that none of the end reactions are negative and the negative live reaction in types B or C at either of the center piers does not exceed the positive dead reaction at that point. If these conditions are not fulfilled, the structure becomes a girder supported at two points if of the class A type and at three points if of types B and C, and the formulas are inapplicable. It should be added that the partially continuous truss has an advantage over the continuous girder upon four points of support in having the negative reaction due to live loads occur at one end, where it may be properly taken care of, instead of at the center support, where it might cause the drum to lift from the rollers.

228. Tables of Reactions for Continuous and Partially Continuous Girders Used for Swing Bridges.—The actual stresses in girders used for swing bridges may be computed by the methods already given for simple girders and trusses, once the reactions are determined. It may be noted that influence tables or influence lines may be employed to advantage.

In order to facilitate the computation of the reactions the following tables have been prepared for girders with equal panels. These will be found sufficient for many structures. For bridges not covered by these tables the formulas previously developed should be employed.

229. Maximum Stresses in Swing Bridge. *Assumed Conditions.*—To determine the maximum stresses in swing-bridge trusses, the stresses due to several conditions of loading must be determined. These conditions may be stated as follows:

a. Dead load with bridge open or closed and end reactions equal to zero plus allowance for impact due to motion.

b. Snow load, bridge open. The weight to be assumed for the snow depends upon the situation of the bridge. An allowance of

TABLE 1.—REACTIONS FOR UNIT LOAD—GIRDER CONTINUOUS OVER TWO EQUAL SPANS

Moment of inertia and modulus of elasticity assumed to be constant.

Positive signs indicate upward reactions.

Formulas used in deriving these values are determined by the three-moment equation and are as follows:

$$R_1 = (1 - k) - \left(\frac{k - k^3}{4}\right), \qquad R_3 = -\left(\frac{k - k^3}{4}\right), \qquad R_2 = 1 - (R_1 + R_3)$$

In this and following tables, k = distance of load from left support divided by length of span and n equals number of panels in one arm of truss.

k	R_1 +	R_2 +	R_3 −
	$n = 10$		
1/10	0.8752	0.1495	0.0247
2/10	0.7520	0.2960	0.0480
3/10	0.6318	0.4365	0.0683
4/10	0.5160	0.5680	0.0840
5/10	0.4062	0.6875	0.0937
6/10	0.3040	0.7920	0.0960
7/10	0.2108	0.8785	0.0893
8/10	0.1280	0.9440	0.0720
9/10	0.0572	0.9855	0.0427
Total.......	3.8812	5.7375	0.6187
	$n = 9$		
1/9	0.8615	0.1659	0.0274
2/9	0.7250	0.3278	0.0528
3/9	0.5926	0.4815	0.0741
4/9	0.4664	0.6228	0.0892
5/9	0.3484	0.7476	0.0960
6/9	0.2407	0.8519	0.0926
7/9	0.1454	0.9314	0.0768
8/9	0.0645	0.9821	0.0466
Total.......	3.4445	5.1110	0.5555
	$n = 8$		
1/8	0.8442	0.1866	0.0305
2/8	0.6914	0.3672	0.0586
3/8	0.5444	0.5362	0.0806
4/8	0.4062	0.6875	0.0937
5/8	0.2798	0.8154	0.0952
6/8	0.1680	0.9140	0.0820
7/8	0.0737	0.9776	0.0513
Total.......	3.0077	4.4845	0.4919

k	R_1 +	R_2 +	R_3 −
	$n = 7$		
1/7	0.8222	0.2128	0.0350
2/7	0.6487	0.4169	0.0656
3/7	0.4840	0.6035	0.0875
4/7	0.3324	0.7638	0.0962
5/7	0.1982	0.8893	0.0875
6/7	0.0860	0.9709	0.0569
Total......	2.5715	3.8572	0.4287
	$n = 6$		
1/6	0.7928	0.2477	0.0405
2/6	0.5926	0.4815	0.0741
3/6	0.4062	0.6875	0.0937
4/6	0.2407	0.8519	0.0926
5/6	0.1030	0.9606	0.0636
Total.......	2.1353	3.2292	0.3645
	$n = 5$		
1/5	0.7520	0.2960	0.0480
2/5	0.5160	0.5680	0.0840
3/5	0.3040	0.7920	0.0960
4/5	0.1280	0.9440	0.0720
Total.......	1.7000	2.6000	0.3000
	$n = 4$		
1/4	0.6914	0.3672	0.0586
2/4	0.4062	0.6875	0.0937
3/4	0.1680	0.9140	0.0820
Total.......	1.2656	1.9687	0.2343
	$n = 3$		
1/3	0.5926	0.4815	0.0741
2/3	0.2407	0.8519	0.0926
Total.......	0.8333	1.3334	0.1667

TABLE 2.—REACTIONS FOR UNIT LOAD—CONTINUOUS GIRDER WITH FOUR SUPPORTS AND EQUAL SIDE SPANS

Moment of inertia and modulus of elasticity assumed to be constant

Center span = $1/n$ side span.

Positive signs indicate upward reactions.

Formulas used in deriving these values are determined by the three-moment equation and are as follows:

$$R_1 = (1 - k) - \frac{(k - k^3)(2n)(n + 1)}{4n^2 + 8n + 3}, \qquad R_4 = \frac{(k - k^3)n}{4n^2 + 8n + 3}$$

$$R_2 = k + \frac{(k - k^3)(n)(2n^2 + 5n + 2)}{4n^2 + 8n + 3}, \qquad R_3 = -\frac{(k - k^3)n(2n^2 + 3n + 1)}{4n^2 + 8n + 3}$$

k	$R_1(+)$	$R_2(+)$	$R_3(-)$	$R_4(+)$
		$n = 10$		
1/10	0.8549	0.6166	0.4735	0.0021
2/10	0.7125	1.2018	0.9183	0.0040
3/10	0.5757	1.7243	1.3057	0.0056
4/10	0.4470	2.1531	1.6070	0.0069
5/10	0.3292	2.4567	1.7936	0.0078
6/10	0.2251	2.6036	1.8367	0.0079
7/10	0.1374	2.5628	1.7075	0.0074
8/10	0.0688	2.3027	1.3774	0.0060
9/10	0.0221	1.7922	0.8179	0.0035
Total	3.3727	17.4138	11.8376	0.0512
		$n = 9$		
1/9	0.8394	0.6285	0.4703	0.0025
2/9	0.6825	1.2181	0.9054	0.0048
3/9	0.5330	1.7301	1.2697	0.0067
4/9	0.3946	2.1257	1.5284	0.0081
5/9	0.2712	2.3660	1.6459	0.0087
6/9	0.1662	2.4126	1.5871	0.0084
7/9	0.0836	2.2261	1.3166	0.0069
8/9	0.0269	1.7683	0.7995	0.0042
Total	2.9974	14.4754	9.5229	0.0503
		$n = 8$		
1/8	0.8202	0.6433	0.4667	0.0031
2/8	0.6455	1.2368	0.8882	0.0058
3/8	0.4813	1.7318	1.2212	0.0080
4/8	0.3272	2.1316	1.4685	0.0096
5/8	0.2052	2.2287	1.4433	0.0094
6/8	0.1037	2.1316	1.2435	0.0081
7/8	0.0336	1.7387	0.7773	0.0051
Total	2.6167	11.8425	7.5087	0.0491

k	$R_1(+)$	$R_2(+)$	$R_3(-)$	$R_4(+)$
		$n = 7$		
1/7	0.7956	0.6615	0.4609	0.0038
2/7	0.5991	1.2581	0.8643	0.0072
3/7	0.4177	1.7250	1.1524	0.0096
4/7	0.2596	1.9975	1.2678	0.0106
5/7	0.1320	2.0107	1.1524	0.0096
6/7	0.0430	1.7001	0.7492	0.0063
Total..	2.2470	9.3529	5.6470	0.0471
		$n = 6$		
1/6	0.7635	0.6852	0.4537	0.0050
2/6	0.5391	1.2814	0.8296	0.0091
3/6	0.3385	1.7000	1.0499	0.0115
4/6	0.1738	1.8518	1.0369	0.0114
5/6	0.0570	1.6480	0.7128	0.0078
Total..	1.8719	7.1664	4.0829	0.0448
		$n = 5$		
1/5	0.7194	0.7169	0.4430	0.0067
2/5	0.4590	1.3046	0.7754	0.0117
3/5	0.2389	1.6338	0.8861	0.0134
4/5	0.0792	1.5754	0.6646	0.0101
Total..	1.4965	5.2307	2.7691	0.0419
		$n = 4$		
1/4	0.6553	0.7612	0.4262	0.0095
2/4	0.3485	1.3182	0.6819	0.0151
3/4	0.1174	1.4660	0.5966	0.0133
Total..	1.1212	3.5454	1.7047	0.0379
		$n = 3$		
1/3	0.5538	0.8272	0.3951	0.0141
2/3	0.1922	1.2840	0.4938	0.0176
Total..	0.7460	2.1112	0.8889	0.0317

TABLE 3.—REACTIONS FOR UNIT LOAD—PARTIALLY CONTINUOUS GIRDER WITH FOUR SUPPORTS AND EQUAL SIDE SPANS

Shear in center panel = 0.

Moment of inertia and modulus of elasticity assumed to be constant.

Center span = $1/n$ side span.

Positive signs indicate upward reactions.

Formulas used in deriving these values are derived by the method of least work and are as follows:

$$R_1 = (1 - k) - \frac{n(k - k^3)}{4n + 6} \qquad R_2 = k + \frac{n(k - k^3)}{4n + 6}$$

$$R_3 = -R_4 = (1 - k) - R_1 = \frac{n(k - k^3)}{4n + 6}$$

k	R_1 +	R_2 +	R_3 +	R_4 −
		$n = 10$		
1/10	0.8785	0.1215	0.0215	0.0215
2/10	0.7583	0.2417	0.0417	0.0417
3/10	0.6406	0.3593	0.0593	0.0593
4/10	0.5269	0.4730	0.0730	0.0730
5/10	0.4185	0.5815	0.0815	0.0815
6/10	0.3165	0.6835	0.0835	0.0835
7/10	0.2224	0.7776	0.0776	0.0776
8/10	0.1374	0.8626	0.0626	0.0626
9/10	0.0628	0.9372	0.0372	0.0376
Total	3.9619	5.0379	0.5379	0.5379
		$n = 9$		
1/9	0.8654	0.1346	0.0235	0.0235
2/9	0.7325	0.2675	0.0453	0.0453
3/9	0.6032	0.3968	0.0635	0.0635
4/9	0.4791	0.5209	0.0764	0.0764
5/9	0.3621	0.6379	0.0823	0.0823
6/9	0.2540	0.7460	0.0794	0.0794
7/9	0.1564	0.8436	0.0658	0.0658
8/9	0.0711	0.9289	0.0400	0.0400
Total	3.5238	4.4762	0.4762	0.4762
		$n = 8$		
1/8	0.8491	0.1509	0.0259	0.0259
2/8	0.7007	0.2993	0.0493	0.0493
3/8	0.5571	0.4428	0.0678	0.0678
4/8	0.4210	0.5789	0.0789	0.0789
5/8	0.2948	0.7052	0.0802	0.0802
6/8	0.1809	0.8191	0.0691	0.0691
7/8	0.0818	0.9182	0.0432	0.0432
Total	3.0854	3.9144	0.4144	0.4144

k	R_1 +	R_2 +	R_3 +	R_4 −
		$n = 7$		
1/7	0.8283	0.1717	0.0288	0.0288
2/7	0.6603	0.3397	0.0540	0.0540
3/7	0.4994	0.5006	0.0720	0.0720
4/7	0.3493	0.6507	0.0792	0.0792
5/7	0.2137	0.7863	0.0720	0.0720
6/7	0.0960	0.9040	0.0468	0.0468
Total	2.6470	3.3530	0.3528	0.3528
		$n = 6$		
1/6	0.8009	0.1991	0.0324	0.0324
2/6	0.6074	0.3926	0.0593	0.0593
3/6	0.4250	0.5750	0.0750	0.0750
4/6	0.2592	0.7407	0.0741	0.0741
5/6	0.1157	0.8842	0.0509	0.0509
Total	2.2082	2.7916	0.2917	0.2917
		$n = 5$		
1/5	0.7631	0.2369	0.0369	0.0369
2/5	0.5354	0.4646	0.0646	0.0646
3/5	0.3262	0.6738	0.0738	0.0738
4/5	0.1446	0.8554	0.0554	0.0554
Total	1.7693	2.2307	0.2307	0.2307
		$n = 4$		
1/4	0.7074	0.2926	0.0426	0.0426
2/4	0.4318	0.5682	0.0682	0.0682
3/4	0.1903	0.8097	0.0597	0.0597
Total	1.3295	1.6705	0.1705	0.1705
		$n = 3$		
1/3	0.6173	0.3827	0.0494	0.0494
2/3	0.2716	0.7284	0.0617	0.0617
Total	0.8889	1.1111	0.1111	0.1111

10 lb. per square foot is probably a reasonable one for the latitude of New York or Boston. A snow load should be used only for highway bridges and railroad bridges with solid floors.

c. Live load only; truss to be considered either continuous or partially continuous as its construction may warrant.

d. Dead load with bridge closed and with upward end reactions due to raising the ends, each equal to $1\frac{1}{2}$ times the maximum negative reaction due to live load including impact. This assumption for the reactions is to prevent end hammer by providing liberally for impact, effect of changes in temperature, and wear in end-lifting apparatus. The end-lifting apparatus should be so designed that neither end of the structure can be lifted a distance greater than the upward deflection at that end due to the application there of a concentrated load equal to the assumed value of the dead reaction.

e. Live load only; truss to be considered as supported at one end and at the center point of support adjoining this end, thus acting as a simple span. This condition may occur if the ends are not raised, after the bridge is closed, by design, carelessness, or because of breakdown in the end-lifting apparatus.[1]

The maximum stress for any given bar will be either that due to Cases *a* and *b* combined or that due to any reasonable combination of live, impact, and dead loading. Such combinations may be as follows:

1. Dead stress, Case *d*, and live stress with impact, Case *c*.
2. Dead stress without impact, Case *a*, and live stress with impact, Case *c*. This condition may exist if the ends are merely supported at the abutments when the bridge is closed and not actually raised and if the live load is so applied that no end uplift will occur. The designer should observe that a live load may be discontinuous and that hence a full panel load may be considered at either end with live panel loads at any or all other points of the structure, and that this end panel load may be sufficient to overcome uplift due to the other loads. The designer must use his judgment in determining whether the conditions described in this paragraph are likely to occur for the loading giving maximum stress in any given bar of the truss under consideration.

[1] If one end only is raised its normal amount, no dead reaction will occur at either end, the bridge being simply tilted.

3. Dead stress without impact, Case *a*, and live stress with impact, Case *e*. This condition may occur if the ends are not raised and if an unbalanced uplift exists at one end.

230. Specifications for Impact and Reversal of Stress.—Opinions as to the proper method of making allowances for impact and for reversal of stress due to motion of bridge vary.

The 1935 specifications of the American Railway Engineering Association give the following rules for railroad bridges:

Stresses in structural parts which vary with the movement of the span (as in the case of a bascule bridge) shall be increased 20 per cent as an allowance for impact. This impact allowance shall not be combined with the live load stresses.

Stresses in structural parts caused by the machinery or by forces applied for moving or stopping the span shall be increased 100 per cent as an allowance for impact.

The end floor beams of the moving span and the adjacent floor beams of the fixed spans shall be proportioned for a concentrated load of 75,000 lb. on each track, in addition to the specified live load and impact.

Structural members and their connections, subject to reversal of stress during the movement of the span, shall be proportioned as follows: Determine the resultant tensile stress and the resultant compressive stress, and increase each by 25 per cent of the smaller; then proportion the member to resist either increased resultant stress.

Secondary stresses occurring in connection with reversal of stress, and those in trusses of unusual form, shall be computed and provided for in proportioning.

231. Computation of Maximum Stresses in Swing Bridges by Approximate Method. *Illustration.*

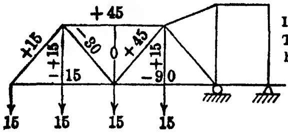


FIG. 375.

Problem: Compute the maximum stress in all bars of the truss shown in Fig. 374 for the following loads.[1] Assume trusses 20′0″ center to center. Bridge is a single track bridge.

[1] Note that a span as short as this would ordinarily be constructed of type A, Art. 224. Type C is used in this problem as it is slightly more complicated to compute and hence furnishes a better illustration of methods of computation.

Dead. 600 lb. per foot all on bottom chord, giving panel loads equal to 15,000 lb.

Snow. No snow load. Bridge has open floor.

Live. 2,000 lb. per foot on bottom chord, giving panel loads equal to 50,000 lb., and locomotive excess of 40,000 lb.

Solution:

Case a. The index stresses in 1,000-lb. units for all bars to left of U_3L_4 are shown in Fig. 375. Since truss is symmetrical stresses in bars to right of center need not be computed. The actual stresses in these bars are given in the table that follows. In determining these stresses, the end panel loads are assumed to equal the panel loads at intermediate points in order to provide for the weight of the end-lifting apparatus.

The stresses or their components in the other bars are given by the following computations:

Bars U_3U_4 and U_4U_5. Use method of moments with L_4 as origin.

$$\text{H.C.} = \frac{15(1 + 2 + 3 + 4)25}{40} = +93.75$$

Bar L_4L_5. Stress $= -93.75$

Bar U_4L_4. Stress = V.C. in U_3U_4

$$= -93.75 \times {}^{10}\!/_{25} = -37.50$$

Bar U_3L_4. Use method of moments with vertical section between L_3 and L_4 and origin at intersection of U_3U_4 and bottom chord which occurs at L_0.

$$\text{V.C.} = -15(1 + 2 + 3)\,{}^{25}\!/_{100} = -22.5$$

DEAD STRESSES, CASE *a*, UNITS OF 1,000 POUNDS

Bar	Index stress or computed component	Ratio	Actual stresses
L_0L_1, L_1L_2	-15.0	${}^{25}\!/_{30}$	$-$ 12.5
L_2L_3, L_3L_4	-90.0	${}^{25}\!/_{30}$	$-$ 75.0
U_4U_5	$+93.8$	....	+ 93.8
U_3U_4	$+93.8$	${}^{27}\!/_{25}$	+101.3
L_4L_5	-93.8	....	$-$ 93.8
U_4L_4	-37.5	....	$-$ 37.5
U_1U_2, U_2U_3	$+45.0$	${}^{25}\!/_{30}$	+ 37.5
L_0U_1	$+15.0$	${}^{39}\!/_{30}$	+ 19.5
U_1L_2	-30.0	${}^{39}\!/_{30}$	$-$ 39.0
L_2U_3	$+45.0$	${}^{39}\!/_{30}$	+ 58.5
U_3L_4	-22.5	${}^{39}\!/_{30}$	$-$ 29.3
U_1L_1, U_3L_3	$+15.0$	....	+ 15.0
U_2L_2	$-$ 0.0	....	0.0

Reactions at L_4 and L_5 are equal to one-half total dead load = 75; other reactions are zero.

Case b. This case need not be considered as snow load is negligible.

Case c. The truss is partially continuous, hence the values in Table 3 will be used in computing the reactions. The position of the loads for any given bar may be readily determined by use of the reactions given in the table, and such few computations as are necessary for this purpose will not generally be given.

The maximum values of uplifts and reactions will first be computed.

Maximum End Uplifts.—Inspection of table shows that maximum uplift at L_0 occurs with full load from L_5 to L_9 and with E at L_7. Its value = $0.171 \times 50 + 0.068 \times 40 = 11.27$.

This equals the maximum uplift at L_9 which will occur with full load from L_0 to L_4 with E at L_2. This is so small that a fraction of the full live panel load at the end would balance it, hence Case *c* may occur even if ends are not raised; that is, Case *a* and Case *c* may be combined.

No uplift will occur at points L_4 and L_5 since a load upon any portion of the structure will cause upward reactions at these supports.

Maximum Reactions.—The maximum gross upward reaction at L_0 will occur with loads from L_0 to L_4 and with E at L_0. Its value = $1.330 \times 50 + 25 + 40 = 131.5$. This equals the maximum gross reaction at L_9.

The maximum gross upward reaction at L_4 occurs with full load from L_0 to L_9 with E at L_4. Its value = $1.841 \times 50 + 50 + 40 = 182.0$. This equals the maximum upward gross reaction at L_5.

The loading and necessary computation for maximum stresses in all the bars of this truss for all possible combinations are given in the following table:

MAXIMUM LIVE STRESSES, CASE *c*, UNITS OF 1,000 POUNDS

Bar	Position of uniform load	Position of locomotive excess	All necessary stress computations R_0 = left reaction (+) when upward V.C. = vertical component (+) when tension H.C. = horizontal component (+) when tension S = maximum stress
L_0L_1 L_1L_2	L_0–L_4 L_5–L_9	L_1 L_7	$R_0 = +1.330 \times 50 + 0.707 \times 40 = +94.8$ $S = 94.8 \times {}^{25}/_{30} = +79.0$ $R_0 = -(0.171 \times 50 + 0.068 \times 40) = -11.3$ $S = -11.3 \times {}^{25}/_{30} = -9.4$
L_2L_3 L_3L_4	L_0–L_4 L_5–L_9	L_3 L_7	$R_0 = 1.330 \times 50 + 0.190 \times 40 = +74.1$ $S = +74.1 \times \frac{75}{30} - \frac{50 \times 75}{30} = +60.3$ $R_0 = -11.3$ $S = -11.3 \times {}^{75}/_{30} = -28.2$
U_3U_4	L_0–L_9	L_7	$R_0 = (1.330 - 0.170)50 - 0.068 \times 40 = +55.3$ H.C. $= +\frac{50 \times 6 \times 25 - 100 \times 55.28}{40} = +49.3$ $S = 49.3 \times {}^{27}/_{25} = +53.2$
U_4U_5	L_0–L_9	L_7	$+49.3$
L_4L_5	L_0–L_9	L_7	-49.3
U_4L_4	L_0–L_9	L_7	$-49.3 \times {}^{10}/_{25} = -19.7$
U_1U_2 U_2U_3	L_0–L_4 L_5–L_9	L_2 L_7	$R_0 = 1.330 \times 50 + 0.432 \times 40 = +83.8$ $S = \frac{83.78 \times 50 - 50 \times 25}{30} = -98.0$ $R_0 = -11.3$ $S = +11.3 \times {}^{50}/_{30} = +18.8$
L_0U_1	L_0–L_4	L_1	$R_0 = +1.330 \times 50 + 0.707 \times 40 = +94.8$ V.C. $= -94.8$ $S = 94.8 \times {}^{39}/_{30} = -123.2$
L_0U_1	L_5–L_9	L_7	$R_0 = -11.3$ V.C. $= +11.3$ $S = 11.3 \times {}^{39}/_{30} = +14.6$

MAXIMUM LIVE STRESSES, CASE *c*, UNITS OF 1,000 POUNDS.—(*Continued*)

Bar	Position of uniform load	Position of locomotive excess	All necessary stress computations R_0 = left reaction (+) when upward V.C. = vertical component (+) when tension H.C. = horizontal component (+) when tension S = maximum stress
U_1L_2	L_2–L_4	L_2	V.C. = $50(0.432 + 0.190) + 40 \times 0.432 = +48.4$
	L_0–L_1		$S = 48.4 \times {}^{39}/_{30} = +62.9$
			V.C. = $90(1.000 - 0.707) + 50 \times 0.170 = -34.9$
	L_6–L_9	L_1	$S = -34.9 \times {}^{39}/_{30} = -45.3$
L_2U_3	L_3–L_4	L_3	V.C. = $90 \times 0.190 = -17.1$
			$S = -17.1 \times {}^{39}/_{30} = -22.2$
	L_0–L_2	L_2	V.C. = $50(1.000 + 0.170 - 0.707) + 90(1.000 - 0.432) = +74.3$
	L_6–L_9		$S = 74.3 \times {}^{39}/_{30} = +96.6$
U_3L_4	L_0–L_3	L_3	V.C. = $50({}^{75}/_{100}) + 90({}^{75}/_{100}) = -105.0$
			$S = -105 \times {}^{39}/_{30} = -136.5$
U_1L_1	L_0–L_2	L_1	$+90$
U_3L_3	L_2–L_4	L_3	$+90$
U_2L_2	......	..	0

Case *d*. The maximum live end uplift has been found to be 11.27, and as impact = 45 per cent a dead reaction of $11.27 \times 1.45 \times 1\frac{1}{2} = 24.5$ should be used. The dead stresses may most readily be determined by subtracting the stresses due to this end reaction from the dead stresses already found. The necessary computations and final results are given in the following table:

DEAD STRESSES, CASE *d*, UNITS OF 1,000 POUNDS

Bar	Components due to reaction	Stresses due to end reaction	Stresses, Case *a*	Stresses, Case *d*
L_0L_1, L_1L_2	H.C. = $24.5 \times \frac{25}{30}$ = +20.4	+20.4	− 12.5	+ 7.9
L_2L_3, L_3L_4	H.C. = +61.2	+61.2	− 75.0	−13.8
U_3U_4	H.C. = $-24.5 \times \frac{100}{40}$ = −61.2	−65.7	+101.3	+35.6
U_4U_5	H.C. = −61.2	−61.2	+ 93.8	+32.6
L_4L_5	H.C. = +61.2	+61.2	− 93.8	−32.6
U_4L_4	V.C. = $+61.2 \times \frac{10}{25}$ = +24.5	+24.5	− 37.5	−13.0
U_1U_2, U_2U_3	H.C. = $-24.5 \times \frac{5}{3}$	−40.8	+ 37.5	− 3.3
L_0U_1	V.C. = −24.5	−31.8	+ 19.5	−12.3
U_1L_2	V.C. = +24.5	+31.8	− 39.0	− 7.2
L_2U_3	V.C. = −24.5	−31.8	+ 58.5	+26.7
U_3L_4	V.C. = 0	0.0	− 29.3	−29.3
U_1L_1, U_3L_3	V.C. = 0	0.0	+ 15.0	+15.0
U_2L_2	0	0.0	0.0	0.0

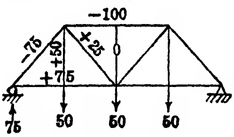


FIG. 376.

Case *e*. Index stresses for this case are shown in Fig. 376. Actual stresses are given in the following table:

MAXIMUM LIVE STRESSES, CASE *e*, UNITS OF 1,000 POUNDS

Bar	Index stress or computed component due to uniform live load	Ratio	Actual stress due to uniform load	Stress due to locomotive excess	Maximum stress
L_0L_1, L_1L_2, L_2L_3, L_3L_4	+ 75	$\frac{25}{30}$	+62.5	$+\frac{3}{4}(40)\frac{25}{30}$ = +25.0	+ 87.5
U_1U_2, U_2U_3	−100	$\frac{25}{30}$	−83.3	$-\frac{1}{2}(40)\frac{50}{30}$ = −33.3	−116.6
L_0U_1, U_3L_4	− 75	$\frac{39}{30}$	−97.5	$-\frac{3}{4}(40)\frac{39}{30}$ = −39.0	−136.5
U_1L_2	$+\frac{3}{4}50$ = +37.5	$\frac{39}{30}$	+48.8	$+\frac{1}{2}(40)\frac{39}{30}$ = +26.0	+ 74.8
L_2U_3	$-\frac{1}{4}50$ = −12.5	$\frac{39}{30}$	−16.3	$-\frac{1}{4}(40)\frac{39}{30}$ = −13.0	− 29.3
U_1L_1	....		+50.0	+40	+ 90.0
U_3L_3	....		+50.0	+40	+ 90.0

TABLE GIVING MAXIMUM STRESSES COMBINED WITH IMPACT*

Bar	Case a, dead	Case c, live + impact	Case d, dead	Case e, live + impact	Combination for maximum	Maximum stress
L_0L_1 L_1L_2	− 12.5	−9.4 × 1.25 = −11.8 +79.0 × 1.25 = +98.8	+ 7.9	+87.5 × 1.45 = +126.9	a and c a and e	− 24.3 +114.4
L_2L_3 L_3L_4	− 75.0	+60.3 × 1.25 = +75.4 −28.2 × 1.25 = −35.5	−13.8	+126.9	a and c c and d	−110.5 + 61.6
U_3U_4	+101.3	+53.2 × 1.25 = +66.5	+35.6	0	a and c	+167.8
U_4U_5	+ 93.8	+49.3 × 1.25 = +61.6	+32.6	0	a and c	+155.4
L_4L_5	− 93.8	−61.6	−32.6	0	a and c	−155.4
U_4L_4	− 37.5	−19.7 × 1.25 = −24.6	−13.0	0	a and c	− 62.1
U_1U_2 U_2U_3	+ 37.5	+18.8 × 1.25 = +23.5 −98.0 × 1.25 = −122.5	− 3.3	−116.6 × 1.45 = −169.1	a and c a and e	+ 61.0 −131.6
L_0U_1	+ 19.5	+14.6 × 1.25 = +18.2 −123.2 × 1.25 = −154.0	−12.3	−136.5 × 1.45 = −198.0	a and c a and e	+ 37.7 −178.5
U_1L_2	− 39.0	+62.9 × 1.25 = +78.6 −45.3 × 1.25 = −56.6	− 7.2	+74.8 × 1.45 = +108.5 −29.3 × 1.45 = −42.5	c and d a and c	+ 71.4 − 95.6
L_2U_3	+ 58.5	−22.2 × 1.25 = −27.7 +96.6 × 1.25 = +120.7	+26.7	−130.9 − 55.7	a and c c and d	+179.2 − 1.00
U_3L_4	− 29.3	−136.5 × 1.25 = −170.6	−29.3	−136.5 × 1.45 = −197.9	a and e	−227.2
U_2L_3 U_1L_1	+ 15.0	+90 × 1.93 = +173.7	+15.0	+157.5	a and c c and d	+188.7
U_2L_2	0.0	0.0	0.0	0.0		0.0
R_0 and R_9	0.0	+131.5 × 1.25 = 164.4	24.5	140 × 1.45 = 203.0	e	203.0
R_4	75	182.0 × 1.25 = 227.5	50.5	165.0 × 1.45 = 239.2	a and c	314.2

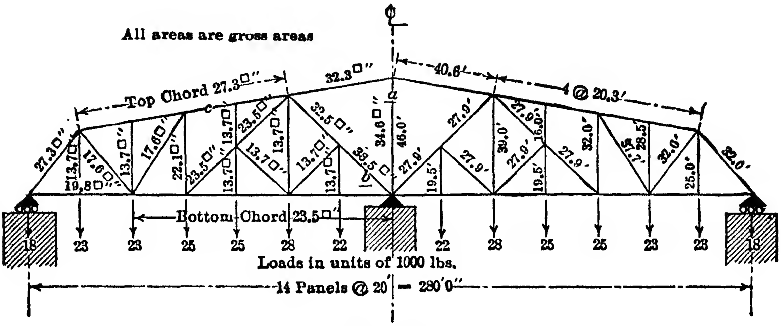
* Live impact by formulas in Art. 16 which give following values for main members and reactions
Case c Truss continuous L = 225 ft. I = $^{100}/_{20}$ + $^{1800}/_{185}$ + 10 = 25
Case e Truss discontinuous L = 100 ft. I = $^{100}/_{20}$ + 100 − 60 = 45
For all hangers L = 20, I = $^{100}/_{20}$ + 100 − 12 = 93

Problems

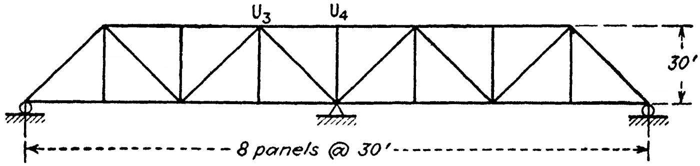
84. *a.* Compute downward deflection due to dead panel loads shown on figure of each end of this truss when bridge is open.

b. Compute upward deflection at each end due to simultaneous application of upward forces of 33,000 lb. at each end.[1]

85. Compute maximum live stresses in bars *a*, *b*, and *c*, of truss of Prob. 84 assuming structure to act as a continuous girder. Use loading of 1,500 lb. per lineal foot.



PROB. 84.



PROB. 86.

86. Determine the force which the end-lifting apparatus should be designed to exert at each end of this truss in order to develop a factor of safety of 1½ against occurrence of uplift due to a uniform live load of 600 lb. per foot of truss and one concentrated live load of 16,000 lb. *I* to be computed by equation given in Art. 17.

References

Movable Bridges, Hovey, John Wiley & Sons, New York, 1926.

Specifications for Movable Railway Bridges, American Railway Engineering Association, Chicago, 1935.

Standard Specifications for Movable Highway Bridges, American Association of State Highway Officials, Washington, D. C., 1938.

[1] Note that this equals downward deflection at center assuming truss to be supported at ends, due to application at center of a downward force of 66,000 lb.

CHAPTER XVIII

MASONRY DAMS

232. Definitions.—A dam is a structure designed to resist water pressure either by its weight alone, in which case it is called a *gravity* dam, or by its weight and resistance to bending combined, as in the case of a *reinforced-concrete* dam. In either type, the resultant of the water pressure and the weight of the dam must pass through the base of the dam at a safe distance from its edge, as explained later. An arched dam is one which is curved in plan and in which arch action as well as gravity may be counted upon to resist the water pressure.

Dams may be divided into two general classes:

a. Reservoir dams with the top of the dam at a level always higher than the water surface at the back of the dam.

b. Overflow dams with the top of the dam at a level lower than the maximum height of the water at the back of the dam.

The former type only will be considered here, but the same general principles are applicable to both classes, the overflow dam differing only in having a head of water at its crest and possibly a vacuum between the sheet of falling water and the downstream surface of the dam.

233. Assumptions for Gravity Dams.—The design of unreinforced gravity dams is ordinarily based upon the following limitations and assumptions:

1. The portions of the structure above and below any assumed horizontal plane[1] act as monoliths.
2. Tension may not exist upon any plane.
3. Plane sections through the dam remain plane during distortion of structure within limits of working stress.
4. Stress varies as strain within limits of working stress.

234. Distribution of Stress over Joints of Masonry Dams.—On the basis of the foregoing assumptions the intensity of the

[1] Such a horizontal plane is commonly called a *joint* regardless of whether it coincides or not with an actual joint in the masonry and will be so designated hereafter.

maximum direct stress at the extreme fiber of a plane horizontal section of a block of homogeneous material capable of resisting both tension and compression is given by the following well-known equation:

$$s = \frac{P}{A} \pm \frac{Mc}{I} \tag{97}$$

in which s = maximum unit stress, lb. per square foot
P = resultant, lb., of the vertical forces above the section
A = area of section, sq. ft.
M = moment, ft.-lb., about neutral axis[1] of the section of the external forces applied to the portion of the structure above the section, combined with the moment about the same axis of the weight of that portion of the structure lying above the given section
c = distance, ft., from neutral axis to the extreme fiber of the section
I = moment of inertia of section, ft.^{4}, about neutral axis

If Eq. (97) is applied to the case of a rectangular joint 1 ft. in width and d ft. in length it becomes

$$s = \frac{P}{d} \pm \frac{6M}{d^2} \tag{98}$$

Equation (98) may be used to determine the distribution of normal stress over a horizontal joint of a masonry structure which is incapable of carrying tension, provided that the value of s is never negative. The limitations that this latter condition imposes are explained later. The stresses thus found are not necessarily the maximum stresses occurring in the dam; the principal stresses that may occur on an oblique section should be computed, particularly if the dam is not rectangular in section.

235. Application of Equations to Dams.—Figure 377 illustrates the forces acting upon a portion of a dam.

[1] The neutral axis as here used is the neutral axis of the cross section with respect to flexure only; *i.e.*, it is the principal axis lying parallel to the longitudinal axis of the dam.

M_1 = resultant moment about central axis of base due to water pressure acting upon the right-hand surface of the entire dam above base of section shown, combined with that upon base if upward pressure is assumed, and P = weight of entire dam above base of section shown. Evidently, $M_1 - Pz$ corresponds to M of previous formulas.

Instead of using the moment M_1 due to the water, the resultant of the water pressure and the weight of the dam may be determined and the point where this resultant cuts the joint located. The moment of the resultant may then be computed and the maximum fiber stress determined as follows:

Let the vertical component of this resultant be V, and let the distance from its point of intersection with the joint to the axis

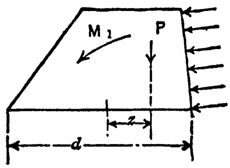


FIG. 377.

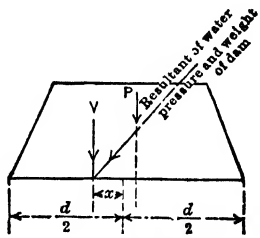


FIG. 378.

passing through center of gravity of the joint be x; then the general formula for maximum intensity of vertical stress becomes,

$$s = \frac{V}{A} \pm \frac{Vxc}{I}$$

For a rectangular joint of width unity and length d, this may be written

$$s = \frac{V}{d} \pm \frac{6Vx}{d^2} \tag{99}$$

This case is illustrated by Fig. 378.

As the assumptions of Art. 233 are not strictly correct for a material like masonry, formulas (97) to (99) are somewhat approximate but are in general use and are probably as accurate as the character of the data available in problems of dam design will warrant.

Examination of formula (99) shows that the vertical stress at one edge of the joint will equal zero when $\frac{V}{d} - \frac{6Vx}{d^2} = 0$, *i.e.*,

when $x = d/6$; and that when x is greater than $d/6$ there will be a tendency for tension to occur at one edge of the joint. The following important theorem may therefore be stated:

In order that tension may not exist at any point of a horizontal rectangular masonry joint, the resultant pressure on the joint must lie within its middle third.

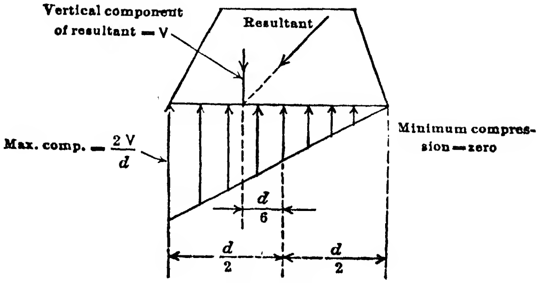


FIG. 379.

Further consideration of this formula shows that when $x = d/6$ the maximum pressure on the joint $= \frac{V}{d} + \frac{6Vd}{6d^2} = \frac{2V}{d} =$ twice the average pressure. That is, for a joint where the resultant passes through the middle third point, the maximum compression is twice the average and the minimum is zero. This is illustrated graphically by Fig. 379.

It is also evident that if the resultant passes outside the middle third of the entire joint and if the material cannot resist tension, the compression will be distributed over only that portion of the joint which has its middle third point at the point of application of the resultant, the remainder of the joint offering no resistance to bending. Such a distribution of stress is shown in Fig. 380.

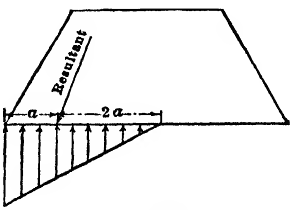


FIG. 380.

236. Outer Forces.—The outer forces to be considered require the most careful study, especially for high dams. In general, it would seem as if two cases only need be treated, reservoir full and reservoir empty, but the question of what, if any, upward[1]

[1] Much difference of opinion exists among engineers as to what allowance should be made for upward pressure, and for discussions on this

water pressure should be assumed under the dam and at the various joints and what ice pressure should be considered must be thoroughly investigated.

To illustrate the loading used for one of the important dams in the United States the conditions assumed in the design of the Wachusett Dam of the Metropolitan Water Works of Massachusetts as given on the official drawing of the accepted section follow:

1. Reservoir empty, *i.e.*, water drained to elevation 300.
2. Reservoir filled to elevation 400. No ice thrust and no upward water pressure included.
3. Reservoir filled to elevation 395. Ice pressure 47,000 lbs. per linear foot of dam, at elevation 395. Upward water pressure corresponding to reservoir head at heel of joint and to backwater head at toe, varying uniformly from heel to toe, uniformly distributed, and considered to be exerted on two-thirds area of joint. Water is assumed to press against dam, where earth is filled against it, in same manner as if earth were not there. Only vertical pressure of earth over the masonry has been included, due allowance being made for diminished weight of particles of earth when submerged.

The same provision for upward pressure was made in the design of the Shasta Dam in California which, it is said, will be the highest gravity dam in existence on its completion. In addition to normal upward pressure, this dam is also designed to resist an upward pressure of 0.10 per cent gravity to allow for earthquake action.

237. Economical Cross Section.—The economical profile of a masonry dam may be investigated in the following manner by use of the assumptions previously stated, provided that the effect of upward pressure is neglected. Let the various profiles shown in Fig. 381 be considered. Let the weight of the masonry per cubic foot $= \gamma$ and the weight of the water per cubic foot $= w$.

point the reader is referred to references at the close of the chapter. It may be safely stated, however, that the absolute exclusion of upward water pressure can only be accomplished by the use of nonporous material laid in the dry in waterproof mortar, on a nonporous base, and that the amount of water pressure which will actually occur varies directly with the porosity of foundation, cementing material, and masonry. Dams may be drained, however, in order to control the upward water pressure, but such drains may become stopped by silt or other debris and dependence should not be placed upon them.

The width of base consistent with no tension may then be computed in terms of h and d. The horizontal component of the water pressure is the same in all cases and is shown on the figure. For case II a vertical component also exists equal to $whd/2$.

The resultant of the dead load acts at the following points:

Case I. Inner middle third point.

Case II. Outer middle third point.

Case III. Center of section.

In order to obtain the maximum economy consistent with no tension on the base, the resultant must evidently pass through the *outer middle third* point (the *outer* side being considered that farthest away from the water). The moment of the resultant under these conditions about this middle third point equals zero, equals the combined moment of the weight of the dam and the water pressure; hence, the width of base may be obtained by putting the expression for the moment about the outer middle third point equal to zero.

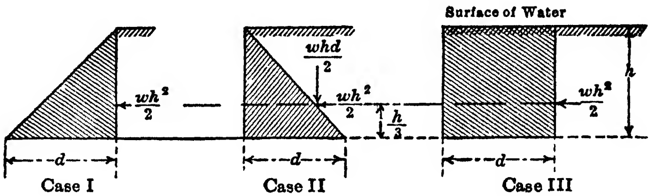


FIG. 381.

The resulting equations are as follows.

$$\text{Case I.}\quad \left(\frac{\gamma hd}{2}\right)\left(\frac{d}{3}\right) - \left(\frac{wh^2}{2}\right)\left(\frac{h}{3}\right) = 0$$

$$\text{Case II.}\quad \left(\frac{whd}{2}\right)\left(\frac{d}{3}\right) - \left(\frac{wh^2}{2}\right)\left(\frac{h}{3}\right) = 0$$

$$\text{Case III.}\quad (\gamma hd)\left(\frac{d}{6}\right) - \left(\frac{wh^2}{2}\right)\left(\frac{h}{3}\right) = 0$$

Solving these equations gives the following results:

$$\text{Case I.}\quad d = h\sqrt{\frac{w}{\gamma}}$$

$$\text{Case II.}\quad d = h$$

$$\text{Case III.}\quad d = h\sqrt{\frac{w}{\gamma}}.$$

Since w is always less than γ, it is evident that Case I gives a more economical section for full reservoir than either of the other cases. It should be noticed that this case also gives the limiting condition when the reservoir is empty, the resultant pressure then passing through the inner middle third point.

γ/w = specific gravity of the masonry; hence, if this is denoted by β we may write for Case I

$$d = \frac{h}{\sqrt{\beta}} \qquad \text{or} \qquad h = d\sqrt{\beta}$$

The section shown by Case I cannot be adopted in practice since an appreciable top width is necessary to resist the action of

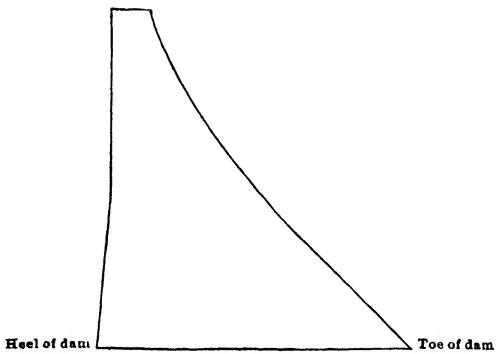


FIG. 382.

waves, ice, and floating material of all sorts and to serve as a foot-path or driveway. Moreover, the top of the dam for the same reasons should extend somewhat above the normal water level. At the bottom of the dam, hydraulic pressure may also occur on the downstream side and on the base. It is necessary to compute the allowable pressure at the base of the dam, which for high dams is often the controlling factor in determining the width of base, and also the intensity of the maximum principal stress that may occur on a diagonal plane. The resistance of the dam to slipping on any joint must also be considered, and the section so proportioned that the resultant pressure at any joint will not make an angle with the vertical greater than the angle of repose at that joint. The shearing strength of the masonry must also be investigated.

Regard for the above-mentioned considerations coupled with the necessity of giving a pleasing section ordinarily results in selecting for high reservoir dams a profile somewhat like that indicated in Fig. 382.

238. Determination of Profile of a Low Dam.—If the dam under consideration is comparatively low, with a narrow top width, a trapezoidal profile may prove most economical. The width of base for such a dam may be readily determined analytically by the following simple method:

Let the width of base be assumed as equal to x, so that the weight of the dam itself as well as the upward water pressure can

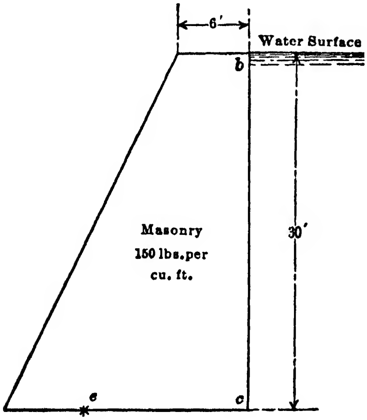


FIG. 383.

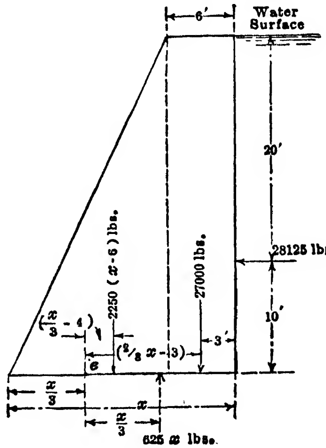


FIG. 384.

be expressed in terms of x. The water pressure on side bc (Fig. 383) can be expressed in terms of the known height h. Since the limiting case for stability will occur when the resultant of all the forces acting on the dam passes through the outer middle third point e, the moment of these forces about e may be placed equal to zero and the resulting equation solved for x. The process is illustrated by the following example:

Problem: *a.* Determine analytically for the dam shown in Fig. 383 the width of base consistent with no tension. Assume the masonry to weigh 150 lb. per cubic foot and upward pressure on the base to equal hydrostatic head at the heel or upstream face reducing uniformly to zero at the toe or downstream face and applied over two-thirds area of the base.

b. Determine the maximum intensity of pressure at base of the dam.

c. Determine the angle between resultant and normal to base.

Solution: Consider a slice of the wall 1 ft. in length perpendicular to the paper, and let the width of base equal x. Then the forces acting upon this slice will be as shown in Fig. 384.

To prevent tension, the moment of the resultant of all the external forces about the middle third point e must equal zero. The position of e may be assumed at any reasonable place, provided that the signs of the moments of the various forces about it are properly used. The equation resulting from applying $\Sigma M = 0$ about e is as follows:

$$27{,}000\left(\frac{2x}{3} - 3\right) - 28{,}125 \times 10 - \frac{625x^2}{3} + 2{,}250(x - 6)\left(\frac{x}{3} - 4\right) = 0$$

Solution of this equation gives $x = 20.1$ ft. which is the width of base required to prevent tension.

b. The maximum intensity of pressure may be obtained by applying Eq. (98), M being the moment of all the applied forces about an axis passing through the center of gravity of the base. The following method is simpler.

The maximum intensity of pressure at the base of the dam equals twice the average pressure since the resultant passes through the middle third point; hence, its value $= \dfrac{2(27{,}000 + 2{,}250 \times 14.1 - 625 \times 20.1)}{20.1} = 4{,}600$ lb. per square foot.[1]

c. The maximum angle between resultant and normal to base occurs when upward pressure exists. Its tangent equals the total horizontal force divided by the total vertical force $= 28{,}125/46{,}160 = 0.61$.

239. Determination of Preliminary Profile of a High Dam.—

The determination of the profile of a high dam may be divided into several clearly defined cases which will be enumerated in order, downward from the top of the dam.

Case 1. Sides vertical. Thickness equals that at top of dam. Limiting conditions: resultant, reservoir full, must not pass outside the outer middle third point of base of section.

Case 2. Inside face vertical, and outside face inclined. Limiting conditions: resultant, reservoir full, must not pass outside the outer middle third point; reservoir empty, must not pass inside the inner middle third point.

Case 3. Same as Case 2 but both faces inclined.

Case 4. Same as Case 3. Limiting conditions: pressure at toe of dam must not exceed the allowable unit stress;[2] resultant

[1] Note that this same value would be obtained if upward pressure were to be neglected in the computations and the intensity computed by the ordinary beam formula, since the intensity of the upward pressure has been assumed as zero at the toe of the dam.

[2] Note that allowable vertical pressure at the toe of the dam is usually taken as somewhat less than that elsewhere owing to the possibility that the

pressure, reservoir empty, must not pass inside the inner middle third point.

Case 5. Same as Case 3. Limiting conditions: pressure at toe, reservoir full, and at heel, reservoir empty, must not exceed allowable unit stresses.

In addition to the limitations imposed by the conditions above it is also necessary to consider *stability against sliding of one portion of the dam on another.* This condition will be satisfied if the angle between the resultant and normal at any section does not exceed the angle of repose of the material, a reasonable factor of safety being allowed. If the base of the dam is stepped, the shearing resistance thus obtained should be added to that due to sliding. As dam failures have occurred through lateral sliding, this limitation should be carefully observed.

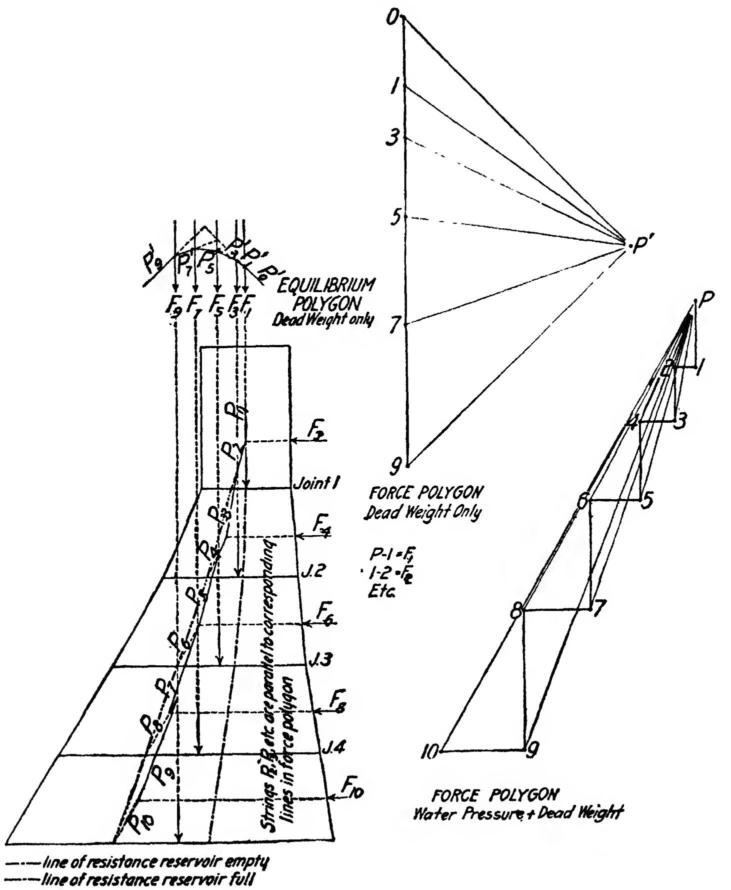
The determination of an exact profile to conform to all these conditions would prove a difficult problem and is not attempted in practice. Instead, the assumed section of the dam may be divided into horizontal slices of trapezoidal section and of reasonable thickness, and the stability of each slice considered independently. The sides of the various trapezoids may then be connected by smooth curves to give the dam a graceful appearance.

The depth of the slices into which the dam should be divided depends upon its height and must be settled by the judgment of the designer. A thickness of 10 ft. for a high dam may, however, be considered reasonable.

240. Graphical Method of Solution.—Though all the various cases of Art. 239 may be treated mathematically, some of the formulas are long and complicated, and for simplicity either a purely graphical method or a combination of graphical and analytical methods may be used. The graphical method involves the construction of a funicular polygon for each condition of loading. The intersections of the appropriate strings with the horizontal planes at top and bottom of each slice into which the dam is assumed to be divided give the points of application of the resultant forces acting on these slices. A line connecting

maximum intensity of pressure occurs on an oblique instead of a horizontal joint. For an important dam the actual maximum intensity of pressure at this point should be determined (see article by Cain, *Trans. Am. Soc. C. E.*, Vol. LXIV, pp. 208 *et seq.*).

these points of intersection is called the *line of resistance* and should keep well within the boundaries of the middle third of the cross section. The resultant upon any joint should not make



LINES OF RESISTANCE
MASONRY DAM
FIG. 385.

an angle with the normal to the joint greater than the angle of repose.

Figure 385 shows the application of this process to a simple case. To construct the line of resistance, reservoir empty, requires the determination of the point of application of the

resultant of the total weight above each joint. For clearness, this is determined by a figure located above the dam profile. The various resultant weights F_1, F_3, etc., are projected up, the equilibrium polygon drawn, and the intersection of the strings P_1', P_3', etc., with P_0', projected downward to the joint above which the forces F_1, F_3, etc., act.

The line of resistance, reservoir full, is constructed by drawing string P_2 through the intersection of F_1 and F_2 till it meets F_3; from this intersection P_3 is drawn till it meets F_4; etc. A line connecting the points of intersection of the funicular polygon strings with the various horizontal joints is the line of resistance. For each joint the point of intersection is that made by the string numbered to correspond to the horizontal force acting on the section of the dam immediately above the joint.

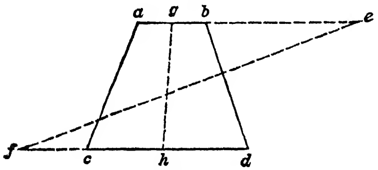


FIG. 386.

If upward pressure is assumed as occurring at the joints, it should be represented by an upward force at each joint, which should be taken into account in constructing the funicular polygon.

The magnitude of the resultant upon any joint may be determined from the force polygon by scale; *e.g.*, the resultant pressure on joint 3, reservoir full, $= P_6$ of the force polygon.

The maximum intensity of pressure at any joint may be found when the position and magnitude of the resultant are known by the methods previously given for the distribution of stress over masonry joints.

It is evident that the graphical method is purely a method of trial. The fact that it is possible to start at the top and work downward, fixing the size of each trapezoidal slice in successsion, makes it much less tedious than if it were necessary to try an entire new profile each time.

In the application of the graphical method, it is often desirable to determine graphically the center of gravity of a trapezoid. This may be found by the following method:

Prolong the parallel sides ab and cd, Fig. 386, and lay off $be = cd$ and $cf = ab$. Connect g and h, the centers of sides ab and cd, respectively, and e and f. The intersection of ef and gh will be at the center of gravity of the trapezoid.

241. Graphical and Analytical Methods Combined.—In order to reduce the number of trials necessary in the application of the graphical method, it is sometimes desirable to apply analytical methods to Cases 1 and 2, Art. 239, and these methods will now be given.

Case 1. The problem here is to determine the depth at which the width must begin to be increased to secure stability against overturning. The formula already deduced for the depth of a rectangular dam, $h = d\sqrt{\beta}$, may be used.

That the allowable compression will not be exceeded in this portion of the dam may be easily shown. Let the weight of the masonry be assumed as 156.25 lb. per cubic foot; β will then have a value of $2\frac{1}{2}$, and h will equal $1.58d$. As this value of β is seldom exceeded and as d is seldom greater than 20 ft., h will rarely exceed 31.6 ft.; hence, the maximum compression on the masonry of this section will not generally exceed $2 \times 156.25 \times 31.6 =$ approximately 10,000 lb. per square foot, a low value for masonry.

The preceding figures are determined for water pressure on the side of the dam only, the water being assumed to reach to the top of the dam. If ice pressure is considered, this result may be modified somewhat but thick ice would probably not occur with the maximum height of water. For magnitude of ice pressure see book by Burr, listed in the references at end of the chapter. An upward water pressure occurring at any section would have the effect of reducing the weight of the masonry, *i.e.*, of reducing β, and the depth of this portion of the dam should be reduced accordingly.

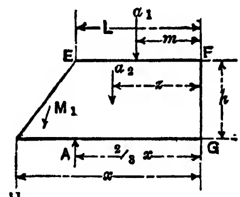


FIG. 387.

Case 2. The problem here is to determine the limit of depth for a vertical inner surface, and the necessary thickness of the dam at this limiting depth.

The following method referring to Fig. 387 may be used, upward water pressure being neglected:

Let a_1 = area, sq. ft., of vertical slice of dam above upper joint (lower joint determined under Case 1).

m = distance from edge of this joint to a vertical line passing through center of gravity of a_1.

L = width of upper joint (previously determined).

a_2 = area of trapezoid $EFGH$ (length = unity).

z = horizontal distance from center of gravity of $EFGH$ to vertical side FG.

h = depth of $EFGH$.

M_1 = moment of horizontal forces applied above HG about any horizontal axis lying in joint HG and normal to trapezoid, divided by the weight of 1 cu. ft. of masonry.

x = width of joint HG.

A = resultant upward pressure on HG divided by the weight of 1 cu. ft. of masonry $= a_1 + a_2$.

Since the limiting condition occurs when the resultant upward pressure acts through the outer middle third point and since $EFGH$ must be in equilibrium under the action of the external forces, we may apply $\Sigma M = 0$ about the point G and solve for x. The following equation results:

$$a_1m + a_2z + M_1 = \tfrac{2}{3}Ax$$

in which a_1, m, and M_1 are known, $a_2 = h\left(\dfrac{L + x}{2}\right)$, and a_2z can be expressed in terms of h, L, and x.

To determine a_2z, make use of the principle that the center of gravity of a triangle is coincident with the point of application of the resultant of a set of parallel forces applied in any direction at the *apices* of the triangle and each equal to one-third its area. To apply this principle, divide the trapezoid into two triangles EFG and EGH, Fig. 388. The apex loads for triangle EFG are each $hL/6$; and for triangle EGH, $hx/6$.

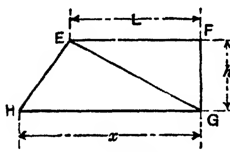


FIG. 388.

The moment of a_2 about G equals the moment of all the foregoing apex loads about G equals $\left(\dfrac{hL}{6} + \dfrac{hx}{6}\right)L + \dfrac{hx}{6} \cdot x = \dfrac{h}{6}(L^2 + Lx + x^2) = a_2z$. By substituting for a_2z the value above, and for A its value, *viz.*, $a_1 + a_2 = a_1 + \left(\dfrac{L + x}{2}\right)h$, the equation $a_1m + a_2z + M_1 = \tfrac{2}{3}Ax$ becomes,

$$a_1m + \frac{h}{6}(L^2 + Lx + x^2) + M_1 = \frac{2x}{3}\left[a_1 + \left(\frac{L + x}{2}\right)h\right]$$

which by reduction gives the following expression for the limiting value:

$$x^2 + x\left(L + \frac{4a_1}{h}\right) = \frac{6}{h}(M_1 + a_1 m) + L^2$$

This equation gives the ratio between x and h for the limiting conditions, provided that M_1 is also expressed in terms of h, as may easily be done. The equation is somewhat complicated, and most designers would prefer to use the graphical method throughout.

The foregoing method is applicable, provided that the compressive strength and resistance to horizontal sliding are not exceeded, until a point is reached where the line of pressure, reservoir empty, passes outside the middle third, from which point on a purely graphical method may be applied.

242. Arched Dams.—The design of high arched dams is complicated and will not be discussed here. It involves the dividing of the arch into horizontal and vertical slices, the former acting as arches and the latter as cantilevers and equating the deflections of the arch and the cantilever at as many points as seems advisable, the so-called trial-load method being used.

The U. S. Bureau of Reclamation (J. L. Savage, Chief Engineer) has designed numerous dams by this method, including the great Hoover Dam in Boulder Canyon, and prepared extensive tables to facilitate the necessary computations.

Problems

87. Determine the width of base of this dam with water standing at level shown, assuming that no tension may exist and that no upward pressure occurs at base. Assume masonry to weigh 150 lb. per cubic foot.

88. *a.* Determine the width of base of this dam, assuming that no tension may exist and that upward water pressure acts on its base corresponding to full hydrostatic head at heel and zero at toe.

b. Compute maximum intensity of pressure at toe, and tangent of angle between normal and resultant at base.

89. *a.* Draw a diagram showing the variation of intensity of pressure on the foundations of the dam shown in Fig. 383 assuming its width to be 24 ft.

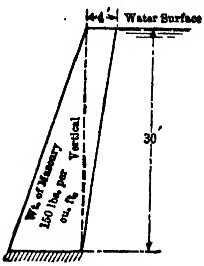
b. Draw a similar diagram, assuming the width of base to be 18 ft.

90. Draw a diagram showing intensity and distribution of stress over the base of this conduit for water at high water level on outside of conduit:

a. When conduit is full.

b. When conduit is empty.

Conduit is exposed to outer water pressure on one side only as shown. Assume upward pressure on the base corresponding to two-thirds of hydrostatic pressure at *a* and zero at *b*.



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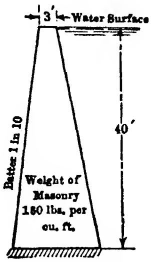

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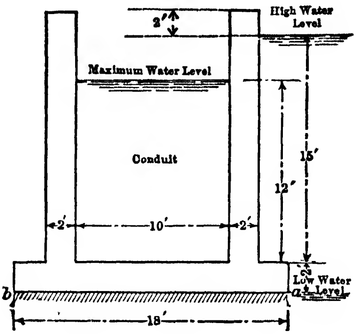

PROB. 90.—Walls and base of reinforced concrete, wt. = 150 lb. per cubic ft.

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CHAPTER XIX

EARTH PRESSURE

243. Cohesion, Friction, and Weight.—In the theoretical treatment of earth pressure, it is commonly assumed that the earth is a granular mass, without cohesion, acting like a pile of pebbles. As to the neglect of cohesion, it may be said that although under some conditions a considerable amount of cohesion may exist in earth, as is shown by the vertical slopes that frequently occur in freshly cut banks, its value is influenced greatly by the effect of moisture and is often entirely destroyed by removing the earth from its original situation; hence, in newly made embankments, cohesion cannot be relied upon.

If cohesion does not exist, the surface slope of a mass of earth will make an angle with the horizontal, the tangent of which equals the coefficient of friction. The value of this coefficient varies with the character of the earth and with the amount of moisture that it contains; in railroad construction, it has been found necessary, for ordinary material, to use a slope of one and one-half horizontal to one vertical to prevent slipping. This is equivalent to using a coefficient of friction of two-thirds corresponding to an angle of repose of 33° 40′, a value which may probably be used with safety for most earth.

The weight of dry earth of various characteristics is given in standard engineering handbooks such as the American Civil Engineer's Pocket Book and the Carnegie Pocket Companion. The weight of saturated earth depends upon the percentage of voids in the dry earth and can be computed as follows:

Let W_d = weight per cu. ft. of dry earth, lb.
W_s = weight per cu. ft. of saturated earth, lb.
P = percentage of voids in the dry earth.
W = weight of water per cu. ft. lb.

Then,

$$W_s = Wd\left(1 - \frac{100 - P}{100}W\right)$$

For example, if $W_d = 100$, $P = 30$ per cent, and

$$W = \text{weight of sea water} = 64$$

Then

$$W_s = 100(1 - {}^{70}\!/\!_{100} \times 64)$$
$$= 100\left(\frac{55.2}{100}\right) = 55.2 \text{ lb. per cu. ft.}$$

The pressure on a retaining wall or other surface due to saturated earth may be obtained by the method that follows by using the weight and coefficient of friction of saturated earth and adding the *hydrostatic pressure* as if the wall were exposed directly to the water pressure.

244. Active and Passive Pressure.—In a fluid like water in which friction between the particles is zero, the resultant pressure

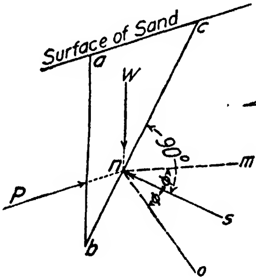


FIG. 389.

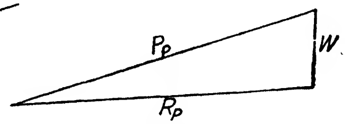


FIG. 390.

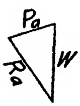


FIG. 391.

on any plane is normal to that plane and can have but one value consistent with equilibrium. In a granular material, on the other hand, the resultant pressure on a given plane may make an angle with the normal less than or equal to the angle of repose and hence may have several values, each of which is consistent with equilibrium in the material. This may be illustrated as follows:

Consider the equilibrium of a triangular prism *abc*, Fig. 389, contained in a mass of granular material like sand, the upper surface of the prism coinciding with the sloping upper surface of the sand. Assume that the pressure P on *ab* is parallel to the surface but of unknown magnitude. For equilibrium the resultant pressure on the surface *bc* must make an angle with the normal *ns*, less than the angle of repose φ of the material. Its

two extreme positions are evidently *mn* and *no*, and it may lie anywhere between these two positions.

The two triangles of force shown by Figs. 390 and 391 represent the forces acting in each of the two extreme conditions of equilibrium.

In each case,

$$
\begin{aligned}
W &= \text{weight of prism} \\
P &= \text{total force on side } ab \\
R &= \text{total force on side } bc
\end{aligned}
$$

Figure 390 shows the relative values of P and R when R corresponds in direction to *mn*, and Fig. 391 shows the same values when R corresponds to *no*.

These diagrams show that the force P may vary considerably in magnitude without overcoming the equilibrium of the particle. P_a, the smaller value of P, is called the *active* pressure; it is the smallest force, consistent with equilibrium of the particle, that can act in the assumed line of action of P. The application of a smaller force permits the resultant on the side *bc* to make an angle below the normal greater than φ and thus allows the prism to slide downward on the plane *bc*. P_a corresponds to the minimum force that a retaining wall holding back the earth to the right of *ab* must be designed to resist; it is called the active pressure since it equals the force that the earth actually exerts upon the wall.

P_p, the larger value of P, shown by Fig. 390 is the maximum value that may be exerted on *ab* without forcing the prism to slide up on the surface *bc*. It is called the *passive* pressure because it is a measure of the passive resistance of the earth to being forced upward. It corresponds to the force that could be counted upon to resist water pressure acting against *ab* or the overturning forces acting on a telegraph pole.

In order to determine the extreme values of P, in any given case, *i.e.*, the minimum and maximum values consistent with equilibrium, it would be necessary to determine the value of the angle *abc* corresponding to these extreme cases as well as the points of application of the forces themselves. A common method in use for doing this is a method of trial which will now be explained.

245. Method of Trial.—By this method, force triangles are drawn for various planes passing through point b, Fig. 389. If the active pressure is desired the minimum value of P consistent with the resultant pressure on any plane bc making the angle φ

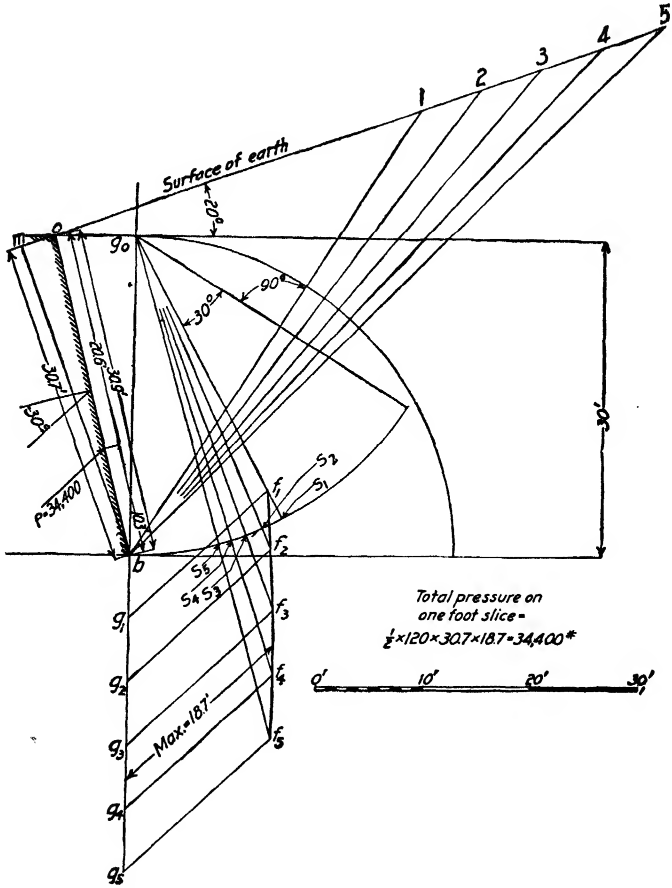


FIG. 392.

below the normal is determined; for the passive pressure the maximum value is obtained for the resultant making the angle φ above the normal. The pressure on any plane is assumed to vary uniformly downward from the top. The application of this method to a simple case is illustrated by Fig. 392 in which the

planes $b1$, $b2$, etc., represent trial planes. The thrust on the back of the wall is in this case assumed to make an angle of 30° with the normal; *i.e.*, the angle of repose between either earth and earth or earth and wall is assumed to be 30°. The arcs shown are drawn in order to simplify the construction. These arcs are swung with equal radii from b and g_0 as centers. g_0s_1, g_0s_2, etc., show the assumed direction of the thrust on the planes $b1$, $b2$, etc. s_1s_2, s_2s_3, etc., are laid off on the arc swung with g_0 as a center and are made equal to the intercepts between the lines $b1$, $b2$, etc., on the arc swung with b as a center. This process is equivalent to laying off angles between the different normals equal to the angles between the planes corresponding to the different normals.

The distances g_0g_1, g_0g_2, etc., represent the weights of the various triangular prisms and may evidently be made equal to their bases 01, 02, etc., provided that the scale is properly chosen.

The lines g_1f_1, g_2f_2, etc., are drawn parallel to the assumed direction of the pressure on the wall, and the points f_1, f_2, etc., are located at the intersection of these lines with the lines g_0s_1, etc. Through the f points a smooth curve is drawn, and the maximum value of a line parallel to the gf lines, intercepted between the vertical through b and this curve, is taken as equal to the maximum thrust. In the case shown, this line as scaled on the scale of distance equals 18.7 ft. This must be multiplied by the scale of force which equals one-half the product of the common altitude bm of the various triangular prisms, and the weight per cubic foot of the earth, assumed as 120 lb. for this case.

This method is consistent with the application to each wedge of earth of the two equations $\Sigma H = 0$ and $\Sigma V = 0$. In order that ΣM may also $= 0$, the point of application and direction of the forces on the wall and on the sloping side of the wedge must meet on the resultant of the weight of the prism; hence, this method gives only an approximate solution.

246. Rankine's Method.—Rankine's method is based upon the following assumptions:

a. The principles governing the distribution of stress in a homogeneous solid body are applicable to a mass of earth.

b. The mass of earth is homogeneous and has a plane upper surface of unlimited extent.

c. The pressure on every plane is a thrust; *i.e.*, the earth is without cohesion, and tension cannot occur on any plane.

d. The pressure on some one plane passing through any given point makes an angle with the normal to that plane just equal to the angle of repose of the material.

If these assumptions are made, it is possible to show that the stresses on two particular planes passing through a given point are conjugate stresses. (Stresses are conjugate when the direction of the stress on a given plane at a given point of a body is parallel to another plane, the direction of the stress upon which at the same point is parallel to the first plane.) That this relation exists between the stress on a vertical plane and that on a plane parallel to the earth surface, on the basis that the foregoing assumptions are correct, may be demonstrated as follows for points on the line of intersection of the two planes:

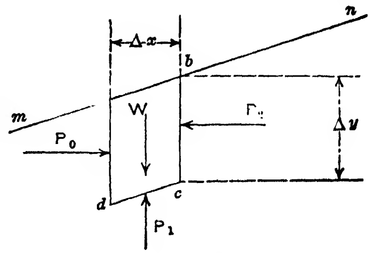


FIG. 393.

Let *mn* in Fig. 393 represent the earth surface, and consider the relation between the stresses on two planes, one vertical and the other parallel to the earth surface, at point *d* at the intersection of the two planes.

Let *abcd* represent a particle of earth having a length of unity at right angles to the plane of the paper and a weight of W acting as shown. Let the direct forces acting upon its sides be represented by P_0, P_1, and P_2, assumed to act as shown. By hypothesis, there can be no other forces acting. For equilibrium, all forces must meet at a point, and this must evidently lie on the line of action of W; hence, P_1 must be applied at center of *dc*. Since the earth mass is considered of infinite extent, conditions on planes *ad* and *bc* must be identical. Hence, P_0 and P_2 must be equal and parallel and act at equal distances from the surface; and as they must intersect on the line of action of W, they must be parallel to the earth's surface. As they are equal and parallel, their horizontal components will be equal; hence, P_1 can have no horizontal component and must, therefore, be vertical.

It is furthermore clear that the conclusions just reached for an infinite mass are very nearly correct for a mass of earth with a plane upper surface of extent such that the conditions on two adjoining planes a slight distance apart are very nearly identical.

It follows that at any point in a mass of earth of reasonable extent and with plane upper surface the pressure on a plane parallel to the surface is vertical, and the pressure on a vertical plane is parallel to the surface; hence, these pressures are conjugate, and the relation between the intensities of the pressures at any point upon two such planes may be expressed by the equations for conjugate forces. Let p and p' represent two such intensities. Their relation will then be expressed by the following equations:[1]

$$\frac{p}{p'} = \frac{\cos\theta - \sqrt{\cos^2\theta - \cos^2\varphi}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\varphi}} \tag{100}$$

in which θ = the angle made by the surface with the horizontal
φ = angle of repose of the earth

This formula is deduced for the case of p less than p'. If the hypothesis is made that p' is less than p, the first term of the equation should be inverted, the second remaining unchanged; hence, we may write for the general case

$$\frac{p}{p'} = \frac{\cos\theta \mp \sqrt{\cos^2\theta - \cos^2\varphi}}{\cos\theta \pm \sqrt{\cos^2\theta - \cos^2\varphi}} \tag{101}$$

If p' represents the intensity of pressure on a plane parallel to the surface at a vertical distance y from it, its value will equal $wy \cos\theta$ lb. per square foot, in which w is the weight of a cubic foot of earth. Substituting this value in Eq. (101) gives the following range of values for p, the intensity of the pressure on a vertical plane at a vertical distance y below the surface:

$$p = wy\cos\theta\frac{\cos\theta \mp \sqrt{\cos^2\theta - \cos^2\varphi}}{\cos\theta \pm \sqrt{\cos^2\theta - \cos^2\varphi}} \tag{102}$$

In this equation the negative sign in the numerator and the positive sign in the denominator should be used to get the active pressure; to get the passive pressure the signs should be reversed. The following inequalities may now be written to cover all possible cases of equilibrium:

$$p \geq wy\cos\theta\frac{\cos\theta - \sqrt{\cos^2\theta - \cos^2\varphi}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\varphi}} \tag{103}$$

[1] See Lanza's Applied Mechanics, 9th ed., p. 889, John Wiley & Sons, Inc., New York.

$$p \leq wy \cos \theta \frac{\cos \theta + \sqrt{\cos^2 \theta - \cos^2 \varphi}}{\cos \theta - \sqrt{\cos^2 \theta - \cos^2 \varphi}} \tag{104}$$

These formulas give the limiting values consistent with equilibrium of the intensities of pressure on a vertical surface; any value of p between these values will therefore be consistent with equilibrium. It is to be remembered that the direction of the pressure is parallel to the surface of the earth.

The total pressure on any vertical plane may be obtained by multiplying the values in formulas (103) and (104) by $y/2$, giving the following formulas for active and passive pressures, respectively:

$$P_a = \frac{wy^2}{2} \cos \theta \frac{\cos \theta - \sqrt{\cos^2 \theta - \cos^2 \varphi}}{\cos \theta + \sqrt{\cos^2 \theta - \cos^2 \varphi}} \tag{105}$$

$$P_p = \frac{wy^2}{2} \cos \theta \frac{\cos \theta + \sqrt{\cos^2 \theta - \cos^2 \varphi}}{\cos \theta - \sqrt{\cos^2 \theta - \cos^2 \varphi}} \tag{106}$$

Since $wy \cos \theta$ varies directly with the distance y, p also varies with y, having the value zero at the surface of the earth. The point of application of the resultant pressure on a vertical plane is, therefore, at a distance from the surface of the earth equal to two-thirds the depth of the plane. When the earth surface is horizontal,

$$p \gtrless wy \frac{1 \mp \sin \varphi}{1 \pm \sin \varphi} \tag{107}$$

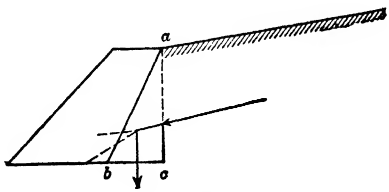


FIG. 394.

when $\theta = \varphi$, $p = wy \cos \theta$, and the active and passive pressures are equal. If $\varphi =$ zero as in the case of water pressure, $p = wy$. By formulas (105) and (106), it is possible to determine the total pressure at any point on a *vertical* plane whether it is a vertical wall surface or a vertical plane of earth. If it is desired to determine the pressure on an inclined wall, it is necessary to compute the pressure on the vertical plane passing through the back corner of the wall and to combine this with the weight of the prism of earth between the wall and this plane. This is illustrated by the problem that follows:

Rankine's method is evidently inapplicable when a wall slopes backward as shown in Fig. 394; since for such a case the resultant

of the weight of the prism *abc* and the pressure on *ac* as determined by Rankine's method may evidently make an angle with the normal to the wall greater than the angle of repose. For such a case the method of trial should be used.

It should be observed that Rankine's method makes no assumption as to the direction of the resultant pressure on an inclined surface. This may easily be determined, however, since the resultant of the pressure on a vertical plane is parallel to the earth surface and the resultant weight between the vertical plane and the surface of the wall is vertical. The following example illustrates the application of this method:

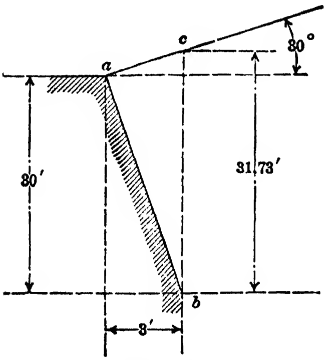


FIG. 395.

Problem: Determine the intensities of the active and passive pressures at point *b* on surface *ab* and the total active pressure per foot in length on that surface for the wall shown in Fig. 395.

Assume weight of earth = 100 lb. per cu. ft.
Assume weight of masonry = 150 lb. per cu. ft.
Assume φ = 35°.

Solution:

$$\cos\theta = 0.866 \qquad \cos^2\theta = 0.750$$
$$\cos\varphi = 0.819 \qquad \cos^2\varphi = 0.671$$
$$\sqrt{\cos^2\theta - \cos^2\varphi} = 0.281$$
$$wy\cos\theta = 3173 \times 0.866 = 2{,}748$$

Therefore, the two extreme conditions consistent with equilibrium are given by the following equations:

$$p = 2{,}748\frac{0.866 - 0.281}{0.866 + 0.281} = 2{,}748 \times \frac{585}{1{,}147} = 1{,}402$$

and

$$p = 2{,}748\frac{0.866 + 0.281}{0.866 - 0.281} = 2{,}748 \times \frac{1{,}147}{585} = 5{,}400$$

These two values of p are the minimum and maximum intensities of pressure at b consistent with equilibrium. The smaller is the active pressure.

The total pressure on *bc* for either condition of earth pressure may now be obtained by multiplying the average pressure on *bc*, *i.e.*, one-half of either of the values just obtained, by the distance *bc*. The total active

pressure on bc per foot of wall is thus found to be

$$31.73 \times \frac{1,402}{2} = 22,250 \text{ lb.}$$

This pressure on the vertical plane may be combined with the weight of the prism abc, and the result will be the pressure on the back of wall.

A graphical solution of Rankine's method is often advantageous to use particularly in determining the pressure upon various sections of a curved surface such as a tunnel wall.[1]

247. Surcharged Wall.—A mass of earth is surcharged when it carries an applied load such, for example, as a building or a railroad track and train, as is illustrated by Fig. 396. In such

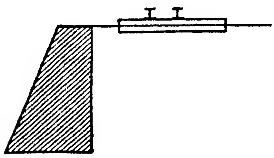


FIG. 396.

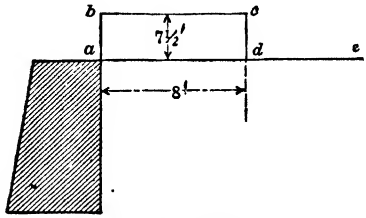


FIG. 397.

a case the retaining wall must resist not only the horizontal pressure due to the earth but also the additional pressure due to the superimposed load coming from the track and the train that it must carry.

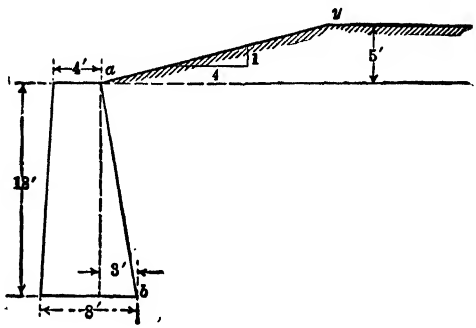
Such cases are commonly treated by reducing the additional weight to an equivalent height of earth. For the case shown in Fig. 396, the train load is assumed to have a maximum value of 4,000 lb. per lineal foot and the weight of track and ballast 2,000 lb. per lineal foot per track. The total superimposed load exclusive of impact, which may safely be neglected, will then be 6,000 lb. per foot. If this is assumed as distributed over a horizontal distance of 8 ft., it would be equivalent to 750 lbs. per square foot which, if the earth weighs 100 lb. per cubic foot would correspond to an additional height of $^{750}/_{100} = 7.5$ ft. The pressure on the wall would therefore correspond very closely to that coming from a mass of earth with an upper surface *abcde.*

[1] For a description of such a method see American Sewerage Practice, Vol. I, Metcalf and Eddy, McGraw-Hill Book Company, Inc., New York, 1914–1915.

Rankine's method would not apply exactly in this case since the upper surface is not plane. The error in applying it would, however, probably be on the safe side. The method of trial can be used with no greater error than in the ordinary case. It should be noted that the pressure on the back of a wall in such a case would be represented by a trapezoid rather than a triangle.

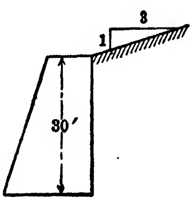
Problems

91. Determine by Rankine's method whether under the worst conditions for overturning the resultant pressure on the base of the conduit shown in Prob. **90** will pass outside the middle third of the base. Assume that

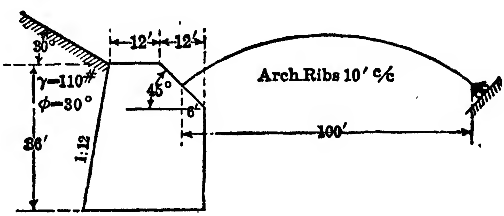


PROB. 92.

earth is filled against outer left side adjoining the conduit, to an elevation of 10 ft. above top of conduit, with a surface slope of 4 to 1 upward from the conduit, and that it has the physical properties given in the following problem.



PROB. 93.



PROB. 94.

92. Determine magnitude, direction, and point of application of active earth pressure per foot of length on side *ab* of this retaining wall by method of trial.

Weight of earth, 110 lb. per cubic feet.

Angle of repose of earth, 30°.

93. The active earth pressure per foot of length exerted on the vertical back of this retaining wall 30 ft. high is 27,000 lb. What would the passive pressure be? $\varphi = 30°$.

94. Determine the magnitude, direction (horizontal and vertical components), and the point of application of the resultant of the pressure upon the back of this wall that can be counted upon to resist the thrust of one arch rib. Use Rankine's theory.

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CHAPTER XX

MASONRY ARCHES WITH FIXED ENDS

248. Determination of Span and Rise.—The span and rise of any given arch are generally determined by the conditions of the problem. If the arch is one of a series, the relative length and rise of each span as compared with the other spans are particularly important from the standpoint of appearance and must be given careful consideration. If the arch is over a stream, adequate waterway and headroom must be provided. An arch span should not be used over streets unless considerable headroom and rise of arch can be obtained; otherwise, the arch has an appearance of clumsiness and oppression rather than of grace and freedom. For such crossings with low headroom, a rigid frame structure or a girder span is more satisfactory (see also Chap. X).

249. Preliminary Determination of Thickness of Arch Rib.—With the span and rise once determined the thickness at crown should next be assumed. In deciding upon this thickness, existing designs may be used as a basis or its value may be computed by an empirical formula such as the following:[1]

$$d_c = \sqrt{L} + \frac{L}{10} + \frac{w_L}{200} + \frac{w_c}{400}$$

where d_c = crown thickness of barrel arch, in.

L = clear span, ft.

w_L = live load, lb., per sq. ft., 50 per cent to be added for impact on railroad bridge

w_c = weight of material over crown, lb. per sq. ft.

After deciding upon the crown thickness, the next step is to make an assumption as to the springing thickness. A reasonable assumption is to make this thickness from 1½ to 3 times the crown thickness, depending upon the flatness of the arch, the larger value corresponding to the flatter arch.

Between the crown and skewback the arch rib should be gradually increased in thickness. Whitney (see References at

[1] Published by Weld, *Eng. News-Record*, Nov. 4, 1905.

end of chapter) gives the following equation for determining the thickness of the rib at intermediate points:

Let d_c = thickness at crown.
d_s = thickness at springing.
d_k = thickness at distance kL from crown.
I_c = moment of inertia at crown.
I_s = moment of inertia at springing.

Other terms are as shown in Fig. 398.

Then

$$d_k = d_c c \sqrt[6]{1 + \tan^2 \theta_k}$$

in which

$$c = \frac{1}{\sqrt[3]{1 - 2(1 - n)k}}$$

and

$$n = \frac{I_c}{I_s \cos \theta_s}$$

The use of the preceding equation requires considerable computing. It is possibly equally satisfactory to determine the shape

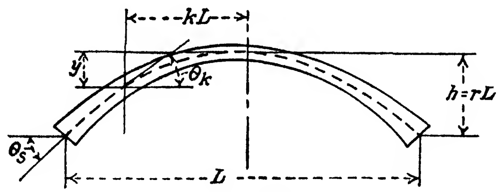

FIG. 398.

θ_k = angle between tangent to arch axis and horizontal

of the rib between crown and springing by eye, it being remembered that the value of the moment, which is the controlling factor in the design, is about the same at the quarter point as at the crown but increases rapidly from quarter point to springing.

250. Shape of Arch Axis.—With thickness at crown and springing assumed, the shape of the arch axis may now be investigated. Various equations for the axes of fixed-ended masonry arches have been developed in recent years, of which those by Cochrane and by Whitney (see References at end of chapter) have received favorable consideration by engineers and will now be given.

Cochrane Equations for Arch Axis (see Fig. 398).
Filled Spandrel Arch.

$$y = \frac{4rL}{1 + 3r}(k^2 + 24k^5r)$$

$$\tan\,\theta_s = \frac{4r}{1 + 3r}(1 + 7.5r)$$

Open Spandrel Arch.

$$y = \frac{8rL}{6 + 5r}(3k^2 + 10k^4r)$$

$$\tan\,\theta_s = \frac{8r}{6 + 5r}(3 + 5r)$$

Whitney Equations for Arch Axis (see Fig. 398). Open or Closed Spandrel Arch.

$$y = \frac{h}{g - 1}(\cosh\,2km - 1)$$

$$\tan\,\theta_k = \frac{2hm}{L(g - 1)}\sinh\,(2km)$$

In the expression above,

$g = \dfrac{w_s}{w_c}$

$m = \cosh^{-1} g$

w_s = dead load per unit length of span at springing

w_c = dead load per unit length of span at crown

If the arch is of the open spandrel type with the load carried to the arch rib by columns or by transverse walls, the values of w_c and w_s may be obtained by considering the panel loads to be distributed over a panel length and assuming the column construction to extend beyond the abutment a distance equal to the distance between the spandrel column and the adjoining column.

The following method of determining the shape of the arch ring for earth-covered reinforced-concrete arches is given by Daniel B. Luten of the National Bridge Company of Indianapolis, Ind., and is reproduced here by permission. It is said to give good results and may be used if desired in making a preliminary design.

Let t = crown thickness, in.
h = rise of arch axis, ft.

L = span, ft.
F = fill over extrados at crown, ft.
P = concentrated load, tons per track, applied to half span (for railroad bridge).
U = uniform live load, lb. per sq. ft.
R_i = radius of intrados curve, ft.
R_e = radius of extrados curve, ft.
C = coefficient of friction of concrete on foundation material.
A = area, sq. ft., of abutment below springing line and adjacent to the back tangent.
H = height of intrados at crown above ground, ft.

a. Locate the intrados curve of the arch as follows:

Draw an ellipse of the required span and rise; pass a segment of a circle through crown and springings of ellipse; bisect the

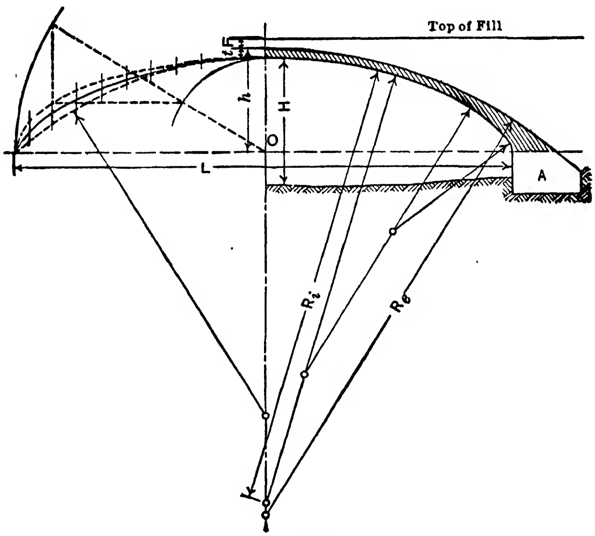

FIG. 399.

vertical distances between the ellipse and the circle; approximate the resulting curve by arcs of circles, adjusting the curve at the springings to become tangent to the verticals. To construct an ellipse, strike two concentric circles with center at O, Fig. 399, passing respectively through crown and springing, and draw radii intersecting these circles. The points of intersection

of horizontal and vertical lines drawn through the points of intersection of each radius with the crown circle and springing circle, respectively, determine points on the ellipse.

b. Apply the formulas that follow to determine the outline of the arch:

$$t = \frac{3L^2(h + 3F)}{4{,}000h - L^2} + \frac{UL^2}{30{,}000h} + \frac{P(L + 5h)}{150h} + 4$$

$$R_e = R_i + \frac{t}{6}$$

$$A = \frac{4(2t - H)}{C}$$

The extrados should be continued by its tangents to the level of the springing lines.

If the uniform and concentrated live loads are not applied to the arch simultaneously, use only the larger of the terms involving these quantities.

The formula for t is for a structure intended to be equally strong in all parts; for short spans, however, the constant 4 in this equation is said to give a crown thickness which, though desirable for practical considerations, is excessive for the given loading. For short spans, therefore, this term may be neglected.

251. Outer Forces.—After the arch ring is assumed, the loads must be determined. To accomplish this, divide the arch ring

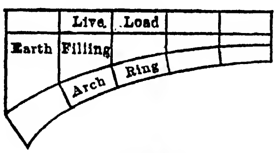

FIG. 400.

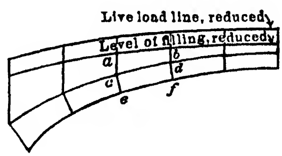

FIG. 401.

into sections of reasonable length as shown in Fig. 400, and compute the magnitude and point of application of the load acting on each section, including the weight of the arch itself.

The dead weight is usually reduced to units corresponding to the weight of the arch ring as shown in Fig. 401; *e.g.*, if the filling weighs 120 lb. per cubic foot and the masonry of the arch ring 150 lb., a 12-ft. height of filling would be plotted above the arch ring as $12 \times {}^{120}\!/\!_{150} = 9.6$ ft. The areas and location of the centers of gravity of the figures *abcd* and *cefd*, Fig. 401, may next

be determined. Both of these areas may be considered as trapezoids. The points of application of the resultant of the weights represented by trapezoids *abcd* and *cdef* must then be determined, either graphically or analytically.[1]

It is sometimes simpler to determine analytically the position of the point of application of the resultant of the weight of the filling and the arch ring rather than to reduce to common units. Impact is generally disregarded in earth-covered masonry arches owing to the deadening effect of the filling.

With respect to the position of the live load, it may be said that full live load over the entire arch does not give the worst condition for stability. The only thoroughly satisfactory method of determining the position of the live loads is by the construction of influence lines, as illustrated later in this chapter. It may be noted that in the case of a symmetrical arch a uniform live load extending from left end approximately three-eighths of the distance to the center gives maximum negative moment at left skewback and maximum positive moment at left quarter point. A load over the remainder of the span consequently gives maximum moments of opposite character at these sections. For maximum positive crown moment the live load should cover approximately the middle quarter of the span, and for maximum negative live crown moment three-eighths of the span from each end, *i.e.*, three-fourths of the span in all.

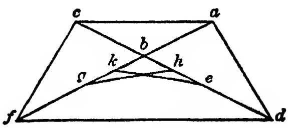

FIG. 402.

With the arch ring assumed and the outer forces determined, the next step is the analysis of the stresses in the assumed arch rib. The expressions developed in the following articles have been found by the author to be simple to understand and apply and are recommended for use.

252. Theory.[2]—An arch span with fixed ends, *i.e.*, fixed in position and direction, is indeterminate to the third degree with

[1] To determine the centroid of a trapezoid, the graphical methods already given may be used, but the following method illustrated by Fig. 402 is often more convenient. Lay off $fg = ab$ and $de = cb$. Bisect bg and be. The center of gravity is at the intersection of ek and gh.

[2] This treatment is based upon a method given by Müller-Breslau in *Z. d. Architekten u. Ingenieur Ver.*, Hanover, 1884. The development of the equation and the conclusions reached are those of the author.

respect to the outer forces. The three unknowns which cannot be determined by statics may either be taken as three of the reaction functions or as a moment, shear, and thrust acting at the crown or at any other convenient section of the arch or at any convenient point in space. Such unknowns are illustrated by Fig. 403 where the arch is shown as divided into two parts which are separated for convenience in representation; these unknowns might equally well be considered as acting at any other point in space.

An exact theory for the arch is complicated and will not be given here. The simple equations that follow are derived by the theorem of least work and are based upon the assumptions

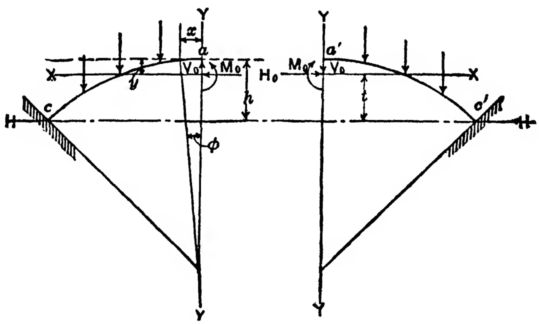

FIG. 403.

below, in addition to those commonly made in the development of the beam formula:

1. The law governing the distribution of stress over the cross section of a curved bar is similar to that over the cross section of a straight bar.

2.* The axial thrust is constant throughout the arch.

3. Work due to shear is negligible.

With respect to these assumptions, it may be stated that the effect of curvature upon the distribution of stress over the cross section is negligible for curved bars having a ratio of thickness to radius of curvature at a given section less than $6/100$, *i.e.*, for practically all bridge arches.

* Computation of several arches with different ratios of span and rise show an error in making this assumption for arches loaded to give maximum moment at crown of from $1/100$ to $26/100$ of 1 per cent of the crown thrust.

The effect of the second assumption is to give an approximate value to the second term in the general equation for work in a bar of varying section, carrying both direct stress and bending, *viz.*:

$$W = \int \frac{M^2}{2EI}ds + \int \frac{S^2}{2AE}ds \text{ (see Art. 200)}$$

As the work due to direct thrust for loading giving maximum stress in a bridge arch is small compared with that due to bending and may ordinarily be entirely neglected without serious error, an approximation in its value is allowable. Moreover the approximation used later is fortunately the least over the central portion of the arch where the arch ring is thinnest and the effect of direct thrust greatest.[1]

Making these approximations, the method of least work may now be applied to the *symmetrical* arch shown diagrammatically in Fig. 403. In all cases, moment is to be considered positive when causing compression in upper fiber of arch.

Let XX and YY = two axes of reference, the former being so located that $\int_0^s \frac{yds}{EI} = 0$ and the latter being an axis of symmetry.

x and y = the ordinates in ft. of any point on the arch axis referred to these axes, x being positive to left and y positive upward.

s = the length, ft., of any portion of the arch axis.

H_0 and V_0 = horizontal and vertical forces, respectively, assumed as acting on each half of arch at intersection of XX and YY.

M_0 = the moment about an axis normal to paper and passing through the intersection of XX and YY.

T = axial thrust at any section of arch equals S in work equation.

[1] For an arch in which the line of resistance coincides with the central axis this approximation is not allowable since for such a case the moment would be that resulting from the axial thrust only, which would therefore be the only term to consider. For a bridge arch subjected to live loads, this condition cannot occur for loading giving maximum stresses.

T_L = axial thrust at any section on left half of arch.

T_R = axial thrust at any section on right half of arch.

m_L = numerical value of moment at any point xy on left half of arch, due to loads applied between that point and crown, left half of arch being assumed to act as a cantilever beam fixed at the abutment.

m_R = similar moment on right half of arch.

M_c = actual bending moment, ft.-lb., about an axis normal to the paper at point a.

I = moment of inertia, ft. units, of a normal cross section of the arch about an axis normal to the paper at point xy.

A = area, sq. ft., of a normal cross section of the arch at point xy.

E = modulus of elasticity, ft. units (E, in. $\times 144$).

d = thickness, ft., of normal section at center of any segment into which arch is divided.

w = dead load per foot in arch.

The resulting equations for a fixed-ended symmetrical arch of homogeneous material loaded with vertical forces are as follows:

$$M_0 = \frac{\int_0^{s/2} (m_L + m_R)\frac{ds}{I}}{2\int_0^{s/2}\left(\frac{ds}{I}\right)} \tag{108}$$

$$H_0 = -\frac{\int_0^{s/2} (m_L + m_R)\frac{yds}{I}}{2\int_0^{s/2}\left(\frac{y^2ds}{I}\right) + 2\int_0^{s/2}\left(\frac{ds}{A}\right)} \tag{109}$$

$$V_0 = \frac{\int_0^{s/2} (m_L - m_R)\frac{xds}{I}}{2\int_0^{s/2} x^2\frac{ds}{I}} \tag{110}$$

Note that m_L and m_R are *numerical* values for the moment due to downward forces; hence, for such forces, M_0 will always be

positive. H_0 will also always be positive since the axis XX is always nearer the crown than the springing line; hence, values of y corresponding to the larger values of m_L and m_R are negative. A negative value for V_0 shows that it will act on each half in the opposite direction to that assumed.

Equation (109) is referred to the axis XX which may be located with reference to any horizontal axis such as HH, Fig. 403, by application of the auxiliary equation

$$t = \frac{\int z\frac{ds}{I}}{\int \frac{ds}{I}} \tag{111}$$

in which z is the distance from center of any arc ds to HH.

The equation $M_c = M_0 - H_0(h - t)$ gives the moment at the crown.

Equations (108) to (110) may be derived as follows: In addition to the nomenclature already used

Let M_L = actual moment and T_L = actual thrust on arch at any point xy on left half of arch.

M_R = corresponding moment and T_R = corresponding thrust on right half of arch.

W = work on entire arch.

Then, it being noted that the moment V_0x is positive when causing compression in upper fiber, regardless of sign of x and V_0, and that

$$\begin{aligned} M_L &= M_0 - m_L - H_0y + V_0x \\ M_R &= M_0 - m_R - H_0y - V_0x \\ T_L &= H_0 \cos\phi + (\Sigma W_L - V_0)\sin\phi \\ T_R &= H_0 \cos\phi + (V_0 + \Sigma w_R)\sin\phi \end{aligned}$$

Now,

$$W = \int_0^{s/2}\left(\frac{M_L{}^2ds}{2EI}\right) + \int_0^{s/2}\left(\frac{M_R{}^2ds}{2EI}\right) + \int_0^{s/2}\left(T_L{}^2\frac{ds}{2AE}\right) + \int_0^{s/2}\left(T_R{}^2\frac{ds}{2AE}\right)$$

Substituting values of M_L and M_R, previously derived, and differentiating with respect to the independent variables give

$$\frac{dW}{dM_0} = \int_0^{s/2}(M_0 - m_L - H_0y + V_0x)\frac{ds}{EI} + \int_0^{s/2}(M_0 - m_R - H_0y - V_0x)\frac{ds}{EI}$$

$$\frac{dW}{dH_0} = \int_0^{s/2}(M_0 - m_L - H_0y + V_0x)\left(-y\frac{ds}{EI}\right) + \int_0^{s/2}(M_0 - m_R - H_0y - V_0x)\left(-y\frac{ds}{EI}\right) + \int_0^{s/2}(T_L + T_R)\cos\phi\frac{ds}{AE}$$

$$\frac{dW}{dV_0} = \int_0^{s/2}(M_0 - m_L - H_0y + V_0x)\left(\frac{xds}{EI}\right) + \int_0^{s/2}(M_0 - m_R - H_0y - V_0x)\left(-x\frac{ds}{EI}\right) + \int_0^{s/2}(T_R - T_L)\sin\phi\frac{ds}{AE}$$

Application of the assumption previously made that T is constant throughout the arch gives $(T_L + T_R)\cos\phi = 2H_0$ and $T_R - T_L = 0$.

Since by construction $\int\frac{yds}{EI} = 0$, either for the whole or for half of a symmetrical arch, all terms consisting of the product of this term and a constant may be placed equal to zero; eliminating such terms therefore and combining other terms give the following expressions:

$$\frac{dW}{dM_0} = 2M_0\int_0^{s/2}\left(\frac{ds}{EI}\right) - \int_0^{s/2}\frac{(m_L + m_R)ds}{EI} = 0$$
$$\frac{dW}{dH_0} = \int_0^{s/2}\frac{(m_L + m_R)yds}{EI} + 2H_0\int_0^{s/2}\left(\frac{y^2ds}{EI}\right) + 2H_0\int_0^{s/2}\left(\frac{ds}{AE}\right)$$
$$\frac{dW}{dV_0} = \int_0^{s/2}\frac{(m_R - m_L)xds}{EI} + 2V_0\int_0^{s/2}\left(\frac{x^2ds}{EI}\right) = 0$$

from which Eqs. (108) to (110) are easily derived.

The auxiliary equation (111) is obtained by locating the center of gravity of a set of hypothetical horizontal forces each having a numerical value equal to ds/I. It is evident that the moment of these about any horizontal axis is $\int z\frac{ds}{I}$ in which z is the distance of the center of each force to the axis. Dividing this by their sum $\int\frac{ds}{I}$ gives the distance t from the reference axis to an axis passing through their center of gravity, *i.e.*, one about which

$$\int\frac{yds}{EI} = 0$$

As integration of Eqs. (108) to (110) is difficult, it is advisable to divide the arch axis into parts of equal lengths Δs and to replace the integration sign by that of summation. Moreover, in arches of masonry or plain concrete I may be replaced by $d^3/12$ if a slice 1 ft. in width is considered. This is also sufficiently correct for reinforced-concrete arches to make its use possible in a preliminary design. Making these substitutions and canceling Δs and E from the equations give the following simple expressions in which the summation sign for all terms except M_L and M_R apply to one-half of arch only.

$$M_0 = \frac{1}{2}\frac{\sum\frac{(m_L + m_R)}{d^3}}{\sum\frac{1}{d^3}} \qquad (112)$$

$$H_0 = -\frac{1}{2}\frac{\sum \frac{(m_L + m_R)y}{d^3}}{\sum \frac{y^2}{d^3} + \sum \frac{1}{12d}} \tag{113}$$

$$V_0 = \frac{1}{2}\frac{\sum \frac{(m_L - m_R)x}{d^3}}{\sum \frac{x^2}{d^3}} \tag{114}$$

$$t = \frac{\sum \frac{z}{d^3}}{\sum \frac{1}{d^3}} \tag{115}$$

253. Formulas for Arches of Constant Cross Section.—For arches of constant cross section, d^3 may be canceled from Eqs. (112) to (115). If n equals the number of equal parts into which one-half the arch is divided, the equations then become

$$M_0 = \frac{1}{2}\frac{\Sigma(m_L + m_R)}{n} \tag{116}$$

$$H_0 = -\frac{1}{2}\frac{\Sigma(m_L + m_R)y}{\Sigma y^2 + n\frac{d^2}{12}} \tag{117}$$

$$V_0 = \frac{1}{2}\frac{\Sigma(m_L - m_R)x}{\Sigma x^2} \tag{118}$$

$$t = \frac{\Sigma z}{n} \tag{119}$$

In the equations above the summations for x, y, z refer to one-half the arch.

The effect of rib shortening due to direct thrust appears only in the term for H_0 and is due to the last term in the denominator of that expression; the effect of this term upon the values of H_0 is small since y is always large compared with d.

254. Comparison of Arch and Fixed-ended Beam.—Examination of formulas (108) to (110) shows that the expressions for M_0 and V_0 do not contain terms in y and hence are independent of the rise of the arch, from which it follows that the value of these terms should be the same as for a fixed-ended beam having a value of ds/I equal to that of the arch. The precision of these

equations may therefore be readily tested by computing M_0 and V_0 for a uniform load applied to a fixed-ended beam of constant cross section in which case $h = 0$ and $t = 0$.

For such a beam, Eqs. (108) and (110) become for the case of a uniform load of w per foot over the left half of the beam

$$M_0 = \frac{\int_0^{L/2} \left(\frac{wx^2}{2}\right) dx}{2\int_0^{L/2} dx} = \frac{wL^2}{48}$$

$$V_0 = \frac{\int_0^{L/2} \left(\frac{wx^2}{2}\right) x dx}{2\int_0^{L/2} x^2 dx} = \frac{3}{32} wL$$

These values are identical with those given by the ordinary formulas for fixed-ended beams of constant section subjected to the same loading.

Had Eqs. (116) and (118) been used, the corresponding values, assuming the half arch to be divided into 10 divisions, would be as follows:

$$M_0 = \frac{wL^2}{48.1}$$

$$V_0 = \frac{wL}{10.5}$$

For uniform load over the entire span, the two values of M_0 and V_0 as obtained by formulas (118) and (116) are

$$M_0 = \frac{wL^2}{24}, \qquad V_0 = 0$$

These agree also with the values obtained for a fixed-ended beam.

Formulas (112) and (114) may be readily used to determine center moment and shear in a fixed-ended beam of variable section.

255. Temperature Stresses.—In an arch with ends fixed in position, the effect of temperature must be considered. The range of temperature to which a masonry arch is subjected is not known precisely, since it depends upon the conductivity of the

materials. Numerous investigations upon the subject have been made and are given in the References at end of chapter. It is generally considered advisable to allow for concrete bridge arches in the latitude of Boston or New York a range of 80°, say 60° drop and 20° rise, the latter being smaller as the arch concrete is usually poured in moderately warm weather and moreover is subject to internal heating during setting.

The change in temperature affects H_0 only, the value of which is given by the following approximate formula in which

ϵ = coefficient of expansion of material
t_0 = change in temperature, degrees
n = number of equal parts into which the half arch is divided

For an arch rib of constant width

$$H_0 = \pm \frac{\epsilon t_0 n E}{\sum \frac{12y^2}{d^3} + \sum \frac{1}{d}} \tag{120}$$

in which the summation refers to one-half the arch.

H_0 will be a thrust if t_0 corresponds to a rise in temperature.

In this formula, no serious error will occur in the ordinary arch if the last term in the denominator is omitted.

For an arch of infinite radius, *i.e.*, a beam, $y = 0$. Therefore, $H_0 = \epsilon t_0 nEd/n = \epsilon t_0 Ed$ and stress per square foot $= H_0/d = \epsilon t_0 E$ as should be the case.

For a concrete arch ϵ may be taken as 0.000006 and E as 288,000,000 lb. per square foot. Hence, Eq. (120) may be written thus for a rib 1 ft. in width.

$$H_0 = \frac{144nt_0}{\sum \frac{y^2}{d^3} + \sum \frac{1}{12d}} \tag{121}$$

Equation (120) may also be derived by the method of deflection as follows:

Let ϵ = coefficient of expansion.
t_0 = increase in temperature.
δ = horizontal deflection to left of center of span equals zero for an arch subjected to a symmetrical change of temperature.

Then,

$$\delta = \int \frac{Mmds}{EI} + \int_1 \frac{H_0 \cos \phi dsdx}{AEds} \pm \frac{\Sigma t_0 L}{2}$$

$$= \int_0^{s/2} (M_0 - H_0 y)(-y)\frac{ds}{EI} + H_0 \int_0^{s/2} \left(\frac{\cos^2 \phi ds}{AE}\right) - \frac{\epsilon t_0 L}{2} = 0$$

But, by construction,

$$\int \frac{M_0 y ds}{EI} = 0$$

Therefore,

$$H_0 = \pm \frac{1}{2} \frac{\epsilon t_0 L}{\int_0^{s/2} \left(\frac{y^2 ds}{EI}\right) + \int_0^{s/2} \left(\frac{\cos^2 \phi ds}{AE}\right)} \tag{122}$$

The last term in the denominator may be written as

$$\int_0^{L/2} \cos \phi \frac{dx}{AE}$$

and since in most arches $\cos \phi$ may be placed equal to unity without making a serious error in this term which is small anyway we may substitute $L/2AE$ for this term without serious error. We may also replace I by $d^3/12$ and A by d.

Making these changes, substituting summations over half arch for integrations, and assuming $L/\Delta s = 2n$, in which n equals number of parts into which the half arch is divided, give Eq. (121) in which the denominator is the same as for H_0 for vertical loads.

256. Effect of Horizontal Movement of Abutments.—The effect of a horizontal movement of an abutment upon the horizontal component of the arch thrust may be determined from the equation for the horizontal deflection of a curved bar. This equation, if the assumptions made in Art. 252 are again made, is as follows:

$$\Delta L = \int_0^L \frac{T dx}{EA} + \int_0^s \frac{M z ds}{EI} \quad \text{(see standard treatises on mechanics)}$$

in which ΔL = change in span due to horizontal component of abutment movement

T = abutment thrust

z = ordinate at any point on the arch axis measured from a horizontal axis corresponding to HH, Fig. 403, *i.e.*, passing through intersection of arch axis and skewback

M = bending moment caused by horizontal thrust about an axis normal to paper at same point

Substituting Hz for M and $H \cos \phi$ for T gives

$$\Delta L = \int_0^L H \cos \phi \frac{ds}{AE} \frac{dx}{ds} + H \int_0^s \frac{z^2 ds}{EI}$$

Substitution of summations for integrations gives the following equation in which the *summation is applied to one-half of the arch* in order to obtain terms corresponding to those developed in previous equations.

$$E\Delta L = 2H \cos^2 \phi \frac{\Delta s}{AE} + 2H \sum \frac{z^2 \Delta s}{I}$$

Hence,

$$H = \frac{E\Delta L}{2\sum \frac{z^2 \Delta s}{I} + 2\sum \frac{\cos^2 \phi \Delta s}{A}} \qquad (123)$$

If I and A are replaced by their values in terms of d, the value of H for a rib 1 ft. in width is given by the following expression:

$$H = \frac{E\Delta L}{2\left(12\sum \frac{z^2 \Delta s}{d^3} + \sum \frac{\cos^2 \phi \Delta s}{d}\right)} \qquad (124)$$

For flat arches, Δs may be placed $= \Delta x$.

For an arch of infinite radius and constant cross section, *i.e.*, a beam, $2\Sigma\Delta x = L$ and $z = 0$; hence, formula (124) becomes

$$H = +E\Delta L \frac{d}{L}$$

This agrees with the value of H for the axial stress in a straight bar corresponding to a change of ΔL in its length. In a similar manner the effect of a settlement of one abutment can be determined.

257. Precision of Formulas.—The formulas developed in this chapter are easy to apply since any draftsman can obtain all

necessary data for a symmetrical arch and fill out tables similar to those given in the illustrative example that follows. Moreover, they can be fitted to an unsymmetrical arch with little trouble by properly selecting inclined axes of reference. These axes should be such that

$$\int \frac{yds}{EI} = 0, \qquad \int \frac{xyds}{EI} = 0$$

and that

$$\int \frac{xds}{EI} = 0$$

as is the case for the horizontal and vertical axes used for the symmetrical arch treated.

258. Line of Resistance.—It is always advisable to construct a line of resistance for a masonry arch after the values of the various unknowns have been determined. This may be done in a manner similar to that employed in the case of a masonry dam. For a symmetrical arch symmetrically loaded, the resultant crown thrust to use in constructing the line of resistance equals H_0, is horizontal, and is located at a distance above the axis XX equal to M_0/H_0.

For unsymmetrical loading, the point of application of the resultant crown thrust may be obtained from the same formula, but the thrust is no longer horizontal, its slope being given by the expression H_0/V_0. Its horizontal component is H_0. With the crown thrust fixed in position and direction, the line of resistance can be readily constructed. It should be kept within the middle third of the arch ring for unreinforced masonry structures but may be allowed to pass outside the middle third for reinforced concrete or steel arches.

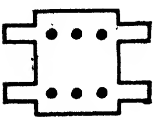

Fig. 404.

259. Distribution of Stress over Cross Section. The maximum fiber stress in an unreinforced arch may be computed by the methods given for masonry dams. Its value depends upon the eccentricity and magnitude of the thrust at a given section, and several sections may have to be tried to determine the limiting condition. If the arch is of reinforced concrete, the maximum stress upon any section, the resultant upon which passes through the middle third, may be found in a similar manner, the steel being assumed as replaced by an equivalent amount

of concrete, *i.e.*, an area of concrete equal to the product of the steel area and the ratio E_s/E_c which may generally be taken as 15. The resulting section to be dealt with is similar to that shown in Fig. 404, the fins being opposite the steel bars. If the resultant at any section of the reinforced-concrete arch passes outside the middle third, special formulas must be applied.[1]

260. Computation of Reactions by Approximate Method. *Illustration.*—The following example illustrates the method of applying the equations of this chapter for the case of a load over a portion of the span. The arch shown in Fig. 405 is considered and the effect of a uniform load of 200 lb. per lineal foot over the left half of the span determined. For this loading the values of

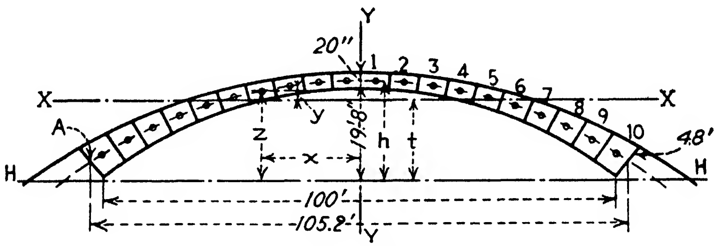

FIG. 405.

m_L at the centers of the various sections may be computed analytically by the formula $m = wx^2/2$, and the value of m_R equals zero for all sections.

Were the problem that of determining the effect of the dead loads the loading should be divided into a series of partial loads as indicated in Art. 251, and the moment at the center of each section into which the arch is divided computed as if these loads were concentrated loads.

For concentrated live loads, the moment at the center of each section would be figured in the usual manner for cantilever beams for any position of the loads.

The division of the arch into sections of equal length measured along the axis can be readily accomplished, as it is unnecessary to work exactly between skewbacks, a variation of a foot or so each way in the total span making little difference in the final results. The number of sections into which to divide the arch

[1] See works upon reinforced concrete mentioned in references at end of chapter.

should be such that no material error will occur from taking the moment throughout each section as constant and equal to that at its center. The ordinates and thickness of the arch ring given in the tables that follow were scaled from a large-sized layout of the arch.

The complete computations are given in the table that follows and require no comment.

With the tabular values once determined, the values of M_0, H_0, and V_0 may be readily obtained from Eqs. (112) to (115), the necessary computations being given at the foot of Table 5.

TABLE 4

Use for determination of value of t

$$t = \sum \frac{z}{d^3} \div \sum \frac{1}{d^3}$$

See numerical value at foot of table

Section	z, feet, scaled	d, feet, scaled	d^3	$1/d^3$	z/d^3	$1/d$
1	20.49	1.70	4.91	0.204	4.18	0.588
2	20.18	1.70	4.91	0.204	4.12	0.588
3	19.45	1.70	4.91	0.204	3.97	0.588
4	18.45	1.74	5.27	0.190	3.51	0.575
5	17.08	1.75	5.36	0.187	3.19	0.571
6	15.37	1.78	5.64	0.177	2.72	0.562
7	13.38	2.00	8.00	0.125	1.67	0.500
8	10.84	2.45	14.71	0.068	0.74	0.408
9	7.98	3.05	28.37	0.035	0.28	0.328
10	4.70	4.00	64.00	0.016	0.07	0.250
			Total	1.410	24.45	4.958

$$t = \frac{24.45}{1.41} = 17.34 \text{ ft.} \qquad h - t = 20.5 - 17.34 = 3.16 \text{ ft.}$$

TABLE 5

For determination of H_0, V_0, and M_0. For equations and values derived therefrom, see foot of table. Moments in 1,000-lb. units.

Section	x scaled	$y = z - t$	x^2	y^2	x/d^3	y/d^3	x^2/d^3	y^2/d^3
1	2.84	+ 3.15	8.0	9.9	0.58	+0.642	1.6	2.02
2	8.48	+ 2.84	71.9	8.1	1.73	+0.579	14.7	1.64
3	14.10	+ 2.11	198.8	4.4	2.88	+0.430	40.5	0.91
4	19.68	+ 1.11	387.3	1.2	3.74	+0.211	73.5	0.23
5	25.18	− 0.26	634.0	0.1	4.70	−0.048	118.2	0.01
6	30.55	− 1.97	933.3	3.9	5.41	−0.349	165.5	0.69
7	35.80	− 3.96	1,281.6	15.7	4.48	−0.495	160.2	1.96
8	40.89	− 6.50	1,672.0	42.2	2.78	−0.442	113.8	2.87
9	45.69	− 9.36	2,087.6	87.6	1.61	−0.329	73.6	3.09
10	50.30	−12.64	2,530.1	159.8	0.79	−0.198	39.5	2.50
						Total	801.1	15.92

Live load over left half of arch

Section	m_L	m_R	(m_L+m_R)	(m_L-m_R)	$(m_L+m_R)\frac{1}{d^3}$	$(m_L-m_R)\frac{x}{d^3}$	$(m_L+m_R)\frac{y}{d^3}$
1	0.8	0	0.8	0.8	0.16	0.5	+ 0.5
2	7.2	0	7.2	7.2	1.46	12.4	+ 4.2
3	19.9	0	19.9	19.9	4.05	57.2	+ 8.5
4	38.7	0	38.7	38.7	7.35	144.9	+ 8.2
5	63.4	0	63.4	63.4	11.81	297.8	− 3.0
6	93.3	0	93.3	93.3	16.52	504.0	− 32.6
7	128.2	0	128.2	128.2	16.05	575.0	− 63.5
8	167.2	0	167.2	167.2	11.37	465.0	− 73.9
9	208.8	0	208.8	208.8	7.36	336.0	− 68.6
10	253.0	0	253.0	253.0	3.95	198.5	− 50.2
				Total	80.08	2591.3	−270.4

$$M_0 = \frac{1}{2}\frac{80{,}080}{1.41} = 28{,}400 \text{ ft.-lb.}$$

$$H_0 = \frac{1}{2}\frac{270{,}400}{15.92 + 0.41} = 8{,}270 \text{ lb.}$$

$$V_0 = \frac{1}{2}\frac{2{,}591{,}300}{801.1} = 1{,}615 \text{ lb.}$$

$$M_c = 28{,}400 - 8{,}270 \times 3.16 = +2{,}270 \text{ ft.-lb.}$$

For temperature stresses, $M_0 = 0$ and $n = 10$; hence, Eq. (121) gives $H_0 = \frac{1{,}440}{16.33} = \pm 88$ lb. per degree Fahrenheit change in temperature and hence $M_c = \mp 88 \times 3.16 = \mp 278$ ft.-lb. per degree Fahrenheit change in temperature.

261. Influence Lines.—Influence lines for moments or thrusts at any section of a fixed-ended arch may be readily constructed and the position of loads for maximum value of the function in question easily determined. With the position of the loads known the value of the function itself can be determined for a uniform loading by multiplying the area between the influence line and the horizontal axis by the uniform load per foot. For a concentrated load system the value of the function can be obtained by placing the loads on the span and getting the combined product of loads and ordinates to the influence line. The determination of the ordinate to the influence line for a typical case is illustrated by the following example:

Problem: Determine the ordinate to the influence line for moment at section 5 (practically the quarter point) of the arch shown in previous example with the load unity at section 5.

Solution: For this case a table similar to Table 5 may be constructed since m_R will be zero in all cases and m_L will also be zero for sections 1 to 5, for in all the cases a load at quarter point will be to the right of the section under consideration. The remaining values are shown in Table 6 which follows:

TABLE 6.—LOAD AT SECTION

Section	m_L	$(m_L + m_R)\frac{1}{d^3}$	$(m_L - m_R)\frac{x}{d^3}$	$(m_L + m_R)\frac{y}{d^3}$
6	5.37	0.95	29.1	− 1.87
7	10.62	1.33	47.6	− 5.26
8	15.71	1.07	43.7	− 6.94
9	20.51	0.72	33.0	− 6.75
10	25.12	0.40	19.8	− 4.97
Totals........	.....	4.47	173.2	−25.79

From the figures in Tables 4, 5, and 6, the values of M_0, H_0 and V_0 are found by inserting the tabular values in Eqs. (112), (113), and (114) and are as follows:

$$M_0 = \frac{1}{2}\left(\frac{4.47}{1.41}\right) = +1.58$$

$$H_0 = -\frac{1}{2}\left(-\frac{25.79}{15.92 + 0.4}\right) = +0.79$$

$$V_0 = \frac{1}{2}\left(\frac{173.2}{801.1}\right) = +0.11$$

The moment at the section under consideration which equals the ordinate to the influence line at this section

$$= M_0 + H_0(17.34 - 17.08) + V_0(25.18) = 4.51$$

The ordinate to the influence line for moment at the point under consideration due to load unity at other points on the arch may be computed in a similar manner, the influence line constructed, and the areas between influence line and axis determined by planimeter or otherwise.

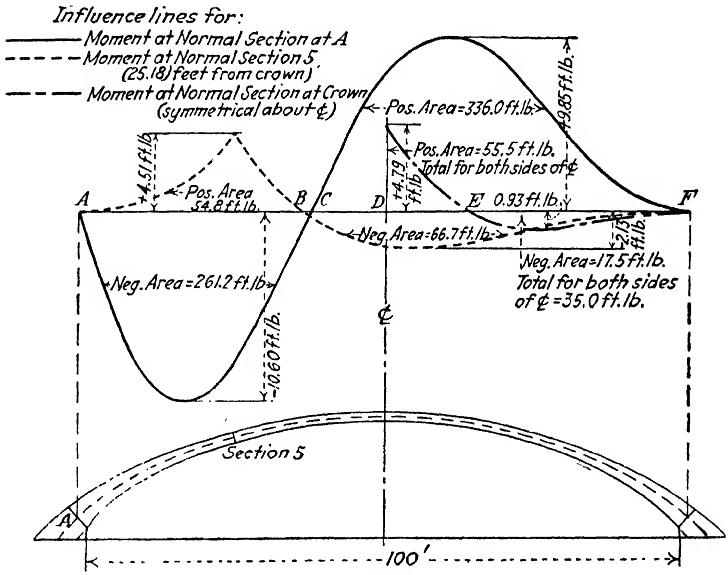

FIG. 406.

Figure 406 shows the complete influence line for moment at section 5 and also for moment at left end and at crown and gives the areas between these various influence lines and the axis.

For uniform live load equal w per ft. the maximum moment at section 5 occurs with load over distance FB and equals $-66.7w$ ft.-lb., the maximum moment at A occurs with loading from C to F and equals $+336.0w$ ft.-lb. and the maximum crown moment occurs with loading over distance equal to DE on each side of crown and equals $55.5w$ ft.-lb.

262. Critical Section.—Since, for appearance, the arch rib is generally made of a constantly increasing thickness from the crown to the skewback where the moment is a maximum, it is

generally sufficient to compute the maximum moments, thrusts, and resulting stresses at crown and skewback. The moment at the quarter point is seldom much larger than the moment at the crown. Influence lines for these three points for various arches are given in the papers by Whitney and Cochrane referred to in the References.

Problems

95. Compute for arch shown in Art. 260, (a) the maximum moment and thrust at crown due to a uniform live load of 200 lb. per linear foot; (b) the value of M_0, H_0, and V_0 for the same uniform live load extending over the central three-tenths of the span.

96. Determine values of H_0, V_0, and M_0 for loading giving maximum moment at section A, Fig. 406, using uniform live load of 200 lb. per linear foot, and compute the moment, shear, and normal component of thrust acting at a section normal to arch axis at A, and the eccentricity of the thrust at this section.

97. Determine by use of the formulas in this chapter expressions for the moment at center of a fixed-ended beam of constant cross section due to a concentrated load P applied at center of beam.

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CHAPTER XXI

WIND STRESSES IN FRAMED BENTS OF HIGH BUILDINGS

263. Definition.—Framed bents of the portal type, illustrated in Fig. 407 and designed to resist lateral as well as vertical forces, are widely used in high buildings. Such bents frequently have no diagonal bracing to resist lateral forces such as wind or vibration due to earthquakes or machinery, since such bracing would interfere with doors, windows, and other openings. The girders are, instead, rigidly fastened to the columns by gusset plates or otherwise, so that the joints are capable of transmitting bending moments; provisions are also made for transmitting bending moments of considerable magnitude from the columns to their foundations. The framed bent differs materially from a truss in that the members must carry bending moment as well as direct stress. Such structures are extremely indeterminate with respect to lateral forces and are usually solved by approximate methods.

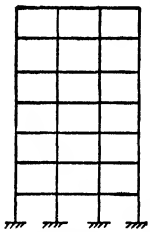


FIG. 407.

264. Approximate Methods of Solution.—The determination of the approximate stresses in framed bents due to wind pressure is generally accomplished by neglecting the stiffening effect of walls and floors and by using *three, and three only,* of the following general assumptions.

a. A point of contraflexure occurs in each column section midway between points of lateral support; *e.g.,* in a building, such points of contraflexure occur midway between floors. Note that a point of contraflexure may be treated in the calculations as a frictionless hinge, since it is a point of zero moment.

b. A point of contraflexure occurs in each girder at midlength; *i.e.,* halfway between columns.

c. The direct unit stress in each column of a row located parallel to the direction of the wind varies in proportion to the

distance of the column axis from the centroid of the cross-sectional areas of all the columns in the row.

d. The shear in any row of columns located parallel to the direction of the wind is divided as follows: Shear in all interior columns is equal; shear in each of the two exterior columns is equal and has a value equal to one-half the shear in each of the interior columns.

The combination of assumptions *a*, *b*, and *c* gives the "cantilever" method and that of *a*, *b*, and *d* the "portal" method.

Assumptions *a* and *b* are based upon the hypothesis that when such a structure is distorted by lateral forces points of contraflexure will occur at or near the centers of the individual members.

Assumption *c* is based on the hypothesis that the framework under lateral forces acts like a cantilever beam; hence, the unit stress at any point varies as the distance of the point from the neutral axis. In connection with this assumption, it should be observed that the determination of the position of the centroid of the group of columns in the given row and of the total stress in each column requires a knowledge of relative areas of the various columns. This can be estimated with sufficient accuracy, however, by considering these areas to vary in proportion to the total vertical live and dead loads that they have to carry.

Assumption *d* corresponds to the condition that would exist if the bent were made up of a set of two-column bents, each independent of the other, each exterior column forming one column of such a bent and each longitudinal half of an intermediate column also forming one column of such a bent, and the shear in each of these hypothetical column bents being divided equally between its two columns. This assumption is more convenient to apply than assumption *c*. It may, however, be observed that for certain special cases, such as a two-column bent or a three-column symmetrical bent, assumptions *c* and *d* give identical results.

The cantilever method was used by H. G. Balcom in the design of the 1,248-ft.-high Empire State Building in New York, N. Y. (see References at end of chapter).

The use of three of the foregoing assumptions combined with the equations of statics makes it possible to compute the shear, direct stress, and bending moment due to lateral forces in each member of a framed bent, a complete approximate solution thus

being obtained. In general, only three of the assumptions above can be used jointly, since, if all four are used, the number of equations would usually be in excess of the number necessary for a solution; hence, the result would be ambiguous. In the case of a two-column bent, any two of the three assumptions are sufficient, with the equations of statics, to give a complete solution. For the case of a symmetrical three-column bent, assumptions *c* and *d* give identical results when either is combined with assumptions *a* and *b*. Assumptions *a* and *b* combined with either *c* or *d* are in common use.

The degree of approximation of these assumptions is considered in certain of the papers mentioned in the References, and will not be discussed here. It should be observed, however, that, generally speaking, the lateral forces in the main members of such a structure are less important than the live and dead stresses, and consequently approximation in their values is not so serious as in the other stresses; also, that the maximum live, dead, and lateral stresses are unlikely to occur simultaneously. Moreover the masonry was found by Prof. J. Charles Rathbun in an extensive series of tests made on the Empire State Building and reported in the *Proceedings* of the American Society of Civil Engineering, September, 1938, to increase the rigidity of the building about 350 per cent above that of the unsupported frame. Prof. Rathbun also found that these tests neither corroborated nor refuted the cantilever or portal method.

The application of the varying assumptions may be simplified by utilizing the following theorems, although the methods of computation in the examples illustrating assumption *c* may be used if desired.

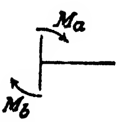


FIG. 408.

265. Theorems. *Applicable to All Assumptions.*

1. The moment at the outer end of any exterior girder equals the arithmetical sum of the moments in the exterior column at the section directly above and below the girder; *viz.*, $M_a + M_b$ (Fig. 408).

2. The increment in direct wind stress at any floor in any column equals the increment in the combined shear in the girders meeting the column at the point in question; *e.g.*, the direct stress T_3 in the lower section of the interior column shown in Fig. 409 equals T_2 plus or minus, as the case may be, the combined shears in the two girders connected to this column at *b*.

3. Direct stress in girders may be found by applying $\Sigma H = 0$ at each joint, it being remembered that the horizontal forces acting at the joint include the *column shears.*

Applicable to Assumption b Only.—The moment at one end of any girder equals the moment at the other end of the same girder and is of opposite character, since no wind forces are applied to the girder except the end moments and shears, and the moment at the center of girder equals zero by hypothesis.

The shear in any girder equals the moment at the end of the girder divided by one-half the girder span.

Applicable to Assumptions b and d Only.—The moment at either end of any interior girder at any floor of a building with adjoining story heights uniform across the bent under consideration equals the moment at the corresponding end of either exterior girder in the same row of girders. The moment is positive at the left end and negative at the right end if the lateral force acts to the right. The truth of this theorem depends upon the fact that the total moment transferred to the exterior girder at *a* (see Fig. 409) from the exterior column is equal to $H_1h_1 + H_2h_2$ and the moment at the right end of the exterior girder is equal and opposite to the foregoing; hence, it equals $-(H_1h_1 + H_2h_2)$. Now the total moment transferred to the girders at *b* from the interior column with the shear distribution, according to assumption *d*, $= 2H_1h_1 + 2H_2h_2$; it follows that the moment transferred to the interior girder at $b = 2H_1h_1 + 2H_2h_2 - H_1h_1 - H_2h_2 = H_1h_1 + H_2h_2$, which equals the moment at the outer end of the exterior girder. The moment at the right end of the interior girder is equal and opposite to the above, hence it equals $-(H_1h_1 + H_2h_2)$ which equals the moment at the right end of the exterior girder. From the foregoing it is evident that, if the girder spans are of equal length, the direct stress in any interior column is zero throughout, since the adjoining girder shears are equal and opposite. It is also evident that in the case of any interior column, if the girder span to the right is greater than that to the left, the increment of stress in the column corresponds to an increase in compression

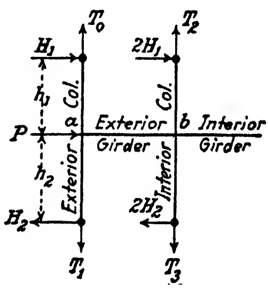


FIG. 409.

in the column, whereas, if the opposite condition obtains, the increment in direct stress in the column corresponds to an increase in tension.

266. Application of Theorems.—The application of the theorems in the previous article to a definite case is clearly illustrated by the following example in which assumptions *a*, *b*, and *d* are applied to the structure shown in Fig. 410. Girder moment $= \frac{H}{10} \times 6 = \frac{3}{5}H$ (+ at left ends of all girders; − at right ends). Shear in girder $G_1 = 3H/25$; $G_2 = 3H/30$; $G_3 = 3H/50$; $G_4 = 3H/25$; $G_5 = 3H/25$. All girder shears are evidently negative; *i.e.*, if a section through the center of any girder is taken the vertical forces applied at that section to the portion

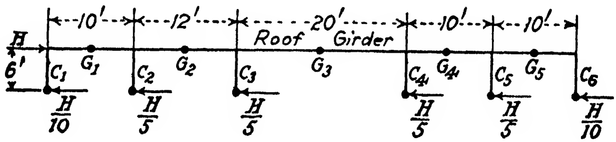


FIG. 410.

of the girder to the left thereof is upward, and that applied to the portion of the girder to the right is downward.

The direct stresses in the columns may now be readily determined from the shears as follows:

Column C_1, direct stress $= (+)\frac{3}{25}H$
Column C_2, direct stress $= \frac{3}{30}H - \frac{3}{25}H = (-)\frac{3}{150}H$
Column C_3, direct stress $= \frac{3}{50}H - \frac{3}{30}H = (-)\frac{6}{150}H$
Column C_4, direct stress $= \frac{3}{25}H - \frac{3}{50}H = (+)\frac{3}{50}H$
Column C_5, direct stress $= \frac{3}{25}H - \frac{3}{25}H = 0$
Column C_6, direct stress $= (-)\frac{3}{25}H$

The foregoing values may be checked by taking moments about any convenient axis, *e.g.*, an axis passing through the point of inflexion in column C_6. This moment $= 6H - \frac{3}{25}H \times 62 + \frac{3}{150} \times 52 + \frac{6}{150}H \times 40 - \frac{3}{50}H \times 20 = 0$.

If the spans were all of equal length, say 20 ft., the direct stresses in all interior columns would each equal zero, the direct stress in left exterior column would be $+\frac{H}{10} \times \frac{6}{10}$ and that in right exterior column would be $-\frac{H}{10} \times \frac{6}{10}$. The moment about point

of inflexion of column C_6 would then equal $-H \times 6 + \frac{H}{10} \times \frac{6}{10} \times 100 = 0$, which checks the results obtained.

With the direct stresses in the columns in the upper story known, the direct stresses in the columns in the story below may be computed in a similar manner, and so on to the columns in the lower story, a general check being then applied to all the computations by taking moments of these lower column stresses and the horizontal forces about any convenient axis.

The computations for the next lower story (shown in Fig. 411) follow:

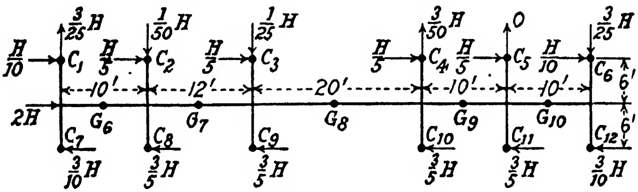


FIG. 411.

Girder moment, $\frac{4H}{10} \times 6$ (+ at left ends; − at right ends)

Girder shears all negative,

$$G_6 = \frac{24}{10}\frac{H}{5}, \qquad G_7 = \frac{4}{10}H, \qquad G_8 = \frac{24}{100}H,$$

$$G_9 = \frac{24}{50}H, \qquad G_{10} = \frac{24}{50}H$$

Direct stress in columns,

$$C_7 = \frac{3}{25}H + \frac{24}{50}H = +\frac{6}{10}H$$

$$C_8 = -\frac{H}{50} - \frac{24}{50}H + \frac{20}{50}H = -\frac{1}{10}H$$

$$C_9 = -\frac{H}{25} - \frac{4}{10}H + \frac{24}{100}H = -\frac{2}{10}H$$

$$C_{10} = +\frac{3}{50}H - \frac{24}{100}H + \frac{24}{50}H = +\frac{3}{10}H$$

$$C_{11} = 0$$

$$C_{12} = -\frac{3}{25}H - \frac{24}{50}H = -\frac{6}{10}H.$$

For a check, take moment about point of inflexion in column C_{12} of direct stresses and shears in columns. The resulting equation which follows shows that this moment equals zero as should be the case:

$$+12H + 12H - \tfrac{12}{25}H \times 62 + \tfrac{4}{50}H \times 52 + \tfrac{4}{25}H \times 40 - \tfrac{12}{50}H \times 20 = 0$$

267. Illustrative Examples.—The various approximate methods of solution are clearly illustrated by the following examples.

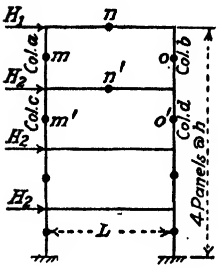


FIG. 412.

In Figs. 412–423, assumed positions of points of contraflexure are shown by solid black circles.

Problem: Determine direct stress and shear due to loads shown, in column section c of structure shown in Fig. 412, using assumptions[1] a and b; also in girder connecting these sections.

Solution: Consider first the equilibrium of the portion of the structure above a horizontal plane through the assumed points of inflexion m and o in columns a and b.

Let H_a, H_b, V_a, and V_b (Fig. 413) equal unknown direct shears and direct stresses in columns a and b at points of inflexion

It follows from $\sum M = 0$ about point o that

$$\frac{H_1h}{2} - V_aL = 0.$$

Therefore,

$$V_a = \frac{H_1h}{2L}$$

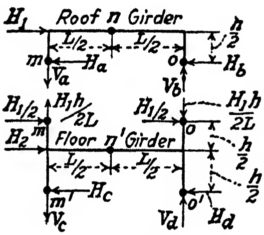


FIG. 413.

Next, apply $\Sigma M = 0$ about point n of forces to left of vertical through n.

This gives

$$\frac{H_ah}{2} - \frac{V_aL}{2} = 0$$

Substituting in the expression above the value of V_a previously found gives

$$\frac{H_ah}{2} - \frac{H_1h}{2L} \cdot \frac{L}{2} = 0$$

Therefore,

$$H_a = \frac{H_1}{2} = H_b$$

[1] The application of assumptions a, b, and c to a general case is given in the Bethlehem Steel Company's handbook entitled Bethlehem Structural Shapes, p. 176 *et. seq.*, 1926.

The foregoing expression is equivalent to that obtained by applying assumption *c*.

Next, consider the portion of the structure lying between two horizontal planes passing respectively through hinges m and o and m' and o'.

Applying $\Sigma M = 0$ about o' gives the following expression:

$$\frac{3H_1h}{2} + \frac{H_2h}{2} - V_cL = 0$$

Therefore,

$$V_c = \frac{3H_1h}{2L} + \frac{H_2h}{2L}$$

The foregoing value may also be computed by taking moments about o' of *all* forces on structure above plane through $m'o'$.

Now, applying $\Sigma M = 0$ about n' of forces to left of vertical through n' gives

$$\frac{H_1}{2}\cdot\frac{h}{2} + \frac{H_1h}{2L}\cdot\frac{L}{2} - \frac{V_cL}{2} + \frac{H_ch}{2} = 0$$

Therefore,

$$H_c = \frac{V_cL}{h} - H_1$$

which, when value of V_c is substituted, gives

$$H_c = \frac{3H_1}{2} + \frac{H_2}{2} - H_1 = \frac{H_1 + H_2}{2}$$

The foregoing expression is equivalent to that obtained by applying assumption *c*.

The direct stresses in the girders may be found by applying $\Sigma H = 0$ to the portion of the structure to the left of the vertical plane through the floor-girder hinge and above the appropriate horizontal plane.

Let S_1 = direct compression in girder.

Then

$$\frac{H_1}{2} + H_2 - S_1 - \frac{H_1 + H_2}{2} = 0$$

Therefore,

$$S_1 = \frac{H_2}{2}$$

The moment at each end of the girder equals the moment in the column at the end of the girder. For the floor girder this moment equals

$$\left(\frac{H_1}{2} + \frac{H_1 + H_2}{2}\right)\frac{h}{2} = (2H_1 + H_2)\frac{h}{4}$$

The shear in this girder equals its moment divided by $L/2$ =

$$(2H_1 + H_2)\left(\frac{h}{4}\right)\left(\frac{2}{L}\right) = (2H_1 + H_2)\frac{h}{2L}$$

The moment curve in the floor girder will be as shown in Fig. 414.

The left column between hinges m and m' will be subjected to the forces shown in Fig. 415 in which the line $mabm'$ is the curve of moments drawn in such a manner that compression in outer fiber occurs when the moment curve is on the same side of the member as the fiber under consideration.

The stresses in the other members can be computed in a similar manner to those already computed.

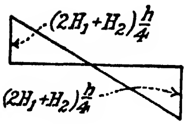


FIG. 414.

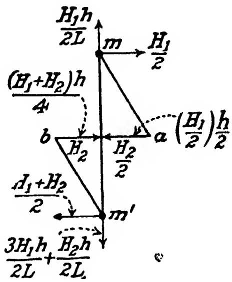


FIG. 415.

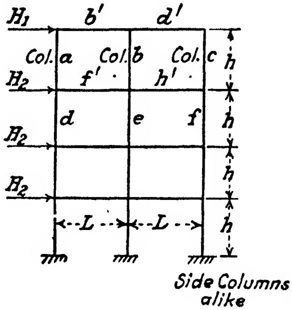


FIG. 416.

Problem: Determine direct stress and shear due to loads shown in columns d, e, and f and in connecting girders f' and h' of the structure shown in Fig. 416, using assumptions a, b, and c.

Solution: Application of assumption c shows that the direct stress in the center column equals zero and that the direct stresses in the side columns are equal. The condition of the portion of the structure above a horizontal plane passing through the points of inflexion in the upper column sections will be as shown in Fig. 417.

Applying $\Sigma M = 0$ about m' of all forces on the structure above the horizontal plane MM' gives

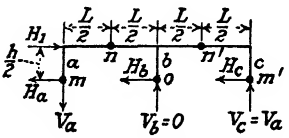


FIG. 417.

$$\frac{H_1h}{2} - 2V_aL = 0$$

Therefore,

$$V_a = \frac{H_1h}{4L}$$

Applying $\Sigma M = 0$ about point n of forces to the left of the vertical plane through n gives

$$\frac{H_ah}{2} - \frac{V_aL}{2} = 0$$

Hence,

$$H_a = \frac{V_aL}{h} = \frac{H_1}{4}$$

Applying $\Sigma M = 0$ about point n' of forces to right of vertical plane through n' gives

$$\frac{H_c h}{2} - \frac{V_c L}{2} = 0$$

Hence,

$$H_c = \frac{V_c L}{h} = \frac{V_a L}{h} = \frac{H_1}{4} = H_a$$

Applying $\Sigma H = 0$ gives $H_b = H_1/2$.

Note that the foregoing values of H_a, H_b, and H_c correspond also to the values which would be obtained by the application of assumption *d*, in place of assumption *c*.

Consider next the portion of the structure between horizontal planes passing through hinges in the upper story and the next story below.

The condition of this portion of the structure will be as shown in Fig. 418.

Applying $\Sigma M = 0$ about m''' of all forces above plane $m''m'''$ gives

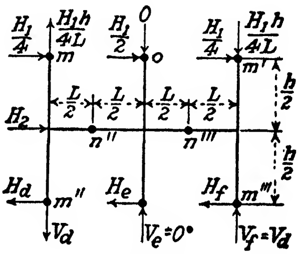


FIG. 418.

$$\frac{3H_1h}{2} + \frac{H_2h}{2} - V_d(2L) = 0$$

Therefore,

$$V_d = \frac{3H_1h + H_2h}{4L}$$

Applying $\Sigma M = 0$ about n'' of all forces to left of vertical section through n'' gives

$$\frac{H_1h}{8} - \frac{3H_1h + H_2h}{8} + \frac{H_1h}{8} + \frac{H_dh}{2} = 0$$

Therefore,

$$H_d = \frac{H_1 + H_2}{4}$$

Similarly, by considering portion of structure to the right of the vertical through n''' we obtain

$$H_f = \frac{H_1 + H_2}{4} = H_d$$

It follows that $H_e = H_1 + H_2 - H_d - H_f = \dfrac{H_1 + H_2}{2}$

Conditions in the column sections in other stories may be investigated in a similar manner.

It is evident that for the case under consideration, *viz.*, a three-column bent with columns equispaced, assumptions *a*, *b*, and *c* and *a*, *b*, and *d* give identical results.

The stresses and shears for the structure shown in Fig. 416 may be found as in the previous example.

With reference to Fig. 418:

Let S_1 = direct compression in girder at n''.

S_2 = direct compression in girder at n'''.

V_1 = vertical force acting up at n'' on portion of girder to left of n''.

V_2 = vertical force acting up at n''' on portion of girder to left of n'''.

Applying $\Sigma H = 0$ to portion of structure to left of vertical plane through n'' gives

$$H_2 + \frac{H_1}{4} - \frac{H_1 + H_2}{4} - S_1 = 0$$

Therefore,

$$S_1 = \frac{3H_2}{4}$$

Applying $\Sigma H = 0$ to portion of structure to right of vertical plane through n''' gives

$$\frac{H_1}{4} + S_2 - \frac{1}{4}(H_1 + H_2) = 0$$

Therefore,

$$S_2 = \frac{H_2}{4}$$

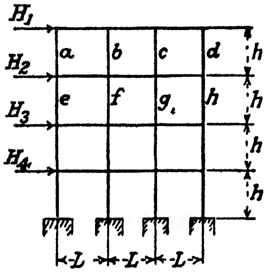
Applying $\Sigma V = 0$ to portion of structure to left of vertical plane through n'' gives

$$\frac{H_1h}{4L} - \frac{3H_1h + H_2h}{4L} + V_1 = 0$$

Therefore, since no vertical force exists in center column

$$V_1 = \frac{H_2h}{4L} + \frac{H_1h}{2L} = V_2$$

The moment at any section of girders or columns may now be readily computed.



Area of center columns is to Area of side columns as 3:2

FIG. 419.

Problem: Compute direct stresses and shears in columns and girders in two upper stories of structure shown in Fig. 419, using assumptions a, b, and c.

Solution: From assumption c the direct unit stress in an exterior column is to the direct unit stress in an interior column as 3 to 1; hence, the total column stress in an exterior column is to that in an interior column as 6 to 3.

Let V = total stress in each intermediate column of upper story.

The conditions in the upper portion of structure will then be as shown in Fig. 420.

Applying $\Sigma M = 0$ about m' of all forces on structure gives the following expression:

$$\frac{H_1h}{2} - 7VL = 0$$

Therefore,

$$V = \frac{H_1h}{14L}$$

Now applying $\Sigma M = 0$ about n of forces to left of vertical section through n gives

$$\frac{2VL}{2} - \frac{H_ah}{2} = 0$$

Therefore,

$$H_a = \frac{2VL}{h} = \frac{H_1}{7}$$

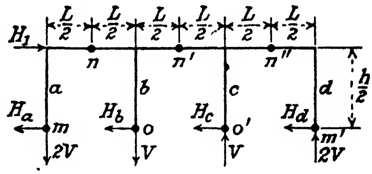


FIG. 420.

and applying $\Sigma M = 0$ about n' of forces to left of vertical section through n' gives

$$2V \cdot \frac{3L}{2} + \frac{VL}{2} - (H_a + H_b)\frac{h}{2} = 0$$

Therefore,

$$H_a + H_b = 7\frac{VL}{h} = \frac{H_1}{2}$$

Therefore,

$$H_b = \frac{H_1}{2} - \frac{H_1}{7} = \frac{5}{14}H$$

Also,

$$H_c + H_d = \frac{H_1}{2}$$

Now, applying $\Sigma M = 0$ about n'' of forces to right of vertical through n'' gives the following:

$$H_d = \frac{2VL}{h} = \frac{H_1}{7}$$

but

$$H_c + H_d = \frac{H_1}{2}$$

Therefore,

$$H_c = \tfrac{5}{14}H_1$$

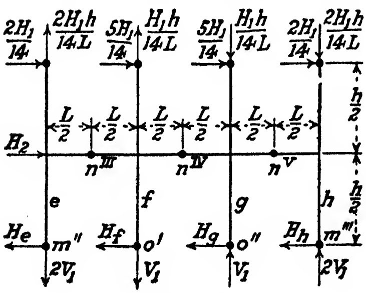


FIG. 421.

With the shears in the columns in the upper story determined, the portion of the structure between horizontal planes through hinges in the columns of the upper story and next story below may be considered. The condition of this portion of the structure will be as shown in Fig. 421.

Applying $\Sigma M = 0$ about m''' of all forces above plane $m''m'''$ gives

$$\frac{3H_1h}{2} + \frac{H_2h}{2} = 7V_1L = 0$$

Therefore,

$$V_1 = \frac{3}{14}\frac{H_1h}{L} + \frac{1}{14}\frac{H_2h}{L}$$

Now, applying $\Sigma M = 0$ about n''' of forces to left of vertical through n''', after substituting above value of V_1, gives

$$\frac{H_eh}{2} + \frac{2}{14}\frac{H_1h}{2} - \frac{4}{14}\frac{H_1h}{2} - \frac{2}{14}\frac{H_2h}{2} = 0$$

Therefore,

$$H_e = \tfrac{2}{4}H_2 + \tfrac{2}{14}H_1 = \tfrac{1}{7}(H_1 + H_2)$$

Similarly, by taking moments about n^v of forces to right of vertical section through n^v we get

$$H_h = \tfrac{1}{7}(H_1 + H_2)$$

Now, by taking moments about n^{iv} of forces to left of vertical through n^{iv} we get, after substituting for V_1 its value as already determined,

$$(H_e + H_f)\frac{h}{2} + \frac{H_1}{2}\cdot\frac{h}{2} - \left(\frac{3}{14}\frac{H_1h}{L} + \frac{1}{14}\frac{H_2h}{L}\right)3\frac{1}{2}L + \frac{H_1h}{14L}\left(3\frac{1}{2}L\right) = 0$$

Therefore,

$$(H_e + H_f)\frac{h}{2} + \frac{H_1h}{4} - \frac{3}{4}H_1h - \frac{1}{4}H_2h + \frac{1}{4}H_1h = 0$$

Therefore,

$$H_e + H_f = \tfrac{1}{2}(H_1 + H_2)$$

Hence,

$$H_f = \tfrac{5}{14}(H_1 + H_2)$$

Application of $\Sigma H = 0$ now gives

$$H_g = \tfrac{5}{14}(H_1 + H_2)$$

The other stories in the bent may now be computed in a similar manner, but, inasmuch as it is clear from the foregoing that the column shears in each story are divided between the interior columns and side columns in the ratio of 5 to 2, their values may be written at once without detailed computation. It is evident, however, that assumptions a, b, and c give different results in this case from assumptions a, b, and d.

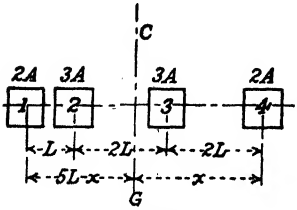


FIG. 422.

Had the column spacing been unequal, it would have been necessary to locate the axis passing through the centroid of the column areas and determine the relative values of the direct stresses in the columns with reference to this axis. For example, if the column area and the spacing had been as shown in Fig. 422, the position of the axis CG

passing through the centroid of the column areas would be given by following equation:

$$2A(5L) + 3A(6L) - 10Ax = 0$$

Therefore,

$$x = {}^{28}\!/_{10}L$$

It follows that, if v equals unit stress at distance L from CG, the total stresses in the various columns would be as follows:

Column 1, $2Av\left(\frac{5L - 2.8L}{L}\right) = 4.4Av$

Column 2, $\frac{3Av(4L - 2.8L)}{L} = 3.6Av$

Column 3, $\frac{3Av(2.8L - 2L)}{L} = 2.4Av$

Column 4, $2Av\frac{2.8L}{L} = 5.6Av$

The remaining computations may now be carried out as before.

Problem: Determine moments, shears, and direct stresses in girders and columns of the steel framework of the portion shown in Fig. 423 of a 30-story building, using assumptions a, b, and d. The bent under consideration is an intermediate bent; *i.e.*, it carries the wind pressure on a surface 20 ft. in length. Entire wind stresses are assumed to be carried by the steel framework.

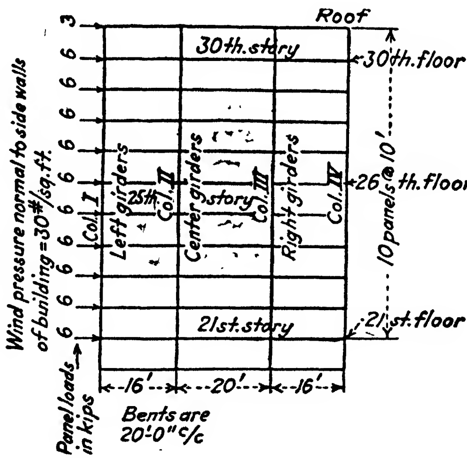


FIG. 423.

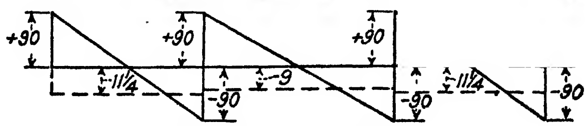
Solution: Wind loads at intermediate panel points equal $30 \times 10 \times 20 = 6{,}000$. The required stresses may most readily be obtained by using tabulated forms such as follow:

COLUMNS—SHEAR AND MOMENT

Story	Total shear in story	Shear in columns = S — All shears positive, *i.e.*, acting to left on portion above section		Moment in columns, ft.-lb. = product of S and one-half story height (positive sign corresponds to compression in fiber in right-hand side of column)			
				Columns I and IV		Columns II and III	
		Columns I and IV	Columns II and III	Top	Bottom	Top	Bottom
30	3	½	1	− 2½	+ 2½	− 5	+ 5
29	9	1½	3	− 7½	+ 7½	− 15	+ 15
28	15	2½	5	−12½	+12½	− 25	+ 25
27	21	3½	7	−17½	+17½	− 35	+ 35
26	27	4½	9	−22½	+22½	− 45	+ 45
25	33	5½	11	−27½	+27½	− 55	+ 55
24	39	6½	13	−32½	+32½	− 65	+ 65
23	45	7½	15	−37½	+37½	− 75	+ 75
22	51	8½	17	−42½	+42½	− 85	+ 85
21	57	9½	19	−47½	+47½	− 95	+ 95
20	63	10½	21	−52½	+52½	−105	+105

GIRDERS—SHEAR, MOMENT, DIRECT STRESS

Floor	Moments for all girders, end of girder		Shears, all minus		Direct stress, all compression		
	Left end (+)	Right end (−)	End girders	Center girders	Left girder	Center girder	Right girder
Roof	2½	2½	5/16	¼	3 − ½ = 2½	2½ − 1 = 1½	½
30	2½ + 7½ = 10	10	10/8 = 1¼	1	6 + ½ − 1½ = 5	5 + 1 − 3 = 3	1½ − ½ = 1
29	7½ + 12½ = 20	20	20/8 = 2½	2	6 + 1½ − 2½ = 5	5 + 3 − 5 = 3	2½ − 1½ = 1
28	12½ + 17½ = 30	30	30/8 = 3¾	3	5	3	1
27	17½ + 22½ = 40	40	40/8 = 5	4	5	3	1
26	22½ + 27½ = 50	50	50/8 = 6¼	5	5	3	1
25	27½ + 32½ = 60	60	60/8 = 7½	6	5	3	1
24	32½ + 37½ = 70	70	70/8 = 8¾	7	5	3	1
23	37½ + 42½ = 80	80	80/8 = 10	8	5	3	1
22	42½ + 47½ = 90	90	90/8 = 11¼	9	5	3	1
21	47½ + 52½ = 100	100	100/8 = 12½	10	5	3	1



Full lines show moment curves
Dotted lines show shear curves

FIG. 424.

The foregoing moments and shears are illustrated by Fig. 424, showing curves of shears and moments for the girder at the twenty-second floor that are typical of corresponding curves for all girders.

COLUMNS—DIRECT STRESS

Story	Column I	Column II	Column III	Column IV
30	$+\ 5/16$	$-\ 1/16$	$+\ 1/16$	$-\ 5/16$
29	$5/16 + 20/16 = +25/16$	$-1/16 - 20/16 + 16/16 = -5/16$	$+5/16$	$-25/16$
28	$25/16 + 40/16 = +65/16$	$-5/16 - 40/16 + 32/16 = -13/16$	$+13/16$	$-65/16$
27	$65/16 + 60/16 = +125/16$	$-13/16 - 60/16 + 48/16 = -25/16$	$+25/16$	$-125/16$
26	$125/16 + 80/16 = +205/16$	$-25/16 - 80/16 + 64/16 = -41/16$	$+41/16$	$-205/16$
25	$205/16 + 100/16 = +305/16$	$-41/16 - 100/16 + 80/16 = -61/16$	$+61/16$	$-305/16$
24	$305/16 + 120/16 = +425/16$	$-61/16 - 120/16 + 96/16 = -85/16$	$+85/16$	$-425/16$
23	$425/16 + 140/16 = +565/16$	$-85/16 - 140/16 + 112/16 = -113/16$	$+113/16$	$-565/16$
22	$565/16 + 160/16 = +725/16$	$-113/16 - 160/16 + 128/16 = -145/16$	$+145/16$	$-725/16$
21	$725/16 + 180/16 = +905/16$	$-145/16 - 180/16 + 144/16 = -181/16$	$+181/16$	$-905/16$

The column stresses in the twenty-first story may be checked by applying $\Sigma M = 0$ around point of inflexion in the right-hand column of twenty-first story. The resulting expression is as follows:

$$3 \times 95 + 6(5 + 15 + 25 + 35 + 45 + 55 + 65 + 75 + 85) - 905/16 \times 52 + 181/16 \times 20 = 285 + 2{,}430 - 2{,}715 = 0$$

268. Stresses Due to Combination of Live, Dead, and Wind Forces.—According to the assumptions previously made, the bending moment in a girder due to wind forces varies from zero at midspan to a maximum at point of intersection of girder and column axis. It follows that the moment to be carried at any given joint between girder and column will be less than the foregoing maximum moment in proportion to the distance from the column axis to the point of application of the girder reaction. The point of connection of the girder to the column must, however, carry into the column the full amount of this maximum bending moment.

In the case of a riveted steel framework, the live and dead moments and shears are often computed as if the girders were discontinuous, and the bending moments at the ends are taken as those due to wind only. This is not the actual condition with respect to live and dead moments as the end connections are capable of carrying bending moments of considerable size even when designed to carry wind stresses only and the actual condi-

tions at a joint correspond to those existing in beams with fixed or partially fixed ends. A joint computed in this manner may, therefore, be overstressed when maximum wind forces are acting, the actual moment being that at the joint due to the wind forces plus that due to such negative moment in the girder as may be developed by the condition of fixedness of its ends.

In case the joint is figured for wind only, and by chance the maximum live and dead loads occur simultaneously with the maximum wind forces assumed in the design, an overstrain of the gusset plate and rivets at the joint might result, causing the elastic limit of the metal to be exceeded, thus distorting the joint, and thereby relieving the stresses somewhat and bringing the girders more nearly into the condition of end-supported beams.

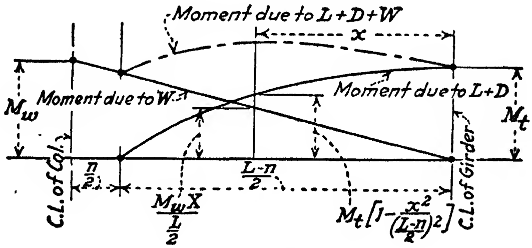


FIG. 425.

The actual moment at any portion of a girder due to a combination of live, dead, and wind forces depends upon conditions;[1] *e.g.*, consider an intermediate beam having a span of L ft. center to center of supports, and assume the maximum moment due to live and dead loads to be equivalent to that on an end-supported span having a length of $L - n$ and uniformly loaded. Now, if we let

M = total moment on girder at any distance x from center,
M_w = moment due to wind forces at end of girder, *i.e.*, at intersection of central axes of girder and column,
M_t = moment at center of girder due to live and dead loads

we may write the following expression for M:

$$M = M_w \frac{x}{\frac{L}{2}} + M_t\left[1 - \frac{x^2}{\left(\frac{L-n}{2}\right)^2}\right]$$

For maximum value of M, $dM/dx = 0$.

[1] See Fig. 425.

Therefore,

$$\frac{M_w}{\frac{L}{2}} - \frac{2M_t x}{\left(\frac{L-n}{2}\right)^2} = 0$$

whence

$$x = \frac{\left(\frac{L-n}{2}\right)^2}{L} \frac{M_w}{M_t}$$

Substituting this value of x in equation for M gives,

$$M_{\text{max.}} = 2\frac{\left(\frac{L-n}{2}\right)^2}{L^2} \frac{M_w^2}{M_t} + M_t - \frac{\left(\frac{L-n}{2}\right)^2}{L^2} \frac{M_w^2}{M_t}$$

$$= \frac{1}{M_t}\left[\frac{\left(\frac{L-n}{2}\right)^2}{L^2} M_w^2\right] + M_t$$

If f_1 = allowable stress due to combined wind and floor load and f = allowable stress due to floor load, and if we let $f_1 = cf$, the section modulus of the beam will have to be whichever of the two following expressions gives the larger value:

$$\frac{\frac{\left(\frac{L-n}{2}\right)^2}{L^2} \frac{M_w^2}{M_t} + M_t}{cf}$$

or

$$\frac{M_t}{f}$$

If the foregoing expressions are equal,

$$\frac{\left(\frac{L-n}{2}\right)^2}{L^2} \frac{M_w^2}{M_t} = M_t(c-1)$$

Therefore,

$$\frac{L-n}{2L} M_w = M_t\sqrt{c-1}$$

Therefore,

$$M_w = \frac{2L}{L-n} \cdot M_t\sqrt{c-1}$$

That is, bending moment due to wind stress need be considered in the design of the girder only when the maximum girder moment due to wind (at intersection of center line of column and girder) is greater than $\left(\frac{2L}{L - n}\right)\sqrt{c - 1}$ times the maximum moment due to live and dead floor loads; *e.g.*, for the case where $L = 16$ ft. and $n = 2$ ft. and $c = 1\frac{1}{2}$ no account need be taken of the bending due to wind when

$$M_w \lessgtr \tfrac{32}{14}\sqrt{\tfrac{1}{2}}M_t$$

or when

$$M_w \lessgtr 1.6M_t$$

Various values of the ratio M_w/M_t are given below for spans of different lengths and for different assumed values of n, it being assumed that $c = 1\frac{1}{2}$.

Span length, feet	Ratio					
	$n = 0$	$n = 1.0$	$n = 1.2$	$n = 1.4$	$n = 1.5$	$n = 1.6$
14	1.42	1.52	1.55	1.57	1.585	1.60
15	1.42	1.515	1.535	1.56	1.57	1.58
16	1.42	1.51	1.53	1.55	1.56	1.57
17	1.42	1.50	1.52	1.54	1.55	1.56
18	1.42	1.50	1.515	1.53	1.54	1.55
19	1.42	1.49	1.51	1.525	1.535	1.54
20	1.42	1.49	1.50	1.52	1.53	1.54

Maximum values of moments for cases where wind force has to be considered can be readily determined from the foregoing equation.

In the case of reinforced-concrete buildings, it is necessary to take full account of the continuity, and in steel frameworks it is desirable; hence, in such cases the girder moment at the column should be taken as the algebraic sum of the wind moment at column plus the negative moment due to continuity. The point of maximum moment can be readily determined by a similar method to that used for steel structures, as well as the value of this moment and the ratio between moment due to wind and that due to floor loads.

With respect to amount of bracing, the upper few stories of a building are sometimes assumed as being supported rigidly by the building partitions and walls, depending, of course, upon the character of these various portions of the structure.

269. Wind Pressure Specified in Building Laws.—Building laws of various cities vary in their requirements, both as to amount of wind pressure, proportions of buildings in which wind bracing is required, allowable unit stresses, etc. Requirements for certain large cities follow:

New York (1938).—All structures or parts of structures, signs and other exposed structures shall be designed to resist, in the structural frame, horizontal wind pressure from any direction.

When the height of a structure is over 100 ft., the assumed wind pressure shall be 20 lb. per square foot of exposed surface from the top of the structure down to the 100-ft. level.

All structures 100 ft. high or less shall be investigated as to the need for wind bracing; but, in general, wind pressure in such structures may be neglected. All structures, 200 ft. or less in height, in which the height is more than 2½ times the least width, mill buildings, shops, roofs over auditoriums or drill sheds, and structures of similar character, shall be designed to withstand an assumed wind pressure of 20 lb. per square foot on the upper 50 per cent of their height.

Boston (1924).—Wind force is to be taken as follows on vertical surfaces:

Building below 40 ft. in height, 10 lb. per square foot.

Buildings between 40 and 80 ft. in height, 15 lb. per square foot.

Buildings over 80 ft. in height, 20 lb. per square foot.

The allowable unit stresses for wind only, or for a combination of live, dead, and wind stresses may exceed by 20 per cent the allowable stresses for live and dead loads only.

Philadelphia (1925).—Wind force on vertical surface for buildings in built-up district, 25 lb. per square foot at tenth story, reduced by 2½ lb. for each succeeding lower story, and increased by 2½ lb. for each succeeding upper story, to a maximum of 35 lb.

Allowable unit stress for combined live, dead, and wind forces may exceed by 30 per cent that for live and dead loads.

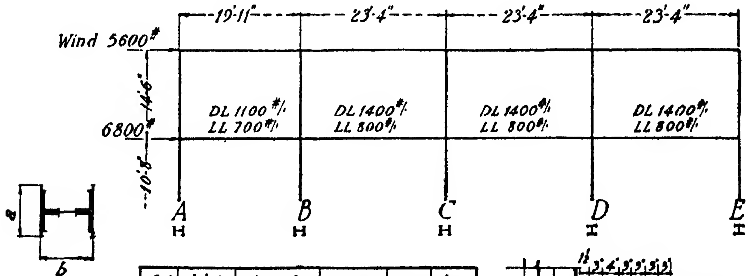
Chicago (July 1, 1924).—Wind force to be taken as 20 lb. per square foot on vertical surface. Allowable unit stress for com-

bined live, dead, and wind forces may exceed by 50 per cent that for live and dead stresses.

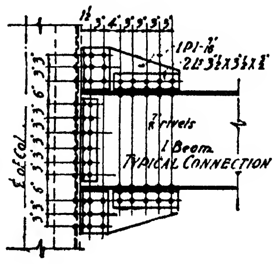
270. Exact Method of Solution.—The stresses in a framed bent may be computed with theoretical exactness by the method of least work or the method of slope deflection, although the solution may be laborious and impracticable to apply in most cases. Solutions by either of these methods may, however, be made for typical cases to determine the degree of approximation of the methods given in the previous articles of this chapter and to obtain, in certain cases, exact methods of solution. The moment-distribution method is now in common use for live and dead loads, and gives results sufficiently precise for practical design. See Chap. XV for the solution of certain simple cases by all these methods.

Problems

98. Determine shears, moments, and direct stresses in columns and girders of the upper five stories of the building shown in Fig. 423, using assumptions a, b, and c, and compare results with those obtained in Art. 267.



Col.	Web	4 angles	2 Plates	a	b
A	$14''\times\frac{5}{8}''$	$6''\times4''\times\frac{5}{8}''$	$16''\times\frac{5}{8}''$	$16''$	$16\frac{1}{2}''$
B	$12''\times\frac{5}{8}''$	$6''\times4''\times\frac{5}{8}''$	$16''\times1\frac{5}{16}''$	$16''$	$16\frac{1}{2}''$
C	$12''\times\frac{5}{8}''$	$6''\times4''\times\frac{5}{8}''$	$16''\times1\frac{7}{16}''$	$16''$	$16\frac{1}{2}''$
D	$16''\times\frac{5}{8}''$	$8''\times6''\times\frac{3}{4}''$	$18''\times\frac{7}{8}''$	$18''$	$18\frac{1}{2}''$
E	$12''\times\frac{5}{8}''$	$6''\times4''\times\frac{5}{8}''$	$16''\times1''$	$16''$	$16\frac{1}{2}''$



PROB. 100.

99. Apply assumptions a, b, and c, also assumptions a, b, and d, to structure shown in Fig. 338, and compare moments at ends of various members for each set of assumptions with those obtained by the slope-deflection method. Assume all columns areas to be equal.

100. Determine the weight of I beams for floor girders in span BC, using beams 18 in. in depth. Allowable unit stress, live and dead loads only, 16,000 lb. per square inch. For live, dead, and wind combined, unit stress may be increased by 33⅓ per cent. Use assumptions a, b, and d.

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